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QUALITATIVE ANALYSIS FOR FRACTIONAL-ORDER NONLOCAL INTEGRAL-MULTIPOINT SYSTEMS VIA A GENERALIZED HILFER OPERATOR

Mohammed N. Alkord[1], Sadikali L. Shaikh[2], Saleh S. Redhwan[3] and Mohammed S. Abdo[4]

[1]Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University Aurangabad, (M.S), 431001, India
e-mail: moh1992alkord@gmail.com

[2]Department of Mathematics, Maulana Azad College of arts, Science and Commerce,
RozaBagh, Aurangabad 431004 (M.S.), India
e-mail: sad.math@gmail.com

[3]Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University Aurangabad, (M.S), 431001, India
e-mail: saleh.redhwan909@gmail.com

[4]Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen
e-mail: msabdo1977@gmail.com

Abstract. In this paper, we consider two types of fractional boundary value problems, one of them is an implicit type and the other will be an integro-differential type with nonlocal integral multi-point boundary conditions in the frame of generalized Hilfer fractional derivatives. The existence and uniqueness results are acquired by applying Krasnoselskii's and Banach's fixed point theorems. Some various numerical examples are provided to illustrate and validate our results. Moreover, we get some results in the literature as a special case of our current results.

1. Introduction

In recent few decades, fractional differential equations (FDEs) have been the point of interest of many studies by many investigators. This is because the theory of FDEs is much significant due to their nonlocal property is suitable

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[0]Corresponding author: Mohammed N. Alkord(moh1992alkord@gmail.com).

to describe memory phenomena in many applied more accurately comparing to classical order differential equations. Therefore, the FDEs styles become more workable and pragmatic comparing to the integer-order samples. FDEs appear in lots of engineering and chemistry, physics, biology, signal and image processing, economics, control theory, biophysics, aerodynamics, blood flow phenomena and so on, see the monographs as [2, 3, 6, 8]. There are several definitions of fractional calculus (FC), like Riemann-Liouville's (RL) definition and Caputo's definition, and there are other less-famous definitions like Erdelyi-Kober's and Hadamard's definitions and so on. In [7], Hilfer was given generalization of fractional derivatives (FDs) of RL and Caputo, which so-called the Hilfer FD of order ϱ_1 and a type $\varrho_2 \in [0,1]$. When we give $\varrho_2 = 0$ and $\varrho_2 = 1$ respectively in the formula of Hilfer FD can get RL's and Caputo's FDs. Such a derivative inset between the RL and Caputo FDs. For more details on this FD above-mentioned can be found in [1, 2]. In Ref [5], the authors introduced the FD with another function in the frame of Hilfer FD, which called ς-Hilfer(or ψ-Hilfer) FD. For some recent results on existence and stability theorems of ς-Hilfer type IVPs, see [9, 10, 12, 14, 16, 18] and for BVPs see [11, 13, 15, 17, 19, 20, 21]. Here we also refer to some recent works [23, 25, 26, 27, 29, 31] dealing with a similar analysis of various problems in this regard.

Encouraged by the researches going on in this direction, in this article, we study the existence and uniqueness theorems for ς-Hilfer type implicit nonlocal integral-multipoint BVPs (for short, implicit-type problem):

$$\begin{cases} {}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma} z(\upsilon) = f(\upsilon, z(\upsilon), {}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma} z(\upsilon)), \quad \upsilon \in \mathbb{J} := [a,b], \\ z(a) = 0, \quad \int_a^b \varsigma'(t) z(t) dt + \hbar = \sum_{r=1}^{m-2} \xi_r z(\theta_r), \end{cases} \tag{1.1}$$

and ς-Hilfer type integrodifferential nonlocal integral-multipoint BVPs (integrodifferential-type problem):

$$\begin{cases} {}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma} z(\upsilon) = f(\upsilon, z(\upsilon), \mathfrak{I}^{\varrho_3;\varsigma} z(\upsilon)), \quad \upsilon \in \mathbb{J}: = [a,b], \\ z(a) = 0, \ \int_a^b \varsigma'(t) z(t) dt + \hbar = \sum_{r=1}^{m-2} \xi_r z(\theta_r), \end{cases} \tag{1.2}$$

where ${}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}$ is the $\varsigma-$Hilfer FD of order (ϱ_1, ϱ_2), $1 < \varrho_1 < 2$, and $0 \leq \varrho_2 \leq 1$, $\mathfrak{I}^{\varrho_3;\varsigma}$ is the Riemann-Liouville fractional integral of order $\varrho_3 > 0$, $f : \mathbb{J} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $a < \theta_1 < \theta_2 < \cdots < \theta_{m-2} < b$, $a \geq 0$, and $\xi_r, \theta_r \in \mathbb{R}$, $r = 1, 2, ..., m-2$.

We focus on the subject of non-local problems due to in many cases the non-local condition in these kinds of problems reflects physical phenomena more than classical initial (boundary) conditions. So, we address the existence and uniqueness theorems of problems (1.1) and (1.2) by applying Banach's and Krasnoselskii's fixed point theorems under the minimum assumptions. The work done in this article is recent and riches the literature, especially the ς-Hilfer type nonlinear problems. The FDEs (1.1) and (1.2) are new to the literature on FDEs and include many problems, as a special case, for $\varsigma(\upsilon) = \upsilon$, the outcomes obtained in this paper incorporates the results of Nuchpong et al. [23]. For $\varsigma(\upsilon) = \log \upsilon$ and $\varsigma(\upsilon) = \upsilon^{\varrho}$, our problems are reduced to Hilfer-Hadamard-type problems and Hilfer-Katugampola problems, respectively. Moreover, if $\varrho_2 = 0$ and $\varrho_2 = 1$, then proposed problems reduce to generalized RL-type problems and generalized Caputo-type problems, respectively.

The content of this article is organized as follows: Section 2, we present some essential fractional calculus definitions and notions about ς-Hilfer FD that will be applied. The existence results and Ulam–Hyers type stability for the problems (1.1) and (1.2) are checked in Section 3. Some illustrative examples are included to illustrate our obtained results in Section 4. Finally, conclusive remarks and suggested future directions are expressed in Section 5.

