Nonlinear Functional Analysis and Applications
Vol. 28, No. 2 (2023), pp. 497-520
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.02.11
http://nfaa.kyungnam.ac.kr/journal-nfaa

INERTIAL EXTRAPOLATION METHOD FOR SOLVING SYSTEMS OF MONOTONE VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS USING BREGMAN DISTANCE APPROACH

Hammed A. Abass[1], Ojen K. Narain[2] and Olayinka M. Onifade[3]

[1]School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban, South Africa;
Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Science University,
P.O. Box 94, Pretoria 0204, South Africa
DSI-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS)
e-mail: AbassH@ukzn.ac.za, hammed.abass@smu.ac.za, hammedabass548@gmail.com

[2]School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban, South Africa
e-mail: naraino@ukzn.ac.za

[3]School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban, South Africa
e-mail: 222121238@stu.ukzn.ac.za, martinsolayinka11@gmail.com

Abstract. Numerous problems in science and engineering defined by nonlinear functional equations can be solved by reducing them to an equivalent fixed point problem. Fixed point theory provides essential tools for solving problems arising in various branches of mathematical analysis, such as split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving the existence of solution of integral and differential equations.The theory of fixed is known to find its applications in many fields of science and technology. For instance, the whole world has been profoundly impacted by the novel Coronavirus since 2019 and it is imperative to depict the spread of the coronavirus. Panda et al. [24] applied fractional derivatives to improve the 2019-nCoV/SARS-CoV-2 models, and by means of fixed point theory, existence and uniqueness of solutions of the models were proved. For more information on applications of fixed point theory to real life problems, authors should (see [6, 13, 24] and the references contained in).

[0]Received September 22, 2022. Revised October 15, 2022. Accepted October 17, 2022.
[0]2020 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25.
[0]Keywords: Monotone variational inclusion problem, Bregman relatively mapping, multi-valued, iterative scheme, fixed point problem.
[0]Corresponding author: H. A. Abass(hammedabass548@gmail.com).

1. INTRODUCTION

Numerous problems in science and engineering defined by nonlinear functional equations can be solved by reducing them to an equivalent fixed point problem. Fixed point theory provides essential tools for solving problems arising in various branches of mathematical analysis, such as split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and problems of proving the existence of solution of integral and differential equations.The theory of fixed is known to find its applications in many fields of science and technology. For instance, the whole world has been profoundly impacted by the novel Coronavirus since 2019 and it is imperative to depict the spread of the coronavirus. Panda et al. [24] applied fractional derivatives to improve the 2019-nCoV/SARS-CoV-2 models, and by means of fixed point theory, existence and uniqueness of solutions of the models were proved. For more information on applications of fixed point theory to real life problems, authors should (see [6, 13, 24] and the references contained in).

Let E be a real Banach space with its dual E^*. The Monotone Variational Inclusion Problem (MVIP) is to find $x^* \in E$ such that

$$0 \in (U+V)x^*, \tag{1.1}$$

where $U : E \to E^*$ is a single-valued monotone mapping and $V : E \to 2^E$ is a multi-valued monotone mapping. We denote the set of solution of (1.1) by $MVIP(U,V) = (U+V)^{-1}(0^*)$. MVIP (1.1) has received considerable attention due to its wide theoretical value in nonlinear analysis or optimization theory and wide spectrum of applications such as image reconstruction, machine learning and signal processing. It is also known that MVIP (1.1) has been an important tool for solving problems arising in mechanics, finance, applied sciences, among others (see [1, 2, 3, 5]). If $U = 0$ in (1.1), we obtain the following Variational Inclusion Problem (VIP), which is to find $x^* \in E$ such that

$$0 \in Vx^*. \tag{1.2}$$

The classical method for solving MVIP (1.2) is the following forward-backward splitting method (see [1, 17, 29]): for any $x_1 \in E$ and $\lambda > 0$

$$x_{n+1} = \mathrm{Res}_\lambda^V \circ U_\lambda(x_n), \ \forall\ n \geq 1,$$

where $\mathrm{Res}_\lambda^V := (I + \lambda V)^{-1}$ is the resolvent of V, $U_\lambda := I - \lambda U$ where I denotes the identity mapping on E. The forward-backward splitting method is known to include as special cases of the proximal point algorithm (PPA), (when $U = 0$) and the gradient method, (see [15, 16]). However, from the numerical point of view, the weak convergence of this method is not desirable and enough to make it efficient. Several results have been presented to solve MVIP in the framework of real Hilbert spaces. For instance, Zhang and Wang [35] introduced a contraction algorithm for solving MVIP (1.1) and proved that the sequence generated by their algorithm converges weakly to the solution of problem (1.1).

In the framework of real Hilbert space, Tianchai *et al.* [30] introduced the following iterative method for approximating solution of (1.1) and fixed point problem of a nonexpansive mappings: find $x_0, x_1 \in C$ and let $\{x_n\} \subset C$ be generated by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = S(\alpha_n f(x_n) + (1-\alpha_n)J_{\lambda_n})(y_n - \lambda_n U y_n + e_n), \end{cases}$$

for all $n \in \mathbb{N}$, where C is a nonempty, closed and convex subset of a real Hilbert space H, U is an α-inverse strongly monotone and V is a maximal monotone operator on H such that the domain of V is included in C. Let $J_{\lambda_n} = (I+V)^{-1}$ be the resolvent of V for $\lambda > 0$ and $S : C \to C$ be a nonexpansive mapping. Then they proved that the sequence $\{x_n\}$ converges strongly to a point x^* in their solution set. Also, Wei and Duan [33] extended the results of Lopez et al. [20] from uniformly convex and q-uniformly smooth Banach spaces to uniformly smooth and uniformly convex Banach spaces.

Furthermore, in the framework of p-uniformly convex real Banach spaces which are also uniformly smooth, Okeke and Izuchukwu [23] studied and analysed a Halpern iterative method for split feasibility problem and zero of the sum of two monotone operators and proved a strong convergence result for approximating the solution of the aforementioned problems when $U : E \to E^*$ is a single-valued nonlinear mapping and $V : E \to 2^{E^*}$ is a multi-valued mapping.

Very recently, Ogbuisi and Izuchukwu [22] introduced the following shrinking iterative method for approximating a point in the set of zeros of the sum of two monotone operators, which is also a solution of a fixed point problems for a Bregman strongly nonexpansive mapping in a real reflexive Banach space. For $u, x_0 \in E$ are arbitrary, the sequence $\{x_n\}$ is generated by

$$\begin{cases} C_0 = C, \\ y_n = \triangledown g^*(\alpha_n \triangledown g(u) + \beta_n \triangledown g(x_n) + \gamma_n \triangledown g(T(x_n))), \\ u_n = (Res_{\lambda V^g} \circ U^g_\lambda) y_n, \\ C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq \alpha_n D_g(z, u) + (1-\alpha_n) D_g(z, x_n)\}, \\ x_{n+1} = P^g_{C_{n+1}}(x_0), \quad n \geq 0, \end{cases}$$

with conditions $\lim_{n\to\infty} \alpha_n = 0, \alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $P^g_{F(T)\cap MVIP(1.1)}(x_0)$. One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term is represented by $\theta_n(x_n - x_{n-1})$ and is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. For growing interests in this direction (see [1, 3, 7, 25]).