2. Preliminaries

In this section, we setting notations and some introductory facts that will be applied in the proofs of the subsequent results.

Let $C(\mathbb{J}, \mathbb{R})$ and $L(\mathbb{J}, \mathbb{R})$ are the Banach spaces of continuous functions and Lebesgue integrable functions from $\mathbb{J}$ into $\mathbb{R}$ with the norms

$$\|z\|_{\infty} = \sup\{|z| : \upsilon \in \mathbb{J}\}$$

and

$$\|z\|_{L} = \int_a^b |z(\upsilon)|\, d\upsilon,$$

respectively.

For $\zeta = \varrho_1 + 2\varrho_2 - \varrho_1\varrho_2$, $1 < \varrho_1 < 2$, and $0 \le \varrho_2 \le 1$. Then $1 < \zeta \le 2$. Let $\varsigma \in C^1(\mathbb{J}, \mathbb{R})$ be an increasing function with $\varsigma'(\upsilon) \neq 0$, for all $\upsilon \in \mathbb{J}$.

Definition 2.1. ([18]) Let $\varrho_1 > 0$ and $g \in L^1(\mathbb{J}, \mathbb{R})$. The ς-RL fractional integral of order ϱ_1 of a function g is given by

$$\mathfrak{I}^{\varrho_1;\varsigma} g(\upsilon) = \frac{1}{\Gamma(\varrho_1)} \int_a^{\upsilon} \varsigma'(t)(\varsigma(\upsilon) - \varsigma(t))^{\varrho_1 - 1} g(t)dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2. ([32]) The $\varsigma-$Hilfer FD of order ϱ_1 and parameter ϱ_2 is defined by

$${}_{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma} g(\upsilon) = \mathfrak{I}^{\varrho_2(n-\varrho_1);\varsigma} \left(\frac{1}{\varsigma'(\upsilon)} \frac{d}{d\upsilon} \right)^n \mathfrak{I}^{(1-\varrho_2)(n-\varrho_1);\varsigma} g(\upsilon),$$

where $n - 1 < \varrho_1 < n$, $0 \le \varrho_2 \le 1$, $\upsilon > a$.

Lemma 2.3. ([18, 32]) *Let* $\varrho_1, \eta, \delta > 0$. *Then*

(1) $\mathfrak{I}^{\varrho_1;\varsigma}\mathfrak{I}^{\eta;\varsigma}g(\upsilon) = \mathfrak{I}^{\varrho_1+\eta;\varsigma}g(\upsilon)$,

(2) $\mathfrak{I}^{\varrho_1;\varsigma}(\varsigma(\upsilon)-\varsigma(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\varrho_1+\delta)}(\varsigma(\upsilon)-\varsigma(a))^{\varrho_1+\delta-1}$.

We note also that ${}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1} = 0$, where $\zeta = \varrho_1 + \varrho_2(n-\varrho_1)$.

Lemma 2.4. ([32]) *Let* $g \in L^1(\mathbb{J},\mathbb{R})$, $\varrho_1 \in (n-1, n]$ $(n \in \mathbb{N})$ *and* $\varrho_2 \in [0,1]$. *Then*

$$(\mathfrak{I}^{\varrho_1;\varsigma}{}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}g)(\upsilon) = g(\upsilon) - \sum_{k=0}^{n}\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-k}}{\Gamma(\zeta-k+1)}g_\varsigma^{[n-k]}\lim_{\upsilon\to a}\left(\mathfrak{I}^{(1-\varrho_2)(n-\varrho_1);\varsigma}g\right)(a),$$

where $g_\varsigma^{[n-k]}(\upsilon) = \left(\frac{1}{\varsigma'(\upsilon)}\frac{d}{d\upsilon}\right)^{[n-k]}g(\upsilon)$.

Here we can suffice to refer to Banach's fixed point theorem [13] and Krasnoselskii's fixed point theorem [13].

3. Main results

We first, prove an auxiliary lemma concerning a linear variant of the $\varsigma-$Hilfer type BVP (1.1).

Lemma 3.1. *Let* $\zeta = \varrho_1 + 2\varrho_2 - \varrho_1\varrho_2$ *where* $1 < \varrho_1 < 2$, *and* $0 \le \varrho_2 \le 1$, *and* $\phi \in C(\mathbb{J},\mathbb{R})$. *If*

$$\Omega = \frac{(\varsigma(b)-\varsigma(a))^{\zeta}}{\zeta} - \sum_{r=1}^{m-2}\xi_r(\varsigma(\theta_r)-\varsigma(a))^{\zeta-1} \neq 0, \tag{3.1}$$

then the function $z \in C(\mathbb{J},\mathbb{R})$ *is a solution of the* $\varsigma-$*Hilfer BVP*

$${}_H\mathfrak{D}^{\varrho_1,\varrho_2,\varsigma}z(\upsilon) = \phi(\upsilon), \quad \upsilon \in \mathbb{J}, \tag{3.2}$$

$$z(a) = 0, \quad \int_a^b \varsigma'(t)z(t)dt + \hbar = \sum_{r=1}^{m-2}\xi_r z(\theta_r), \tag{3.3}$$

if and only if

$$z(\upsilon) = \mathfrak{I}^{\varrho_1,\varsigma}\phi(\upsilon) + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left[\sum_{r=1}^{m-2}\xi_r\mathfrak{I}^{\varrho_1,\varsigma}\phi(\theta_r) - \left(\mathfrak{I}^{1+\varrho_1,\varsigma}\phi(t)\right)(b) - \hbar\right]. \tag{3.4}$$

Proof. Assume that z is a solution of (3.2) and (3.3). Operating fractional integral $\mathfrak{I}^{\varrho_1,\varsigma}$ on (3.2). It follows from Lemma (2.4) that

$$z(\upsilon) = c_0\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-2}}{\Gamma(\zeta-1)} + c_1\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Gamma(\zeta)} + \mathfrak{I}^{\varrho_1,\varsigma}\phi(\upsilon),$$

since $(1-\varrho_2)(2-\varrho_1)=2-\zeta$, where c_0,$c_1 \in \mathbb{R}$. From (3.3), the condition $z(a)=0$ gives $c_0=0$. Hence

$$z(\upsilon)=c_1\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Gamma(\zeta)}+\mathfrak{J}^{\varrho_1,\varsigma}\phi(\upsilon). \tag{3.5}$$