The idea of inertial extrapolation method was first introduced by Polyak [25] and was inspired by an implicit discretization of a second-order-in-time dissipative dynamical system, so -called "Heavy Ball with Friction"

$$v''(t) + \gamma v'(t) + \bigtriangledown f(v(t)) = 0, \tag{1.3}$$

where $\gamma > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. System (1.3) is discretized so that, having the terms x_{n-1} and x_n, the next term x_{n+1} can be determined using

$$\frac{x_{n-1} - 2x_n + x_{n-1}}{j^2} + \gamma\frac{x_n - x_{n-1}}{j} + \bigtriangledown f(x_n) = 0,\ n \geq 1, \tag{1.4}$$

where j is the step-size. Equation (1.4) yields the following iterative algorithm:

$$x_{n+1} = x_n + \beta(x - n - x_{n-1}) - \alpha \bigtriangledown f(x_n),\ n \geq 1, \tag{1.5}$$

where $\beta = 1 - \gamma_j, \alpha = j^2$ and $\beta(x_n - x_{n-1})$ is called the inertial extrapolation term which is intended to speed up the convergence of the sequence generated by (1.5).

Motivated by the aforementioned results and other results in the literature, we introduced a modified inertial Halpern method for approximating solution of systems of monotone variational inclusion problem and fixed point problem of a finite family of multi-valued Bregman relatively nonexpansive mapping in a reflexive Banach space. We establish a strong convergence result for approximating common solutions for systems of zeros of sum of maximal monotone operators involving a Bregman inverse strongly monotone operators and fixed point equation for finite family of a multi-valued relative nonexpansive mapping in a reflexive Banach space. Several consequences and application to other optimization problems were discussed. Our result improves and extends some important results presented by authors in literature.

We state our contributions in this article as follows:

(1) We were able to dispense with the condition $\sum_{n=1}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty$ which is often used when employing the inertial method during the course of obtaining our strong convergence result, (see [1]).

(2) The class of mapping employed in our iterative algorithm generalizes the ones in [1].
(3) The result discussed in this article extends and generalizes the results of [1, 14, 19, 34] from Hilbert spaces and 2-uniformly convex Banach spaces to reflexive Banach spaces.
(4) Our algorithm defined does not require at each step of the iteration process, the computation of subsets of $C_n,\ Q_n$ and D_n (or C_{n+1}) as in the case in [2] and the computation of the projection of the initial point onto their intersection, which leads to a high computational cost of iteration processes. The removal of all these restrictions makes our work applicable to more real world problems.
(5) We will also like to emphasize that the sequence generated by our iterative method converges strongly, which is more desirable to the weak convergence result obtained in [36].

2. Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. Let E be a reflexive Banach space with E^* its dual and C be a nonempty, closed and convex subset of E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then the Fenchel conjugate of g denoted by $g^* : E^* \to (-\infty, +\infty]$ is defined by

$$g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) : x \in E\},\ x^* \in E^*.$$

Let the domain of g be denoted as $\text{dom}g = \{x \in E : g(x) < +\infty\}$, hence for any $x \in \text{int}(domg)$ and $y \in E$, the right-hand derivative of g at x in the direction of y is defined by

$$g^0(x, y) = \lim_{t\to 0^+} \frac{g(x + ty) - g(x)}{t}.$$

The function g is said to be:

(i) Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x, y)$ coincides with $\nabla g(x)$ (the value of the gradient ∇g of g at x);
(ii) Gâteaux differentiable, if it is Gâteaux differentiable for any $x \in int(domg)$;
(iii) Fréchet differentiable at x, if its limit is attained uniformly in $||y|| = 1$;
(iv) Uniformly Fréchet differentiable on a subset C of E, if the above limit is attained uniformly for $x \in C$ and $||y|| = 1$.

Let $g : E \to (-\infty, +\infty]$ be a function. Then g is said to be:

(i) essentially smooth, if the subdifferential of g denoted as ∂g is both locally bounded and single-valued on its domain, where $\partial g(x) = \{w \in E : g(x) - g(y) \geq \langle w, y - x\rangle,\ y \in E\}$;
(ii) essentially strictly convex, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of dom ∂g;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex. See [8, 9] for more details on Legendre functions.

Alternatively, a function g is said to be Legendre if it satisfies the following conditions:

(i) The intdom g is nonempty, g is Gâteaux differentiable on intdom g and dom ∇g = intdom g;
(ii) The intdom g^* is nonempty, g^* is Gâteaux differentiable on intdom g^* and dom∇ g^* = intdom g.

Let $g : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of g at $x \in$ dom g is the function $\upsilon_g(x, .) : [0, +\infty) \to [0, +\infty)$ defined by

$$\upsilon_g(x, t) := \inf\{D_g(y, x) : y \in \text{dom } g, ||y - x||\}.$$

The function g is totally convex at x if $\upsilon_g(x, t) > 0$, whenever $t > 0$. The function g is totally convex if it is totally convex at any point $x \in$ int(dom g) and is said to be totally convex on bounded sets if $\upsilon_g(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function g on the set B is the function υ_g : int(dom g) $\times [0, +\infty) \to [0, +\infty)$ defined by

$$\upsilon_g(B, t) := \inf\{\upsilon_g(x, t) : x \in B \cap \text{dom } g\}.$$

We know that g is totally convex on bounded sets if and only if g is uniformly convex on bounded sets (see [12], Theorem 2.10).

If E is a Banach space and $B_s := \{z \in E : ||z|| \leq s\}$ for all $s > 0$, then, a function $g : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E, [35, see pp. 203 and 221] if $\rho_s t > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha g(x) + (1 - \alpha)g(y) - g(\alpha(x) + (1 - \alpha)y)}{\alpha(1 - \alpha)},$$

for all $t \geq 0$, with ρ_s denoting the gauge of uniform convexity of g. The function g is also said to be uniformly smooth on bounded subsets of E, [35, see pp. 221], if $\lim_{t\downarrow 0} \frac{\sigma_s}{t}$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B, y\in S_E, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)ty) + (1-\alpha)g(x-\alpha ty) - g(x)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function g is said to be uniformly convex if the function $\delta g : [0,+\infty) \to [0,+\infty)$ defined by

$$\delta g(t) := \sup\left\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g(\frac{x+y}{2}) : ||y-x|| = t\right\},$$

satisfies $\lim_{t\downarrow 0} \frac{\delta g(t)}{t} = 0$.

Definition 2.1. ([10]) Let $g : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. Then, the function $D_g : E \times E \to [0,+\infty)$ defined by

$$D_g(x,y) := g(x) - g(y) - \langle \nabla g(y), x-y\rangle \tag{2.1}$$

is called the Bregman distance with respect to g, where $x, y \in E$.

It is well known that Bregman distance D_g does not satisfy the properties of a metric function because D_g fail to satisfy the symmetric and triangular inequality property. However, the Bregman distance satisfies the following so-called three point identity: for any $x \in \text{dom } g$ and $y, z \in \text{int}(\text{dom } g)$,

$$D_g(x,y) + D_g(y,z) - D_g(x,z) = \langle \nabla g(z) - \nabla g(y), x-y\rangle. \tag{2.2}$$

The relationship between D_g and $||.||$ is guaranteed when g is strongly convex with strong convexity constant $\rho > 0$, that is,

$$D_g(x,y) \geq \frac{\rho}{2}||x-y||^2, \quad \forall\ x \in \text{dom } g,\ y \in \text{int(dom g)}. \tag{2.3}$$

Let $T : C \to \text{int}(\text{dom } g)$ be a mapping. A point $p \in C$ is called a fixed point of T if $Tp = p$. However, if T is a multi-valued mapping, then an element $p \in C$ is called a fixed point of T if $p \in Tp$. We denote by $F(T)$ the set of all fixed points of T. Furthermore, a point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim\limits_{n\to\infty} ||Tx_n - x_n|| = 0$. In the case of a multi-valued mapping, a point $p \in C$ is called an asymptotic fixed point of T, if there exist $\{x_n\} \subset C$ which converges weakly to p such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

Let C be a nonempty, closed and convex subset of int(dom g). Then an operator $T : C \to \text{int(dom g)}$ is said to be

(i) Bregman nonexpansive, if

$$D_g(Tx,Ty) \leq D_g(x,y),\ \ \forall\ x,y \in C.$$

(ii) Bregman relatively nonexpansive, if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \ \forall \ p \in F(T), \ x \in C \text{ and } F(\hat{T}) = F(T). \tag{2.4}$$

(iii) Bregman firmly nonexpansive (BFNE) if

$$\langle \nabla g(Tx) - \nabla g(Ty), Tx - Ty \rangle \leq \langle \nabla g(x) - \nabla g(y), Tx - Ty \rangle, \ \ \forall x, y \in C.$$

(iv) Bregman strongly nonexpansive (BSNE) with $\hat{F}(T) \neq \emptyset$ if

$$D_g(y, Tx) \leq D_g(y, x), \ \ \forall \ x \in C, y \in \hat{F}(T),$$

for any bounded sequence $\{x_n\}_{n \geq 1} \subset C$,

$$\lim_{n \to \infty} (D_g(y, x_n) - D_g(y, Tx_n)) = 0$$

implies

$$\lim_{n \to \infty} D_g(Tx_n, x_n) = 0.$$

(v) quasi-Bregman nonexpansive if $F(T) \neq \emptyset$ and for all $x \in C, \ q \in F(T)$

$$D_g(q, Tx) \leq D_g(q, x).$$

Let $K(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C, respectively. Let $\mathcal{H}$ be the pompieu-Hausdorf metric on $CB(C)$ defined by

$$\mathcal{H}(X, Y) := \max \left\{ \sup_{x \in X} d(x, Y), \sup_{b \in B} d(y, X) \right\},$$

for all $X, Y \in CB(C)$, where $d(x, Y) := \inf\{\|x - y\| : y \in Y\}$ is the distance from the point x to the subset of Y.

Definition 2.2. A mapping $T : C \to CB(C)$ is called Bregman relatively nonexpansive mapping if

(1) $F(T) \neq \emptyset$,
(2) $D_g(x^*, u) \leq D_g(x^*, x), \ \forall \ u \in Tx, \ x \in C$ and $x^* \in F(T)$,
(3) $F(T) = \hat{F}(T)$.

Definition 2.3. A function $g : E \to \mathbb{R}$ is said to be strongly coercive if

$$\lim_{||x_n|| \to \infty} \frac{g(x_n)}{||x_n||} = \infty.$$

Lemma 2.4. ([31]) *Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subset of E. Then,*

(i) *For any* $x, y \in B_s$ *and* $\alpha \in (0,1)$, *we have*

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(||x-y||),$$

where $B_s := \{z \in E : ||z|| \leq s\}$.

(ii) *For any* $x, y \in B_s$,

$$\rho_s(||x-y||) \leq D_g(x,y),$$

where $B_s := \{z \in E : ||z|| \leq s\}$.

Lemma 2.5. ([12]) *Let* E *be a reflexive Banach space,* $g : E \to \mathbb{R}$ *be a strongly coercive Bregman function and* V *be a function defined by*

$$V(x, x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*), \; x \in E, \; x^* \in E^*.$$

Then V *is convex in the second variable and* $V(x, x^*) = D_g(x, \nabla g^*(x^*))$, *for all* $x \in E$ *and* $x^* \in E^*$.

Lemma 2.6. ([12]) *Let* E *be a Banach space and* $g : E \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* E. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in* E. *Then,*

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0 \;\; \Rightarrow \;\; \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 2.7. ([12]) *Let* E *be a Banach space and* $g : E \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* E. *If* $x_0 \in E$ *and the sequence* $\{D_g(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is also bounded.*

Lemma 2.8. ([11]) If dom g contains at least two points, then the function g is totally convex on bounded sets if and only if the function g is sequentially consistent.

Definition 2.9. Let E be a reflexive Banach space and C be a nonempty, closed and convex subset of E. A Bregman projection of $x \in \text{int}(\text{dom } g)$ onto $C \subset \text{int}(\text{dom } g)$ is the unique vector $\text{Proj}_C^g(x) \in C$ satisfying

$$D_g(P_C^g(x), x) = \inf\{D_g(y,x) : y \in C\}.$$

Lemma 2.10. ([26]) *Let* C *be a nonempty, closed and convex subset of a reflexive Banach space* E *and* $x \in E$. *Let* $g : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function. Then,*

(i) $z = P_C^g(x)$ *if and only if* $\langle \nabla g(x) - \nabla g(z), y - z \rangle \leq 0, \; \forall \; y \in C$.

(ii) $D_g(y, P_C^g(x)) + D_g(P_C^g(x), x) \leq D_g(y, x), \; \forall \; y \in C$.

Lemma 2.11. ([26]) *Let E be a real Banach space and $g : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇g is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.12. ([32]) *Let $\{a_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n n a_{n-1} + t_n s_n + \delta_n, \ \forall n \geq 0,$$

where $\sum_{n=n_0}^{\infty} t_n = +\infty$, $\sum_{n=n_0}^{\infty} \delta_n < +\infty$, for each $n \geq n_0$ (where n_0 is a positive integer) and $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup_{n\to\infty} s_n \leq 0$. Then, the sequence $\{a_n\}$ converges weakly to zero.

Lemma 2.13. ([26]) *Let $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then $f^* : X \to (-\infty, +\infty]$ is a proper convex weak* lower semicontinuous function. Thus, for all $z \in X$; we have*

$$D_f\Bigg(z, \nabla f^*\Bigg(\sum_{i=1}^{N} t_i \nabla f(x_i)\Bigg)\Bigg) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.5}$$

where $\{x_i\} \subseteq X$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^{N} t_i = 1$.

Lemma 2.14. ([22]) *Let $f : E \to \mathbb{R}$ be a strongly coercive, bounded and Frchet differentiable Legendre function which is totally convex on bounded subsets of E. let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping and $B : E \multimap E^*$ be a maximal monotone operator. Then the following statements hold:*

(i) *$D_f(z, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) + D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x), x) \leq D_f(z, x)$ for all $z \in (A + B)^{-1}0, x \in E$ and $\lambda > 0$;*

(ii) *$\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda$ is a BSNE operator such that*

$$F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) = \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)).$$

Lemma 2.15. ([17]) *Assume that $f : X \to \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of X. Let $T_i : 1 \leq i \leq N$ be BSNE operators which satisfy $\hat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$ and let $T := T_N T_{N-1} \cdots T_1$. If $\{F(T_i) : 1 \leq i \leq N\}$ is nonempty, then T is also BSNE with $F(T) = \hat{F}(T)$.*

Lemma 2.16. ([17]) *Let X be a reflexive real Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of X. Let $B : X \to 2^{X^*}$ be a maximal monotone mapping and T be a QBFNE mapping on X. Suppose that $F(\mathrm{Res}^f_{\lambda B}) \cap (T) \neq \emptyset$, then $\mathrm{Res}^f_{\lambda B} \circ T$ is also a QBFNE mapping.*

Lemma 2.17. ([17]) *Let X be a reflexive real Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of X. Let $T_i, i = 1, 2, \cdots, N$ be QBFNE on X and $\mathbb{T}_N = T_N \circ T_{N-1} \circ \cdots \circ T_1$. Assume that $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $F(\mathbb{T}_N) = \bigcap_{i=1}^{N} F(T_i)$.*

Remark 2.18. Set $T^i_\lambda = \mathrm{Res}^f_{\lambda B_i} \circ A^f_{i\lambda}$, where $i = 1, 2, \cdots, N$ and $\lambda > 0$. If $(\bigcap_{i=1}^{N} F(Res^f_{\lambda B_i}) \cap (\bigcap_{i=1}^{N} F(A^f_{i\lambda}))$ is nonempty for each $i = 1, 2, \cdots, N$, then by Lemma 2.16, we obtain that T^i_λ is QBFNE for each $i = 1, 2, \cdots, N$. Thus, by Lemma 2.17, we obtain that

$$F(T^N_\lambda \circ T^{N-1}_\lambda \circ \cdots T^1_\lambda) = \bigcap_{i=1}^{N} F(T^i_\lambda). \tag{2.6}$$

Lemma 2.19. ([27]) *Let $B : X \to 2^{X^*}$ be a maximal monotone mapping such that $B^{-1}(0^*) \neq \emptyset$. Then*

$$D_f\Big(u, \mathrm{Res}^f_{\lambda B}(x)\Big) + D_f\Big(\mathrm{Res}^f_{\lambda B}(x), x\Big) \leq D_f(u, x) \tag{2.7}$$

for all $\lambda > 0$, $u \in B^{-1}(0^*)$ and $x \in X$. Furthermore, $B^{-1}(0^*) = F(\mathrm{Res}^f_{\lambda B})$ and $\mathrm{Res}^f_{\lambda B}$ is single-valued.