Also, the condition $\int_a^b \varsigma'(t)z(t)dt+\hbar=\sum_{r=1}^{m-2}\xi_r z(\theta_r)$ brings us to

$$c_1=\frac{\Gamma(\zeta)}{\Omega}\left[\sum_{r=1}^{m-2}\xi_r\mathfrak{J}^{\varrho_1,\varsigma}\phi(\theta_r)-\left(\mathfrak{J}^{1+\varrho_1,\varsigma}\phi(t)\right)(b)-\hbar\right]. \tag{3.6}$$

Substituting (3.6) in (3.5), we get the solution (3.4). The converse follows by direct computation. □

Now, according to Lemma 3.1, we define the operator $\mathbb{T}:C(\mathbb{J},\mathbb{R})\to C(\mathbb{J},\mathbb{R})$ by

$$(\mathbb{T}z)(\upsilon)=\mathfrak{J}^{\varrho_1,\varsigma}\mathfrak{F}_z(\upsilon)+\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left(\sum_{r=1}^{m-2}\xi_r\mathfrak{J}^{\varrho_1,\varsigma}\mathfrak{F}_z(\theta_r)-\mathfrak{J}^{\varrho_1+1;\varsigma}\mathfrak{F}_z(b)-\hbar\right), \tag{3.7}$$

where $\mathfrak{F}_z\in C(\mathbb{J},\mathbb{R})$ with $\mathfrak{F}_z(\upsilon):=f(\upsilon,z(\upsilon),\mathfrak{F}_z(\upsilon))$.

It should be observed that the implicit-type problem (1.1) has a solution if and only if $\mathbb{T}$ has fixed points. Hence, for convenience purpose, we are setting two constants:

$$\begin{aligned}\Lambda \ : \ &=\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\left[\sum_{r=1}^{m-2}|\xi_r|\left(\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right)\right.\\ &\left.+\ \frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}\right]\end{aligned} \tag{3.8}$$

and

$$\Lambda^\star:=\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\left[\sum_{r=1}^{m-2}|\xi_r|\left(\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right)+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}\right]. \tag{3.9}$$

Clearly, $\Lambda=\Lambda^\star+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}$.

In the following, we demonstrate the results of the existence and uniqueness of (1.1) by employing the Banach's and Krasnoselskii's fixed point theorems.

3.1. **Implicit-type problem (1.1).** Some essential assumptions are presented as follows:

(H$_1$): There exists $\kappa \in (0,1)$ such that

$$|f(\upsilon, z_1, z_1^*) - f(\upsilon, z_2, z_2^*)| \leq \kappa \left(|z_1 - z_2| + |z_1^* - z_2^*|\right),$$

for any $z_1, z_1^*, z_2, z_2^* \in \mathbb{R}$ and $\upsilon \in \mathbb{J}$.

(H$_2$): Let $f \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R})$ and $\Theta \in C(\mathbb{J}, \mathbb{R}^+)$ such that

$$|f(\upsilon, z, z^*)| \leq \Theta(\upsilon), \quad \forall(\upsilon, z, z^*) \in \mathbb{J} \times \mathbb{R}^2.$$

Theorem 3.2. *Assume that* (H_1) *holds. If*

$$\frac{\kappa}{1-\kappa}\Lambda < 1, \tag{3.10}$$

then the implicit-type problem (1.1) *has a unique solution on* $\mathbb{J}$.

Proof. We convert (1.1) into a fixed point problem, that is, $z = \mathbb{T}z$, as $\mathbb{T}$ is defined by (3.7). Note that the fixed points of $\mathbb{T}$ are solutions of (1.1). We shall show that $\mathbb{T}$ has a unique fixed point by using Banach theorem [13]. Indeed, we set $\sup_{\upsilon \in \mathbb{J}} |f(\upsilon, 0, 0)| = \mathbb{N} < \infty$ and choose

$$\varepsilon \geq \frac{\frac{\mathbb{N}}{1-\kappa}\Lambda + (\varsigma(b) - \varsigma(a))^{\zeta-1}\left|\frac{\hbar}{\Omega}\right|}{1 - \frac{\kappa}{1-\kappa}\Lambda}.$$

First, we show that $\mathbb{T}\mathcal{A}_\varepsilon \subset \mathcal{A}_\varepsilon$, where $\mathcal{A}_\varepsilon = \{z \in C(\mathbb{J}, \mathbb{R}) : \|z\| \leq \varepsilon\}$. By using (H_1), we obtain

$$\begin{aligned}
|\mathfrak{F}_z(\upsilon)| &= |f(\upsilon, z(\upsilon), \mathfrak{F}_z(\upsilon))| \\
&\leq |f(\upsilon, z(\upsilon), \mathfrak{F}_z(\upsilon)) - f(\upsilon, 0, 0)| + |f(\upsilon, 0, 0)| \\
&\leq \kappa\,|z(\upsilon)| + \kappa\,|\mathfrak{F}_z(\upsilon)| + \mathbb{N},
\end{aligned}$$

which gives

$$|\mathfrak{F}_z(\upsilon)| \leq \frac{\kappa}{1-\kappa}\,|z(\upsilon)| + \frac{\mathbb{N}}{1-\kappa}.$$