Lemma 2.20. ([18]) *Let $f : X \to (-\infty, +\infty]$ be a Legendre function and let $A : X \to 2^{X^*}$ be a BISM mapping such that $A^{-1}(0^*)$. Then for any $\lambda > 0$, we have the following:*

(i) *$A^{-1}(0^*) = F(A^f_\lambda)$ and A^f_λ is single valued;*
(ii) *For any $u \in A^{-1}(0^*)$ and $x \in (\mathrm{dom}\ A^f_\lambda)$, we have*

$$D_f(u, A_f x) + D_f(A_f x, x) \leq D_f(u, x).$$

Remark 2.21. It follows that

$$(A + B)^{-1}(0^*) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda), \tag{2.8}$$

where A and B are single-valued and multi-valued mappings, respectively. If in addition, A and B are BISM and maximal monotone mappings, respectively, then it follows from Lemma 2.19 and Lemma 2.20 that the composition $Res^f_{\lambda B}\circ A^f_\lambda$ is also single-valued for any $\lambda > 0$.

Lemma 2.22. ([12]) *Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then*

(a) $z = P^f_C(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0$, $\forall x \in E$ *and* $y \in C$;
(b) $D_f(y, P^f_C(x)) + D_f(P^f_C(x), x) \leq D_f(y, x)$, $\forall x \in E, y \in C$.

Lemma 2.23. ([32]) *Let $\{a_n\}$, $\{x_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - t_n - \gamma_n)a_n + \gamma_n n a_{n-1} + t_n s_n + \delta_n, \quad n \geq 0,$$

where $\sum\limits_{n=n_0}^{\infty} t_n = +\infty$; $\sum\limits_{n=n_0}^{\infty} \gamma_n < +\infty$ *for each* $n \geq n_0$ *(where n_0 is a positive integer) and* $\{\gamma_n\} \subset [0, \frac{1}{2}]$, $\limsup\limits_{n\to\infty} s_n \leq 0$. *Then, the sequence $\{a_n\}$ converges to zero.*

Lemma 2.24. ([21]) *Let $\{\Gamma_k\}$ be a sequence of real numbers such that there exists a subsequence $\{\Gamma_{k_j}\}_{j\geq 0}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} \leq \Gamma_{k_j+1}$ for all $j \geq 0$. Define a sequence of integers $\{\tau(k)\}_{k\geq k^*}$ defined by:*

$$\tau(k) = \max\{n \leq k : \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(k)\}_{k\geq k^*}$ is a nondecreasing sequence satisfying $\lim\limits_{k\to\infty} \tau(k) = \infty$, *and for all $k \geq k^*$, we have that $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$.*

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a reflexive Banach space E with its dual E^* and $g : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Freichet differentiable and totally convex on bounded subsets of E. For each $1 \leq r \leq N$, let $U_r : E \to E^*$ be a finite family of BISM mappings, $V_r : E \to 2^{E^*}$ be a finite family of maximal monotone mappings and $S_r : C \to CB(C)$ be a multi-valued Bregman relatively nonexpansive mapping. Suppose that* $\Omega = \bigcap\limits_{r=1}^{N} \big(F(S_r) \cap (U_r + V_r)^{-1}(0)\big) \neq \emptyset$. *Define* *a sequence* $\{x_n\}_{n=1}^{\infty}$ *generated arbitrarily by chosen* $x_0, x_1 \in E$ *and any fixed* $u \in E$*:*

$$\begin{cases} u_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n))), \\ w_n = \nabla g^*(J_\phi^N \circ J_\phi^{N-1} \cdots \circ J_\phi^1 u_n), \\ y_n = \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r})), \quad z_{n,r} \in S_r w_n, \\ x_{n+1} = \nabla g^*(\alpha_n \nabla g(n) + \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n)), \end{cases} \tag{3.1}$$

where $\{\theta_n\} \subset [0, \frac{1}{2}]$*,* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_{n,r}\}$ *are sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$*,* $J_\phi^r = \mathrm{Res}_{\phi V_r}^g \circ U_{r\phi}^g$*,* $r = 1, 2, \cdots N$*,* $\phi > 0$ *and the following conditions are satisfied*

(A1) $0 < e \leq \theta_n < \gamma_n \leq \frac{1}{2}$, $\forall n \geq 1$;

(A2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(A4) $\sum_{r=0}^{N} \beta_{n,r} = 1, \liminf_{n\to\infty} \delta_{n,0}\delta_{n,r} > 0$ *for all* $1 \leq r \leq N$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ *is bounded.*

Proof. Let $x^* \in \Omega$. Then from (3.1), we obtain that

$$\begin{aligned} D_g(x^*, u_n) &= D_g(x^*, \nabla g^*(\nabla g(x_n)) + \theta_n(\nabla g(x_{n-1})) - \nabla g(x_n)) \\ &\leq (1-\theta_n)D_g(x^* - x_n) + \theta_n D_g(x^*, x_{n-1}). \end{aligned} \tag{3.2}$$

From (3.1), Lemma 2.13 and Lemma 2.14, we get

$$\begin{aligned} D_g(x^*, y_n) &= D_g(x^*, \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r}))) \\ &= \delta_{n,0} D_g(x^*, w_n) + \sum_{r=1}^{N} \delta_{n,r} D_g(x^*, z_{n,r}) \\ &\leq \delta_{n,0} D_g(x^*, w_n) + \sum_{r=1}^{N} \delta_{n,r} D_g(x^*, S_r w_n) \\ &= D_g(x^*, w_n) \\ &= D_g(x^* \nabla g^*(J_\phi^N \circ J_\phi^{N-1} \circ \cdots \circ J_\phi^1 u_n)) \\ &\leq D_g(x^*, u_n) \\ &\leq (1-\theta_n)D_g(x^* - x_n) + \theta_n D_g(x^*, x_{n-1}). \end{aligned} \tag{3.3}$$

Let $\rho_s : E \to \mathbb{R}$ be a guage function of uniform convexity of the conjugate function g^*. By (3.3), (3.1) and Lemma 2.13, we get

$$
\begin{aligned}
D_g(x^*, x_{n+1}) &\leq D_g(x^*, \nabla g^*(\beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u))) \\
&= V_g(x^*, \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u))) \\
&= g(x^*) - \langle x^*, \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u) \rangle \\
&\quad + y^*(\beta_n \nabla g(x_n) + \gamma_n(y_n) + \alpha_n \nabla g(u)) \\
&\leq \beta_n g(x^*) + \gamma_n g(x^*) + \alpha_n g(x^*) - \beta_n \langle x^*, \nabla g(x_n) \rangle \\
&\quad - \gamma_n \langle x^*, \nabla g * (y_n) \rangle - \alpha_n \langle x^*, \nabla g(u) \rangle + \beta_n g^*(\nabla g(x_n)) \\
&\quad + \gamma_n g^*(\nabla g(y_n)) + \alpha_n g^*(\nabla g(u)) - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&\quad - \beta_n \alpha_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(u)|| \Big) - \gamma_n \alpha_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(u)|| \Big) \\
&\leq \beta_n \Big[g(x^*) - \langle x^*, \nabla g(x_n) \rangle + g^*(\nabla g(x_n)) \Big] \\
&\quad + \gamma_n \Big[g(x^*) - \langle x^*, \nabla g(y_n) \rangle + g^*(\nabla g(y_n)) \Big] \\
&\quad + \alpha_n \Big(g(x^*) - \langle x^*, \nabla g(u) \rangle + g^*(\nabla g(u)) \Big) \Big] \\
&\quad - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&= \beta_n V_g \Big(x^*, \nabla g(x_n) \Big) + \gamma_n V_g \Big(x^*, \nabla g(y_n) \Big) + \alpha_n V_g \Big(x^*, \nabla g(u) \Big) \\
&\quad - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&\leq \beta_n D_g(x^*, x_n) + \gamma_n D_g(x^*, y_n) + \alpha_n D_g(x^*, u) \\
&\quad - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&\leq \beta_n D_g(x^*, x_n) + \gamma_n (1 - \theta_n D_g(x^*, x_n) + \gamma_n \theta_n D_g(x^*, x_{n-1}) \\
&\quad + \alpha_n D_g(x^*, u) - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&= \beta_n D_g(x^*, x_n) + \gamma_n D_g(x^*, x_n) - \gamma_n \theta_n D_g(x^*, x_n) \\
&\quad + \gamma_n \theta_n D_g((x^*, x_{n-1}) + \alpha_n D_g(x^*, u) \\
&\quad - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&\leq (1 - \alpha_n - \gamma_n \theta_n) D_g(x^*, x_n) + \gamma_n \theta_n D_g(x^*, x_{n-1}) + \alpha_n D_g(x^*, u) \\
&\quad - \beta_n \gamma_n \rho_s^* \Big(||\nabla g(x_n) - \nabla g(y_n)|| \Big) \\
&\leq (1 - \alpha_n - \gamma_n \theta_n) D_g(x^*, x_n) + \gamma_n \theta_n D_g(x^*, x_{n-1}) + \alpha_n D_g(x^*, u)
\end{aligned}
$$

$$\leq \max\{D_g(x^*, x_n), D_g(x^*, x_{n-1}), D_g(x^*, u)\}$$
$$\vdots$$
$$\leq \max\{D_g(x^*, x_1), D_g(x^*, x_0), D_g(x^*, u)\}. \tag{3.4}$$

Hence, $\{D_g(x^*, x_n)\}$ is bounded by applying Lemma 2.7, it implies that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are bounded. □

Now, we state and prove the following strong convergence theorem

Theorem 3.2. *Assume that Lemma 3.1 and assumptions* (A1)-(A4) *holds. Then* $\{x_n\}$ *converges strongly to* $z = P^g_\Omega(u)$ *where* P^g_Ω *is the Bregman projection of* E *onto* Ω.

Proof. Let $x^* \in \Omega$, then by applying (3.3) and Lemma 2.13, we obtain that

$$\begin{aligned}
D_g(x^*, y_n) &= V_g(x^*, \delta_{n,0}\nabla g(w_n) + \sum_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r})) \\
&= g(x^*) - \delta_{n,0}\langle x^*, \nabla g(w_n)\rangle - \sum_{r=1}^{N} \delta_{n,r}\langle x^*, \nabla g(z_{n,r})\rangle \\
&\quad + \delta_{n,0} g^*(\nabla g(w_n)) + \sum_{r=1}^{N} \delta_{n,r} g^*(\nabla g(z_{n,r})) \\
&\quad - \delta_{n,0}\delta_{n,r}\rho^*_s(||\nabla g(w_n) - \nabla g(z_{n,r})||) \\
&= \delta_{n,0} D_g(x^*, w_n) + \sum_{r=1}^{N} \delta_{n,r} D_g(x^*, z_{n,r}) \\
&\quad - \delta_{n,0}\delta_{n,r}\rho^*_s(||\nabla g(w_n) - \nabla g(z_{n,r})||) \\
&\leq D_g(x^*, w_n) - \delta_{n,0}\delta_{n,r}\rho^*_s(||\nabla g(w_n) - \nabla g(z_{n,r})||).
\end{aligned} \tag{3.5}$$

On substituting (3.5) into (3.4), we get

$$\begin{aligned}
D_g(x^*, x_{n+1}) &\leq \beta_n D_g(x^*, x_n) + \gamma_n (D_g(x^*, w_n) \\
&\quad - \delta_{n,0}\delta_{n,r}\rho^*_s(||\nabla g(w_n) - \nabla g(z_{n,r})||) \\
&\quad - \beta_n\gamma_n\rho^*_s(||\nabla g(x_n) - \nabla g(y_n)||) \\
&\leq \beta_n D_g(x^*, x_n) + \gamma_n D_g(x^*, x_n) - \gamma_n\theta_n D_g(x^*, x_n) \\
&\quad + \gamma_n\theta_n D_g(x^*, x_{n-1}) + \alpha_n D_g(x^*, u) \\
&\quad - \beta_n\gamma_n\rho^*_s(||\nabla g(x_n) - \nabla g(y_n)||) \\
&\quad - \gamma_n\delta_{n,0}\delta_{n,r}\rho^*_s(||\nabla g(w_n) - \nabla g(z_{n,r})||)
\end{aligned}$$

$$= (1-\alpha_n)D_g(x^*, x_n) - \gamma_n\theta_n D_g(x^*, x_n) + \gamma_n\theta_n D_g(x^*, x_{n-1})$$
$$+ \alpha_n D_g(x^*, u) - \beta_n\gamma_n\rho_s^*(||\nabla g(x_n) - \nabla g(y_n)||)$$
$$- \gamma_n\delta_{n,0}\delta_{n,r}\rho_s^*(||\nabla g(w_n) - \nabla g(z_{n,r})||). \quad (3.6)$$

We consider two cases:

Case 1: Assume that $\{D_g(x^*, x_n)\}$ is monotone decreasing, that is,

$$D_g(x^*, x_{n_1}) \leq D_g(x^*, x_n).$$

Since $D_g(x^*, x_n) \leq M$, for all $n \geq 1$, where

$$M := \max\{D_g(x^*, u), D_g(x^*, x_1), D_g(x^*, x_0)\},$$

which implies that $\{D_g(x^*, x_n)\}$ is bounded. Therefore, $\{D_g(x^*, x_n)\}$ is convergent. Thus,

$$\lim_{n\to\infty}(D_g(x^*, x_n) - D_g(x_{n+1})) = \lim_{n\to\infty}(D_g(x^*, x_{n-1}) - D_g(x^*, x_n)) = 0. \quad (3.7)$$

From (3.6), we obtain that

$$\gamma_n\delta_{n,0}\delta_{n,r}\rho_s^*(||\nabla g(w_n) - \nabla g(z_{n,r})||) + \beta_n\gamma_n\rho_s^*(||\nabla g(x_n) - \nabla g(y_n)||)$$
$$\leq (1-\alpha_n)D_g(x^*.x_n) - D_g(x^*, x_{n+1}) + \gamma_n\theta_n(D_g(x^*, x_{n-1})$$
$$- D_g(x^*, x_n)) + \alpha_n D_g(x^*, u).$$

By applying (3.7) and $(A3)$, we obtain that

$$\lim_{n\to\infty}\rho_s^*(||\nabla g(w_n) - \nabla g(z_{n,r})||) = 0 = \lim_{n\to\infty}\rho_s^*(||\nabla g(w_n) - \nabla g(y_n)||). \quad (3.8)$$

Using the property of ρ^* in Lemma 2.13, we obtain that

$$\lim_{n\to\infty}||\nabla g(w_n) - \nabla g(z_{n,r})|| = 0 \neq \lim_{n\to\infty}||\nabla g(w_n) - \nabla g(y_n)||. \quad (3.9)$$