For any $z \in \mathcal{A}_\varepsilon$, we get

$$\begin{aligned}
|(\mathbb{T}z)(\upsilon)| \leq \sup_{\upsilon\in\mathbb{J}} \mathfrak{I}^{\varrho_1,\varsigma}\,|\mathfrak{F}_z(\upsilon)| + \sup_{\upsilon\in\mathbb{J}} \frac{(\varsigma(\upsilon) - \varsigma(a))^{\zeta-1}}{|\Omega|} \\
\times \left(\sum_{r=1}^{m-2} |\xi_r|\,\mathfrak{I}^{\varrho_1,\varsigma}\,|\mathfrak{F}_z(\theta_r)| + \mathfrak{I}^{\varrho_1+1;\varsigma}\,|\mathfrak{F}_z(b)| + |\hbar|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathfrak{I}^{\varrho_1;\varsigma}\frac{\kappa}{1-\kappa}\|z\|\right)(b)+\left(\mathfrak{I}^{\varrho_1;\varsigma}\frac{\mathbb{N}}{1-\kappa}\right)(b)\\
&\quad+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\\
&\quad\times\sum_{r=1}^{m-2}|\xi_r|\left(\left(\mathfrak{I}^{\varrho_1;\varsigma}\frac{\kappa}{1-\kappa}\|z\|\right)(\theta_r)+\left(\mathfrak{I}^{\varrho_1;\varsigma}\frac{\mathbb{N}}{1-\kappa}\right)(\theta_r)\right)\\
&\quad+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\\
&\quad\times\left(\left(\mathfrak{I}^{\varrho_1+1;\varsigma}\frac{\kappa}{1-\kappa}\|z\|\right)(b)+\left(\mathfrak{I}^{\varrho_1+1;\varsigma}\frac{\mathbb{N}}{1-\kappa}\right)(b)+|\hbar|\right)\\
&\leq\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\left(\frac{\kappa}{1-\kappa}\varepsilon+\frac{\mathbb{N}}{1-\kappa}\right)\\
&\quad+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\sum_{r=1}^{m-2}|\xi_r|\left(\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\left(\frac{\kappa}{1-\kappa}\varepsilon+\frac{\mathbb{N}}{1-\kappa}\right)\right)\\
&\quad+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+\zeta}}{|\Omega|\,\Gamma(\varrho_1+2)}\left(\frac{\kappa}{1-\kappa}\varepsilon+\frac{\mathbb{N}}{1-\kappa}\right)+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}|\hbar|\\
&\leq\frac{\kappa}{1-\kappa}\Lambda\varepsilon+\frac{\mathbb{N}}{1-\kappa}\Lambda+(\varsigma(b)-\varsigma(a))^{\zeta-1}\left|\frac{\hbar}{\Omega}\right|\\
&\leq\varepsilon.
\end{aligned}
$$

This means that $\mathbb{T}\mathcal{A}_\varepsilon\in\mathcal{A}_\varepsilon$, that is, $\mathbb{T}\mathcal{A}_\varepsilon\subset\mathcal{A}_\varepsilon$.

Next, For each $z,z^*\in\mathbb{R}$ and $\upsilon\in\mathbb{J}$, we have

$$
\begin{aligned}
&|(\mathbb{T}z)(\upsilon)-(\mathbb{T}z^*)(\upsilon)|\\
&\quad\leq\mathfrak{I}^{\varrho_1;\varsigma}|\mathfrak{F}_z(\upsilon)-\mathfrak{F}_{z^*}(\upsilon)|+\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\\
&\qquad\times\left(\sum_{r=1}^{m-2}\xi_r\mathfrak{I}^{\varrho_1,\varsigma}|\mathfrak{F}_z(\theta_r)-\mathfrak{F}_{z^*}(\theta_r)|+\mathfrak{I}^{\varrho_1+1;\varsigma}|\mathfrak{F}_z(b)-\mathfrak{F}_{z^*}(b)|\right).
\end{aligned}
$$

From (H_1), one has

$$
\begin{aligned}
|\mathfrak{F}_z(\upsilon)-\mathfrak{F}_{z^*}(\upsilon)| &= |f(t,z(t),\mathfrak{F}_z(t))(\upsilon)-f(t,z^*(t),\mathfrak{F}_{z^*}(t))(\upsilon)|\\
&\leq \kappa|z(\upsilon)-z^*(\upsilon)|+\kappa|\mathfrak{F}_z(\upsilon)-\mathfrak{F}_{z^*}(\upsilon)|,
\end{aligned}
$$

which implies

$$
|\mathfrak{F}_z(\upsilon)-\mathfrak{F}_{z^*}(\upsilon)|\leq\frac{\kappa}{1-\kappa}|z(\upsilon)-z^*(\upsilon)|.
$$

Consequently,

$$\begin{aligned}
&|(\mathbb{T}z)(\upsilon)-(\mathbb{T}z^{*})(\upsilon)| \\
&\leq \frac{\kappa}{1-\kappa}\left(\ \mathfrak{J}^{\varrho_1;\varsigma}\|z-z^{*}\|\right)(\upsilon)+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|} \\
&\quad\times\left(\sum_{r=1}^{m-2}|\xi_r|\frac{\kappa}{1-\kappa}\left(\ \mathfrak{J}^{\varrho_1;\varsigma}\|z-z^{*}\|\right)(\theta_r)+\frac{\kappa}{1-\kappa}\left(\ \mathfrak{J}^{\varrho_1+1;\varsigma}\|z-z^{*}\|\right)(b)\right) \\
&\leq \frac{\kappa}{1-\kappa}\left[\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\sum_{r=1}^{m-2}|\xi_r|\left(\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right)\right. \\
&\quad\left.+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+\zeta}}{|\Omega|\,\Gamma(\varrho_1+2)}\right]\|z-z^{*}\| \\
&\leq \frac{\kappa}{1-\kappa}\Lambda\|z-z^{*}\|,
\end{aligned}$$

which leads us to $\|\mathbb{T}z-\mathbb{T}z^{*}\|\leq\frac{\kappa}{1-\kappa}\Lambda\|z-z^{*}\|$. By (3.10), we realize that $\mathbb{T}$ is a contraction. Then, a unique solution exists on $\mathbb{J}$ to (1.1) by virtue of the Banach's fixed point theorem [13], and this completes the proof. □

Second, we will use the Krasnoselskii's fixed point theorem [13] to prove the existence result for the implicit-type problem (1.1).

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\frac{\kappa}{1-\kappa}\Lambda^{\star}<1, \tag{3.11}$$

where $\Lambda^{\star}$ *is defined by* (3.9)*, then the implicit-type problem* (1.1) *has at least one solution on* $\mathbb{J}$*.*

Proof. Consider the ball $\mathcal{A}_\sigma=\{z\in C(\mathbb{J},\mathbb{R}):\|z\|\leq\sigma\}$ where $\sigma>0$ with

$$\sigma\geq\Lambda\|\Theta\|+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}|\hbar|, \tag{3.12}$$

and $\sup_{\upsilon\in\mathbb{J}}|\Theta(\upsilon)|=\|\Theta\|$, where Λ is defined by (3.8). Then we build the operators $\mathbb{T}_1$, $\mathbb{T}_2$ on $\mathcal{A}_\sigma$ by

$$(\mathbb{T}_1z)(\upsilon)=\mathfrak{J}^{\varrho_1;\varsigma}\mathfrak{F}_z(\upsilon),\quad \upsilon\in\mathbb{J}$$

and

$$(\mathbb{T}_2z)(\upsilon)=\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left(\sum_{r=1}^{m-2}\xi_r\mathfrak{J}^{\varrho_1,\varsigma}\mathfrak{F}_z(\theta_r)-\mathfrak{J}^{\varrho_1+1;\varsigma}\mathfrak{F}_z(b)-\hbar\right).$$

For any $z, z^* \in \mathcal{A}_\sigma$, we get

$$\begin{aligned}
&|(\mathbb{T}_1 z)(\upsilon) + (\mathbb{T}_2 z^*)(\upsilon)| \\
&\leq \sup_{\upsilon \in \mathbb{J}} \Bigg\{ \mathfrak{I}^{\varrho_1;\varsigma} |\mathfrak{F}_z(\upsilon)| + \frac{(\varsigma(\upsilon) - \varsigma(a))^{\zeta - 1}}{|\Omega|} \\
&\quad \times \left[\sum_{r=1}^{m-2} |\xi_r| \, \mathfrak{I}^{\varrho_1;\varsigma} |\mathfrak{F}_{z^*}(\theta_r))| + \mathfrak{I}^{\varrho_1+1;\varsigma} |\mathfrak{F}_{z^*}(b)| + |\hbar| \right] \Bigg\} \\
&\leq \left[\frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} + \frac{(\varsigma(b) - \varsigma(a))^{\zeta - 1}}{|\Omega|} \sum_{r=1}^{m-2} |\xi_r| \left(\frac{(\varsigma(\theta_r) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} \right) \right. \\
&\quad \left. + \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1 + \zeta}}{|\Omega| \, \Gamma(\varrho_1 + 2)} \right] \|\Theta\| + \frac{(\varsigma(b) - \varsigma(a))^{\zeta - 1}}{|\Omega|} |\hbar| \\
&= \Lambda \|\Theta\| + \frac{(\varsigma(b) - \varsigma(a))^{\zeta - 1}}{|\Omega|} |\hbar| \\
&\leq \sigma.
\end{aligned}$$

This proves that $\mathbb{T}_1 z + \mathbb{T}_2 z^* \in \mathcal{A}_\sigma$. It is easy to find, by using (3.11) that $\mathbb{T}_2$ is a contraction map.

$\mathbb{T}_1$ is continuous, due to $\mathfrak{F}_z(\cdot) = f(\cdot, z(\cdot), \mathfrak{F}_z(\cdot)) \ \in C(\cdot \times \mathbb{R}^2, \mathbb{R})$. Also, $\mathbb{T}_1$ is uniformly bounded on $\mathcal{A}_\sigma$ because we have from (H_2) that

$$\|\mathbb{T}_1 z\| \leq \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} \|\Theta\| .$$

In addition, we prove the compactness of $\mathbb{T}_1$ as follows.

Let $\upsilon_1, \upsilon_2 \in \mathbb{J}$ such that $\upsilon_1 < \upsilon_2$. Then

$$\begin{aligned}
&|(\mathbb{T}_1 z)(\upsilon_2) - (\mathbb{T}_1 z)(\upsilon_1)| \\
&\leq \frac{1}{\Gamma(\varrho_1)} \left| \int_a^{\upsilon_1} \varsigma'(t)[(\varsigma(\upsilon_2) - \varsigma(t))^{\varrho_1 - 1} - ((\varsigma(\upsilon_1) - \varsigma(t))^{\varrho_1 - 1}] \mathfrak{F}_z(t)) dt \right. \\
&\quad \left. + \int_{\upsilon_1}^{\upsilon_2} \varsigma'(t)((\varsigma(\upsilon_2) - \varsigma(t))^{\varrho_1 - 1} \mathfrak{F}_z(t) dt \right| \\
&\leq \frac{\|\Theta\|}{\Gamma(\varrho_1 + 1)} \left[2(\varsigma(\upsilon_2) - \varsigma(\upsilon_1))^{\varrho_1} + |(\varsigma(\upsilon_2) - \varsigma(a))^{\varrho_1} - (\varsigma(\upsilon_1) - \varsigma(a))^{\varrho_1}| \right].
\end{aligned}$$

The last inequality with $\upsilon_2 - \upsilon_1 \to 0$, gives

$$|(\mathbb{T}_1 z)(\upsilon_2) - (\mathbb{T}_1 z)(\upsilon_1)| \to 0, \quad \text{for } l \ \ |\upsilon_2 - \upsilon_1| \to 0, \ z \in \mathcal{A}_\sigma.$$

Then, $\mathbb{T}_1$ is relatively compact on $\mathcal{A}_\sigma$. An application of the Arzel-Ascoli theorem, $\mathbb{T}_1$ is compact on $\mathcal{A}_\sigma$. Hence, all the assumptions of Krasnoselskii's fixed point theorem are satisfied. So, we infer that (1.1) has at least one solution on $\mathbb{J}$. □

3.2. **Integrodifferential-type problem (1.2).**

Theorem 3.4. *Assume that* (H_1) *holds. If*

$$\kappa\kappa_1\Lambda < 1, \tag{3.13}$$

where Λ *is defined by* (3.8) *and* $\kappa_1 = 1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}$, *then the integrodifferential-type problem* (1.2) *has a unique solution on* $\mathbb{J}$.