Since ∇g^* is norm to norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty}||w_n - z_{n,r}|| = 0 = \lim_{n\to\infty}||x_n - y_n||. \quad (3.10)$$

Since $d(w_n, S_r w_n) \leq ||w_n - z_{n,r}||$, for each $r \in (1, 2, \cdots, N)$, we have

$$\lim_{n\to\infty} d(w_n, S_r w_n) = 0. \quad (3.11)$$

Also from (3.1), we get

$$||\nabla g(x_{n+1}) - \nabla g(x_n)|| = \alpha_n||\nabla g(u) - \nabla g(x_n)|| + \gamma_n||\nabla g(y_n) - \nabla g(x_n)||.$$

Using $(A2)$ and (3.9), we obtain

$$\lim_{n\to\infty}||\nabla g(x_{n+1}) - \nabla g(x_n)|| = 0. \quad (3.12)$$

Since ∇g^* is a norm to norm uniformly continuous on bounded subsets of E^*, we get

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.13}$$

By applying (3.12) in (3.1), we have

$$||\nabla g(u_n) - \nabla g(x_n)|| = \theta_n ||\nabla g(x_{n-1}) - \nabla g(x_n)|| = 0 \text{ as } n \to \infty. \tag{3.14}$$

Since ∇g^* is a norm to norm uniformly continuous on bounded subsets of E^*, we have

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.15}$$

From (3.1) and (3.9), we have

$$||\nabla g(y_n) - \nabla g(w_n)|| = \sum_{r=1}^{N} \delta_{n,r} ||\nabla g(z_{n,r}) - \nabla g(w_n)|| = 0 \text{ as } n \to \infty. \tag{3.16}$$

Since ∇g^* is a norm to norm uniformly continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} ||y_n - w_n|| = 0. \tag{3.17}$$

By combining (3.10) and (3.17), we have

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{3.18}$$

Also, from (3.15) and (3.18), we get

$$\lim_{n\to\infty} ||u_n - w_n|| = 0 = ||u_n - J_\phi^N \circ J_\phi^{N-1} \circ J_\phi' u_n||. \tag{3.19}$$

By applying (3.17) and (3.18), we obtain

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{3.20}$$

Since $\{x_n\}$ is bounded and E is a reflexive Banach space, there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$, such that $x_{n_m} \rightharpoonup z$. Using (3.15), there exists a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ which converges weakly to z. Also, from (3.18), there exists a subsequence $\{w_{n_m}\}$ of $\{w_n\}$ which converges to z. Hence, using (3.11) and $F(S_r) = \hat{F}(S_r)$, we obtain that $z \in \bigcap_{r=1}^{N} F(S_r)$. Also, from [18] and (3.19) we obtain that $z \in \acute{F}(J_\phi^N \circ J_\phi^{N-1} \circ J_\phi') = F(J_\phi^N \circ J_\phi^{N-1} \circ J_\phi')$, which implies from Remark 2.18 and Remark 2.21 that

$$z \in \bigcap_{r=1}^{N} F(J_\phi^r) = \bigcap_{r=1}^{N} F\left(\mathrm{Res}_{\phi V_r}^g \circ U_{r\phi}^g\right),$$

that is, $z \in \Omega$.

Now, we show that $\{x_n\}$ converges strongly to $z \in \Omega$. Note that,

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - z, \nabla g(u) - \nabla g(z)\rangle &= \lim_{n\to\infty}\langle x_{n_m} - z, \nabla g(u) - \nabla g(z)\rangle \\ &= \langle x^* - z, \nabla g(u) - \nabla g(z)\rangle.\end{aligned}$$

Applying Lemma 2.21, we obtain

$$\langle x^* - z, \nabla g(u) - \nabla g(z)\rangle \leq 0,$$

and hence

$$\limsup_{n\to\infty}\langle x_n - z, \nabla g(u) - \nabla g(z)\rangle = \langle x^* - z, \nabla g(u) - \nabla g(z)\rangle \leq 0. \tag{3.21}$$

From (3.1), (3.3), (3.4) and Lemma 2.14

$$\begin{aligned}D_g(z^*, x_{n+1}) &\leq D_g(z^*, \nabla g^*(\beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u))) \\ &= V_g(z^*, \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u)) \\ &= V_g(z^*, \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n) + \alpha_n \nabla g(u) \\ &\quad - \alpha_n(\nabla g(u)\nabla g(z))) + \alpha_n\langle x_{n+1} = z, \nabla g(u) - \nabla g(z)\rangle \\ &= \beta_n D_g(z, x_n) + \gamma_n D_g(z, y_n) + \alpha_n\langle x_{n+1}, z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq \beta_n D_g(z, x_n) + \gamma_n D_g(z, w_n) + \alpha_n\langle x_{n+1}, z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq \beta_n D_g(z, x_n) + \gamma_n((1-\theta_n)D_g(z, x_n) + \theta_n D_g(z, x_{n-1})) \\ &\quad + \alpha_n\langle x_{n+1}, z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq (1 - \alpha_n - \gamma_n\theta_n)D_g(z, x_n) + \alpha_n\theta_n D_g(z, x_{n-1}) \\ &\quad + \alpha_n\langle x_{n+1}, z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned} \tag{3.22}$$

Now, by (3.22) and Lemma 2.23, we prove that $x_n \to z$.

Case 2: Suppose that $\{D_g(z, x_n)\}$ is not monotone decreasing sequence. Then, there exists a subsequence $\{D_g(z, x_{n_m})\}$ of $\{D_g(z, x_n)\}$ such that

$$D_g(z, x_{n_m}) < d_g(z, x_{n_m+1})$$

for all $m \in \mathbb{N}$. Set Γ_n of Lemma 2.24, as $\Gamma_n = D_g(z, x_n)$ and $\tau : \mathbb{N} \to \mathbb{N}$ is a mapping for all $n \geq n_0$ (for some n_0 large enough), defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$. Thus

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \ \forall\ n \geq n_0.$$

This implies

$$D_g(z, x_{\tau(n)}) \leq D_g(z, x_{\tau(n)+1}), \ n \geq n_0.$$

Since $\{D_g(z, x_{\tau(n)})\}$ is bounded, $\lim_{n\to\infty} D_g(z, x_{\tau(n)})$ exists.

Following the same arguments as in Case 1, we have the following estimates

$$\begin{cases} \lim\limits_{\tau(n)\to\infty} ||u_{\tau(n)} - x_{\tau(n)}|| = 0; \\ \lim\limits_{\tau(n)\to\infty} ||w_{\tau(n)} - u_{\tau(n)}|| = 0; \\ \lim\limits_{\tau(n)\to\infty} ||y_{\tau(n)} - w_{\tau(n)}|| = 0; \\ \lim\limits_{\tau(n)} d(w_{\tau(n)}, S_{r(w_{\tau(n)})}); \\ \lim\limits_{\tau(n)} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0; \\ \limsup\limits_{\tau(n)\to\infty} \langle |x_{\tau(n)+1} - z, \nabla g(u) - \nabla g(z)\rangle \leq 0. \end{cases} \tag{3.23}$$

From (3.23) and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we get

$$\begin{aligned} D_g(z, x_{\tau(n)+1}) &\leq (1 - \alpha_{\tau(n)} - \gamma_{\tau(n)}\theta_{\tau(n)})D_g(z, x_{\tau(n)}) \\ &\quad + \gamma_{\tau(n)}\theta_{\tau(n)}D_g(z, x_{\tau(n)-1}) \\ &\quad + \alpha_{\tau(n)}\langle |x_{\tau(n)+1} - z, \nabla g(u) - \nabla g(z)\rangle \\ &\leq (1 - \alpha_{\tau(n)})D_g(z, x_{\tau(n)+1}) \\ &\quad + \alpha_{\tau(n)}\langle |x_{\tau(n)+1} - z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned}$$