Proof. We convert (1.2) into a fixed point problem,that is, $z = \mathbb{T}^\star z$ such that $\mathbb{T}^\star : C(\mathbb{J},\mathbb{R}) \to C(\mathbb{J},\mathbb{R})$ defined by

$$\begin{aligned}
&(\mathbb{T}^\star z)(\upsilon)\\
&= \mathfrak{I}^{\varrho_1,\varsigma} f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(\upsilon) + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\\
&\quad\times\left(\sum_{r=1}^{m-2}\xi_r\ \mathfrak{I}^{\varrho_1,\varsigma} f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(\theta_r) - \mathfrak{I}^{\varrho_1+1;\varsigma}\ f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(b) - \hbar\right).
\end{aligned} \tag{3.14}$$

Note that the fixed points of $\mathbb{T}^\star$ are solutions of (1.2). We will show that $\mathbb{T}^\star$ has a unique fixed point by using Banach theorem [13]. Indeed, we select

$$\beta \geq \frac{\Lambda\mathbb{N} + (\varsigma(b)-\varsigma(a))^{\zeta-1}\left|\frac{\hbar}{\Omega}\right|}{1-\kappa\kappa_1\Lambda},$$

where $\mathbb{N}$ is as in Theorem 3.2. First, we show that $\mathbb{T}^\star\mathcal{S}_\beta \subset \mathcal{S}_\beta$, where $\mathcal{S}_\beta = \{z \in C(\mathbb{J},\mathbb{R}) : \|z\| \leq \beta\}$. By using (H_1), we obtain

$$\begin{aligned}
|f(\upsilon,z(\upsilon),\mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon))| &\leq |f(\upsilon,z(\upsilon),\mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon)) - f(\upsilon,0,0)| + |f(\upsilon,0,0)|\\
&\leq \kappa|z(\upsilon)| + \kappa|\mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon)| + \mathbb{N}\\
&\leq \kappa\|z\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}\right) + \mathbb{N}.
\end{aligned}$$

For any $z \in \mathcal{S}_\beta$, we get

$$\begin{aligned}
|(\mathbb{T}^\star z)(\upsilon)| \leq \sup_{\upsilon\in\mathbb{J}}\Bigg\{ &\quad \mathfrak{I}^{\varrho_1,\varsigma}|f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(\upsilon)| + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{|\Omega|}\\
&\times\left(\sum_{r=1}^{m-2}|\xi_r|\,\mathfrak{I}^{\varrho_1,\varsigma}|f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(\theta_r)|\right.\\
&\left.+\,\mathfrak{I}^{\varrho_1+1;\varsigma}|f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))(b)| + |\hbar|\right)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathfrak{I}^{\varrho_1;\varsigma}\left(\kappa\|z\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}\right)+\mathbb{N}\right)(b)\\
&\quad+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\left[\sum_{r=1}^{m-2}|\xi_r|\,\mathfrak{I}^{\varrho_1;\varsigma}\left(\kappa\|z\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}\right)+\mathbb{N}\right)(\theta_r)\right.\\
&\quad\left.+\,\mathfrak{I}^{\varrho_1+1;\varsigma}\left(\kappa\|z\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}\right)+\mathbb{N}\right)(b)+|\hbar|\right]\\
&\leq\left\{\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\left[\sum_{r=1}^{m-2}|\xi_r|\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right.\right.\\
&\quad\left.\left.+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}\right]\right\}\\
&\quad\times\left(\kappa\|z\|\left(1+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_3}}{\Gamma(\varrho_3+1)}\right)+\mathbb{N}\right)+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}|\hbar|\\
&\leq\Lambda\left(\kappa\kappa_1\beta+\mathbb{N}\right)+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}|\hbar|\\
&\leq\beta.
\end{aligned}
$$

This means that $\mathbb{T}^\star\mathcal{S}_\beta\in\mathcal{S}_\beta$, that is, $\mathbb{T}^\star\mathcal{S}_\beta\subset\mathcal{S}_\beta$.

Next, for each $z,z^\star\in C(\mathbb{J},\mathbb{R})$ and $\upsilon\in\mathbb{J}$, we have

$$
\begin{aligned}
&|(\mathbb{T}^\star z)(\upsilon)-(\mathbb{T}^\star z^\star)(\upsilon)|\\
&\leq\mathfrak{I}^{\varrho_1;\varsigma}\left|f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))-f(t,z^\star(t),\mathfrak{I}^{\varrho_3;\varsigma}z^\star(t))\right|(b)+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{\Omega}\\
&\quad\times\left\{\sum_{r=1}^{m-2}|\xi_r|\,\mathfrak{I}^{\varrho_1,\varsigma}\left|f(\theta_r,z(\theta_r),\mathfrak{I}^{\varrho_3;\varsigma}z(\theta_r))-f(\theta_r,z^\star(\theta_r),\mathfrak{I}^{\varrho_3;\varsigma}z^\star(\theta_r))\right|\right.\\
&\quad+\mathfrak{I}^{\varrho_1+1;\varsigma}\left|f(t,z(t),\mathfrak{I}^{\varrho_3;\varsigma}z(t))-f(t,z^\star(t),\mathfrak{I}^{\varrho_3;\varsigma}z^\star(t))\right|(b)\\
&\leq\kappa\kappa_1\left\{\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(b)-\varsigma(a))^{\zeta-1}}{|\Omega|}\right.\\
&\quad\left.\times\left[\sum_{r=1}^{m-2}|\xi_r|\frac{(\varsigma(\theta_r)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}\right]\right\}\|z-z^\star\|\\
&=\kappa\kappa_1\Lambda\|z-z^\star\|,
\end{aligned}
$$

which leads us to $\|\mathbb{T}^\star z-\mathbb{T}^\star z^\star\|\leq\kappa\kappa_1\Lambda\|z-z^\star\|$. By (3.13), $\mathbb{T}$ is a contraction. Then, a unique solution exists on $\mathbb{J}$ for (1.2) by virtue of the Banach's fixed point theorem [13], and this completes the proof. □

Second, we will use the Krasnoselskii's fixed point theorem [13] to prove the existence result for the integrodifferential-type problem (1.1).

Theorem 3.5. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\kappa\kappa_1\Lambda^\star < 1, \tag{3.15}$$

where $\Lambda^\star$ *is defined by* (3.9)*, then the integrodifferential-type problem* (1.2) *has at least one solution on* $\mathbb{J}$.

Proof. Consider the ball $\mathcal{S}_\Pi = \{z \in C(\mathbb{J}, \mathbb{R}) : \|z\| \le \Pi\}$ where $\Pi > 0$ with

$$\Pi \ge \Lambda \|\Theta\| + (\varsigma(b) - \varsigma(a))^{\zeta-1} \left|\frac{\hbar}{\Omega}\right|.$$

Then we build the operators $\mathbb{T}_1^\star$, $\mathbb{T}_2^\star$ on $\mathcal{S}_\Pi$ by

$$(\mathbb{T}_1^\star z)(\upsilon) = \mathfrak{I}^{\varrho_1,\varsigma} f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t))(\upsilon), \quad \upsilon \in \mathbb{J}$$

and

$$\begin{aligned}(\mathbb{T}_2^\star z)(\upsilon) \quad = \quad & \frac{(\varsigma(\upsilon) - \varsigma(a))^{\zeta-1}}{\Omega} \left(\sum_{r=1}^{m-2} \xi_r \; \mathfrak{I}^{\varrho_1,\varsigma} f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t))(\theta_r) \right. \\ & \left. - \, \mathfrak{I}^{\varrho_1+1;\varsigma} \; f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t))(b) - \hbar \right).\end{aligned}$$

For any $z, z^\star \in \mathcal{S}_\Pi$, we get

$$\begin{aligned}
&|(\mathbb{T}_1^\star z)(\upsilon) + (\mathbb{T}_2^\star z^\star)(\upsilon)| \\
&\quad \le \sup_{\upsilon \in \mathbb{J}} \Big\{ \quad \mathfrak{I}^{\varrho_1;\varsigma} \left| f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t)) \right| (\upsilon) \\
&\qquad + \frac{(\varsigma(\upsilon) - \varsigma(a))^{\zeta-1}}{|\Omega|} \left[\sum_{r=1}^{m-2} |\xi_r| \, \mathfrak{I}^{\varrho_1;\varsigma} \left| f(t, z^\star(t), \mathfrak{I}^{\varrho_3;\varsigma} z^\star(t)) \right| (\theta_r) \right. \\
&\qquad + \left. \quad \mathfrak{I}^{\varrho_1+1;\varsigma} \left| \; f(t, z^\star(t), \mathfrak{I}^{\varrho_3;\varsigma} z^\star(t)) \right| (b) + |\hbar| \right] \Big\} \\
&\quad \le \left\{ \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} + \frac{(\varsigma(b) - \varsigma(a))^{\zeta-1}}{|\Omega|} \right. \\
&\qquad \times \left. \left[\sum_{r=1}^{m-2} |\xi_r| \frac{(\varsigma(\theta_r) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} + \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1 + 2)} \right] \right\} \|\Theta\| \\
&\qquad + \frac{(\varsigma(b) - \varsigma(a))^{\zeta-1}}{|\Omega|} |\hbar| \\
&\quad = \Lambda \|\Theta\| + (\varsigma(b) - \varsigma(a))^{\zeta-1} \left| \frac{\hbar}{\Omega} \right| \\
&\quad \le \Pi.
\end{aligned}$$

This proves that $\mathbb{T}_1^\star z + \mathbb{T}_2^\star z^\star \in \mathcal{S}_\Pi$. It is easy to find that $\mathbb{T}_2^*$ is a contraction by using (3.15).

$\mathbb{T}_1^\star$ is continuous, due to $f(\cdot, z(\cdot), \mathfrak{I}^{\varrho_3;\varsigma} z(\cdot)) \in C(\cdot \times \mathbb{R}^2, \mathbb{R})$. Also, $\mathbb{T}_1^\star$ is uniformly bounded on $\mathcal{S}_\Pi$ because we have from (H_2) that

$$\|\mathbb{T}_1^\star z\| \leq \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} \|\Theta\| .$$

The compactness of $\mathbb{T}_1$ holds.

Indeed, we define $\sup_{(t,z_1,z_2)\in\mathbb{J}\times\mathcal{S}_\Pi\times\mathcal{S}_\Pi} |f(t, z_1, z_2)| = \Pi_f$ and let $\upsilon_1, \upsilon_2 \in \mathbb{J}$ such that $\upsilon_1 < \upsilon_2$. Then

$$\begin{aligned}
&|(\mathbb{T}_1^\star z)(\upsilon_2) - (\mathbb{T}_1^\star z)(\upsilon_1)| \\
&\leq \frac{1}{\Gamma(\varrho_1)} \left| \int_a^{\upsilon_1} \varsigma'(t)[(\varsigma(\upsilon_2) - \varsigma(t))^{\varrho_1 - 1} - ((\varsigma(\upsilon_1) - \varsigma(t))^{\varrho_1 - 1}] f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t)) dt \right. \\
&\quad \left. + \int_{\upsilon_1}^{\upsilon_2} \varsigma'(t)((\varsigma(\upsilon_2) - \varsigma(t))^{\varrho_1 - 1} f(t, z(t), \mathfrak{I}^{\varrho_3;\varsigma} z(t)) dt \right| \\
&\leq \frac{\Pi_f}{\Gamma(\varrho_1 + 1)} \left[2(\varsigma(\upsilon_2) - \varsigma(\upsilon_1))^{\varrho_1} + |(\varsigma(\upsilon_2) - \varsigma(a))^{\varrho_1} - (\varsigma(\upsilon_1) - \varsigma(a))^{\varrho_1}| \right].
\end{aligned}$$

Since $\upsilon_2 - \upsilon_1 \to 0$, we obtain

$$|(\mathbb{T}_1^\star z)(\upsilon_2) - (\mathbb{T}_1^\star z)(\upsilon_1)| \to 0, \text{ for any } z \in \mathcal{S}_\Pi.$$

Therefore, $\mathbb{T}_1^\star$ is equicontinuous. Consequently, $\mathbb{T}_1^\star$ is relatively compact on $\mathcal{S}_\Pi$. An implementation of the Arzelá-Ascoli theorem, $\mathbb{T}_1^\star$ is compact on $\mathcal{S}_\Pi$. So, all the assumptions of Krasnoselskii's fixed point theorem are fulfilled. Thus, we infer that (1.2) has at least one solution on $\mathbb{J}$. □