Hence, we obtain

$$D_g(z, x_{\tau(n)}) \leq D_g(z, x_{\tau(n)+1}) \leq \langle |x_{\tau(n)+1} - z, \nabla g(u) - \nabla g(z)\rangle, \tag{3.24}$$

which yields from (3.23) that

$$\lim_{\tau(n)} D_g(z, x_{\tau(n)}) = 0.$$

Hence

$$\lim_{\tau(n)\to\infty} D_g(z, x_{\tau(n)+1}) = 0$$

and therefore

$$\lim_{\tau(n)\to\infty} \Gamma_{\tau(n)} = \lim_{\tau(n)\to\infty} \Gamma_{\tau(n)+1} = 0, \tag{3.25}$$

for all $n \geq n_0$, we have that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_{k+1} \leq \Gamma_k$ for $\tau(n) \leq h \leq n$. This yields for all $n \geq n_0$

$$0 \leq \Gamma_n \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Therefore, $\lim\limits_{n\to\infty} \Gamma_n = 0$ which also implies that $\lim\limits_{n\to\infty} D_g(z, x_{\tau(n)}) = 0$. Hence $x_n \to z = P^g_\Omega u$, as $n \to \infty$. □

By setting $N = 1$, in Theorem 3.2, we obtain the following results.

Corollary 3.3. *Let C be a nonempty closed convex subset of a reflexive Banach space E with its dual E^* and $g : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. Let $U_r : E \to E^*$ be a BISM mapping, $V_r : E \to 2^{E^*}$ be a maximal monotone mapping and $S_r : C \to CB(C)$ be a multi-valued Bregman relatively nonexpansive mapping. Suppose that $\Omega = \bigcap_{r=1}^{N} F(S_r) \cap (U_r + V_r)^{-1}(0) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ generated arbitrarily by chosen $x_0, x_1 \in E$ and any fixed $u \in E$:*

$$\begin{cases} u_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n)), \\ w_n = \mathrm{Res}^g_{\phi_v} \circ U^g_{\phi} u_n, \\ y_n = \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum\limits_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r})), \quad z_{n,r} \in S_r w_n, \\ x_{n+1} = \nabla g^*(\alpha_n \nabla g(u) + \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n)), \end{cases}$$

Assume that Lemma 3.1 and assumptions (A1)-(A4) *holds. Then $\{x_n\}$ converges strongly to $z = P^g_{\Omega}(u)$, where P^g_{Ω} is the Bregman projection of E onto Ω.*

When S_r is a single-valued Bregman relatively nonexpansive mapping we obtain the following result:

Corollary 3.4. *Let C be a nonempty closed convex subset of a reflexive Banach space E with its dual E^* and $g : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. For each $1 \leq r \leq N$, let $U_r : E \to E^*$ be a finite family of BISM mappings, $V_r : E \to 2^{E^*}$ be a finite family of maximal monotone mappings and $S_r : C \to C$ be a Bregman relatively nonexpansive mapping. Suppose that $\Omega = \bigcap_{r=1}^{N} \big(F(S_r) \cap (U_r + V_r)^{-1}(0)\big) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ generated arbitrarily by chosen $x_0, x_1 \in E$ and any fixed $u \in E$:*

$$\begin{cases} u_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n)), \\ w_n = \nabla g^*(J^N_{\phi} \circ J^{N-1}_{\phi} \circ \cdots \circ J'_{\phi} u_n), \\ y_n = \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum\limits_{r=1}^{N} \delta_{n,r}\nabla g(S_r w_n), \\ x_{n+1} = \nabla g^*(\alpha_n \nabla g(u) + \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n)). \end{cases}$$

Assume that Lemma 3.1 and assumptions (A1)-(A4) *holds. Then $\{x_n\}$ converges strongly to $z = P^g_{\Omega}(u)$, where P^g_{Ω} is the Bregman projection of E onto Ω.*

By setting $U = 0$, we obtain the following result:

Corollary 3.5. *Let C be a nonempty closed convex subset of a reflexive Banach space E with its dual E^* and $g : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Frechet differentiable and totally convex on bounded subsets of E. For each $1 \leq r \leq N$, let $U_r : E \to E^*$ be a finite family of BISM mappings, $V_r : E \to 2^{E^*}$ be a finite family of maximal monotone mappings and $S_r : C \to CB(C)$ be a multi-valued Bregman relatively nonexpansive mapping. Suppose that $\Omega = \bigcap_{r=1}^{N} \big(F(S_r) \cap (V_r)^{-1}(0)\big) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ generated arbitrarily by chosen $x_0, x_1 \in E$ and any fixed $u \in E$:*

$$\begin{cases} u_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n))), \\ w_n = \nabla g^*(J_\phi^N \circ J_\phi^{N-1} \circ \cdots \circ J_\phi' u_n), \\ y_n = \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum\limits_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r})), \quad z_{n,r} \in S_r w_n, \\ x_{n+1} = \nabla g^*(\alpha_n \nabla g(u) + \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n)). \end{cases}$$

Assume that Lemma 3.1 and assumptions (A1)-(A4) *holds. Then $\{x_n\}$ converges strongly to $z = P_\Omega^g(u)$, where P_Ω^g is the Bregman projection of E onto Ω.*

Hence $J_\phi^r = \mathrm{Res}_{\phi V_r}, r = 1, 2, \cdots N$ and $\phi > 0$.

4. Application

Variational Inequality Problem: Let E be a reflexive Banach space and E^* be its dual, let $U : E \to E^*$ be a BISM mapping and C be a nonempty, closed and convex subset of dom U. The variational inequality problem (VIP) is to find $z \in C$ such that

$$\langle x - z, Uz \rangle \geq 0, \quad \forall x \in C. \tag{4.1}$$

We denote by $VI(C, A)$ the solution set of VIP (4.1). Recall that the indication function of C is given by

$$i_C(x) = \begin{cases} 0, & \text{if} \quad x \in C, \\ \infty, & \text{if} \quad x \notin C. \end{cases}$$

It is known that i_C is a proper, lower semicontinuous and convex function and its subdifferential ∂i_C is maximal monotone. Furthermore, from [4], we know that

$$\partial i_C(x) = \begin{cases} N_C(x), & \text{if} \quad x \in C, \\ \phi, & \text{if} \quad x \notin C, \end{cases}$$

where N_C is the normal curve of C given by

$$N_C(x) = \{z^* \in E^* : \langle y - x, z\rangle \leq 0, \ \ \forall y \in C\}.$$

Thus, we can define the resolvent associated with ∂i_C for $\phi > 0$ by

$$\mathrm{Res}^g_{\phi\partial i_C}(x) = (\nabla g + \phi\partial i_C)^{-1} \circ \nabla g(x), \ \ \forall x \in E.$$

Hence, we have for any $x \in E$ and $y \in C$,

$$\begin{aligned} z = \mathrm{Res}^g_{\phi\partial i_C}(x) &\Leftrightarrow \nabla g(x) \in \nabla g(z) + \phi\partial i_C(z) \\ &\Leftrightarrow \nabla g(x) \in \nabla g(z) + \phi N_C(z) \\ &\Leftrightarrow \nabla g(x) - \nabla g(z) \in \phi N_C(z) \\ &\Leftrightarrow \frac{1}{\phi}\langle y - z, \nabla g(x) - \nabla g(z)\rangle \leq 0 \ \forall y \in C \\ &\Leftrightarrow \langle y - z, \nabla g(x) - \nabla g(z)\rangle \leq 0 \ \forall y \in C \\ &\Leftrightarrow z = P^g_C(x), \end{aligned}$$

where P^g_C is the Bregman projection from E onto C. Thus, it follows that ([28], Proposition 8) that $F(\mathrm{Res}^g_{\phi\partial i_C} \circ U^g_\phi) = F(P^g_C \circ U^g_\phi) = VIP(U, C)$. Therefore, by setting $V = \partial i_{C_r}, r = 1, 2, \cdots N$ in Theorem 3.2, we obtain the following result:

$$\begin{cases} u_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n))), \\ w_n = \nabla g^*(J^N_\phi \circ J^{N-1}_\phi \circ \cdots \circ J'_\phi u_n), \\ y_n = \nabla g^*(\delta_{n,0}\nabla g(w_n) + \sum\limits_{r=1}^{N} \delta_{n,r}\nabla g(z_{n,r})), \quad z_{n,r} \in S_r w_n, \\ x_{n+1} = \nabla g^*(\alpha_n \nabla g(u) + \beta_n \nabla g(x_n) + \gamma_n \nabla g(y_n)), \end{cases}$$

where $J^r_\phi = \mathrm{Res}^g_{\phi\partial i_C} \circ U^g_\phi \circ U^g_\phi$.

Acknowledgments: The first author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa, Centre of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

[1] H.A. Abass, C. Izuchukwu, O.T. Mewomo and Q.L. Dong, *Strong convergence of an inertial forward-backward splitting method for accretive operators in real Banach spaces*, Fixed Point Theory, **21**(2) (2020), 397-412.

[2] H.A. Abass, A.A. Mebawondu, O.K. Narain and J.K. Kim, *Outer approximation method for zeros of sum of monotone operators and fixed point problems in Banach spaces*, Nonlinear Funct. Anal. Appl., **26**(3) (2021), 451-474.

[3] H.A. Abass, G.C. Ugwunnadi, O.K. Narain and V. Darvish, *Inertial Extragradient Method for Solving Variational Inequality and Fixed Point Problems of a Bregman Demigeneralized Mapping in a Reflexive Banach Spaces*, Numer. Funct. Anal. Optim., **42**(8) (2022), 933-960.

[4] R.P. Agarwal, D. ORegan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications,* Topological Fixed Point Theory and Its Applications, **6**, Springer, New York (2009).

[5] F. Akutsah, O.K. Narain, H.A. Abass and A.A. Mebawondu, *Shrinking approximation method for solution of split monotone variational inclusion and fixed point problems in Banach spaces*, Int. J. Nonlinear Anal. Appl., **12**(2) (2021), 825-842.

[6] W. Ali, A. Turab and J.J. Nieto, *On the novel existence results of solutions for a class of fractional boundary value problems on the cyclohexane graph*, J. Ineq. Appl., **5**(19) (2022), https://doi.org/10.1186/s13660-021-02742-4.

[7] F. Alvarez and H. Attouch, *An Inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal., **9** (2001), 3-11.

[8] H.H. Bauschke and J.M. Borwein, *Legendre functions and method of random Bregman functions*, J. Convex Anal., **4** (1997), 27-67.

[9] H.H. Bauschke, J.M. Borwein and P.L. Combettes, *Essentially smoothness, essentially strict convexity and Legendre functions in Banach spaces*, Commun. Contemp. Math., **3** (2001), 615-647.

[10] L.M. Bregman, *The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming*, U.S.S.R Comput. Math. Phys., **7** (1967), 200-217.

[11] D. Butnariu and A.N. Iusem, *Totally convex functions for fixed points computation and infinite dimensional optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

[12] D. Butnairu and E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstr. Appl. Anal., **2006** (2006), 1-39, Art. ID 84919.

[13] G. Cacciapaglia and F. Sannino, *Evidence for complex fixed points in pandemic data*, Front. Appl. Math. Stat., **7** (2021), https://doi.org/10.3389/fams.2021.659580.

[14] G. Cai, Q.L. Dong and Y. Peng, *Strong convergence theorems for inertial Tseng's extragradient method for solving variational inequality and fixed point problems*, Optim. Lett., **15** (2021), 1457-1474.

[15] G.H. Chen and R.T. Rockafellar,*Convergence rates in forward-backward splitting*, SIAM J. Optim., **7** (1997), 421444.

[16] P.L. Combettes and V.R. Wajs, *Signal recovery by proximal forward-backward splitting*, Multiscale Model. Simul., **4** (2005), 11681200.

[17] C. Izuchukwu, G.N. Ogwo, A.A. Mebawondu and O.T. Mewomo, *On finite family of monotone variational inclusion problems in reflexive Banach spaces*, U. P. B. Sci. Bull. series A, **82**(2) (2020).

[18] G. Kassay, S. Riech and S. Sabach, *Iterative methods for solving systems of variational inequalities in Reflexive Banach spaces*, J. Nonlinear Convex Anal., **10** (2009), 471-485.

[19] G.M. Korpelevich, *The extragradient method for finding for finding saddle points and other problems*, Ekonomikai Matematicheskie Melody, **12** (1976), 747-756.

[20] G. Lopez, V. Martin-Marquez, F. Wang and H.K. Xu, *Forward-Backward splitting methods for accretive operators in Banach spaces*, Abstr. Appl. Anal., (2012), Article ID 109236.

[21] P.E. Mainge, *Viscosity approximation process for quasi nonexpansive mappings in Hilbert space*, Comput. Math. Appl., **59** (2010), 74-79.

[22] F.U. Ogbuisi and C. Izuchukwu, *Approximating a zero of sum of two monotone operators which solves a fixed point problem in reflexive Banach spaces*, Numer. Funct. Anal. Optim., **41**(3) (2020), 322-343.

[23] C.C. Okeke and C. Izuchukwu, *Strong convergence theorem for split feasibility problems and variational inclusion problems in real Banach spaces*, Rendiconti del circolo Matematica di Palermo series 2, **70**(1) (2021), 457-480.

[24] S.K. Panda, A. Atangana and J.J. Nieto, *New insights on novel coronavirus 2019-nCoV/SARS-CoV-2 modelling in the aspect of fractional derivatives and fixed points*, Math. Biosci. Eng., **18**(6) (2021), 8683-8726.

[25] B.T. Polyak, *Some methods of speeding up the convergence of iterates methods*, U.S.S.R Comput. Math. Phys., **4** (5) (1964), 1-17.

[26] S. Reich and S. Sabach, *A strong convergence theorem for a proximal-type algorithm in reflexive Banach spaces*, J. Nonlinear Convex Anal., **10** (2009), 471-485.

[27] S. Reich and S. Sabach, *Two strong convergence theorems for a proximal method in reflexive Banach spaces*, Numer. Funct. Anal. Optim., **31** (2010), 24-44.

[28] S. Reich and S. Sabach, *Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces*, Nonlinear Anal., Theory Methods Appl., Ser. A, **73** (2010), 122135.

[29] Y. Tang, R. Promkam, P. Cholamjiak and P. Sunthrayuth, *Convergence results of iterative algorithms for the sum of two monotone operators in reflexive Banach spaces*, Appl. Math., **67** (2022), 129-152.

[30] P. Tianchai, *The zeros of monotone operators for the variational inclusion problem in Hilbert spaces,* J. Ineq. Appl., **2021** (2021), https://doi.org/10.1186/s13660-021-02663-2.

[31] S. Timnak, E. Naraghirad and N. Hussain, *Strong convergence of Halpern iteration for products of finitely many resolvents of maximal monotone operators in Banach spaces*, Filomat, **31**(15) (2017), 4673-4693.

[32] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(1) (2002), 240-256.

[33] L. Wei and L. Duan, *A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach space and its connection with capillarity equation*, Fixed Point Theory Appl., **2015** (2015), https://doi.org/10.1186/s13663-015-0269-6.

[34] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algo., **80** (2019), 741-752.

[35] C. Zalinescu, *Convex Analysis in General Vector spaces*, World Scientific Publishing Co. Inc., River Edge NJ, 2002.

[36] C. Zhang and Y. Wang, *Proximal algorithm for solving monotone variational inclusion*, Optimization, **67**(8) (2018), 1197-1209.