Remark 3.6. In Theorem 3.2, we can reciprocity the roles of the operators $\mathbb{T}_1^\star$ and $\mathbb{T}_2^\star$ to get other result along with following condition:

$$\Upsilon_1 := \frac{\kappa}{1-\kappa} \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} < 1. \tag{3.16}$$

Remark 3.7. In Theorem 3.4, we can reciprocity the roles of the operators $\mathbb{T}_1^\star$ and $\mathbb{T}_2^\star$ to get other results along with following condition:

$$\Upsilon_2 := \kappa\kappa_1 \frac{(\varsigma(b) - \varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1 + 1)} < 1. \tag{3.17}$$

Corollary 3.8. *Assume that* (H_1), (H_2) *and condition* (3.16) *hold. Then the implicit-type problem* (1.1) *has at least one solution on* $\mathbb{J}$.

Corollary 3.9. *Assume that* (H_1), (H_2) *and condition* (3.17) *hold. Then the integrodifferential-type problem* (1.2) *has at least one solution on* $\mathbb{J}$.

4. UH Stability Analysis

In this section, we discuss the UH and GUH stability of the problem (1.2).

Definition 4.1. The problem (1.2) is UH stable if there exists a constant $\Upsilon_f > 0$ such that for each $\varepsilon > 0$ and every solution $\overline{z} \in C(\mathbb{J}, \mathbb{R})$ of the inequalities

$$\left|{}_{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}\overline{z}(\upsilon) - f(\upsilon, \overline{z}(\upsilon), \mathfrak{I}^{\varrho_3;\varsigma}\overline{z}(\upsilon))\right| \leq \varepsilon, \text{ for all } \upsilon \in \mathbb{J}, \tag{4.1}$$

there exists a solution $z \in C(\mathbb{J}, \mathbb{R})$ of the problem (1.2) that satisfies

$$|z(\upsilon) - \overline{z}(\upsilon)| \leq \Upsilon_f \varepsilon. \tag{4.2}$$

Remark 4.2. *$\overline{z} \in C(\mathbb{J}, \mathbb{R})$ satisfies the inequality (4.1) if and only if there exists a function $\Pi \in C(\mathbb{J}, \mathbb{R})$ with*

(1) $|\Pi(\upsilon)| \leq \varepsilon$, $\upsilon \in \mathbb{J}$,
(2) *for all* $\upsilon \in \mathbb{J}$,

$${}_{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}\overline{z}(\upsilon) = f(\upsilon, \overline{z}(\upsilon), \mathfrak{I}^{\varrho_3;\varsigma}\overline{z}(\upsilon)) + |\Pi(\upsilon)|.$$

Lemma 4.3. *If $z \in C(\mathbb{J}, \mathbb{R})$ is a solution to inequality (4.1), then z satisfies the following inequality*

$$|z(\sigma) - \Theta_z| \leq \varepsilon\Delta,$$

where

$$\Theta_{\overline{z}} = \mathfrak{I}^{\varrho_1,\varsigma}\mathfrak{F}_z(\upsilon) + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left(\sum_{r=1}^{m-2}\xi_r\mathfrak{I}^{\varrho_1,\varsigma}\mathfrak{F}_z(\theta_r) - \mathfrak{I}^{\varrho_1+1;\varsigma}\mathfrak{F}_z(b) - \hbar\right)$$

and

$$\begin{aligned}\Delta &= \frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)} + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left\{\sum_{r=1}^{m-2}\xi_r\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right.\\ &\quad \left. -\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)} - \frac{\hbar}{\varepsilon}\right\}.\end{aligned}$$

Proof. In view of Remark 4.2, we have

$$\begin{cases} {}_{H}\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}z(\upsilon) = f(\upsilon, z(\upsilon), \mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon)) + |\Pi(\upsilon)| \\ z(a) = 0,\ \int_a^b \varsigma'(t)z(t)dt + \hbar = \sum_{r=1}^{m-2}\xi_r z(\theta_r) \end{cases}.$$

Then, by Lemma 3.1, we get

$$\begin{aligned} z(\upsilon) &= \Theta_z + \mathfrak{I}^{\varrho_1,\varsigma}\Pi(\upsilon) \\ &\quad + \frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left(\sum_{r=1}^{m-2}\xi_r\mathfrak{I}^{\varrho_1,\varsigma}\Pi(\theta_r) - \mathfrak{I}^{\varrho_1+1;\varsigma}\Pi(b) - \hbar\right), \end{aligned}$$

which implies

$$
\begin{aligned}
|z(\upsilon)-\Theta_z| &\leq \varepsilon\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\varepsilon\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega} \\
&\quad\times\left(\sum_{r=1}^{m-2}\xi_r\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}-\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}-\frac{\hbar}{\varepsilon}\right) \\
&= \varepsilon\left(\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}+\frac{(\varsigma(\upsilon)-\varsigma(a))^{\zeta-1}}{\Omega}\left\{\sum_{r=1}^{m-2}\xi_r\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1}}{\Gamma(\varrho_1+1)}\right.\right. \\
&\quad\left.\left.-\frac{(\varsigma(b)-\varsigma(a))^{\varrho_1+1}}{\Gamma(\varrho_1+2)}-\frac{\hbar}{\varepsilon}\right\}\right) \\
&\leq \varepsilon\Delta.
\end{aligned}
$$

□

Theorem 4.4. *Assume that (H_1)-(H_2) hold. Under the Lemma* 4.3, *the following equation*

$$
{}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}z(\upsilon)=f(\upsilon,z(\upsilon),\mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon)),\ \ \upsilon\in\mathbb{J}, \tag{4.3}
$$

is UH stable as well as GUH provided that $\kappa\kappa_1\Lambda<1$.

Proof. Let $\overline{z}\in C(\mathbb{J},\mathbb{R})$ be a function satisfies (4.1), and $z\in C(\mathbb{J},\mathbb{R})$ be unique solution of the following problem

$$
\begin{cases}
{}_H\mathfrak{D}^{\varrho_1,\varrho_2;\varsigma}z(\upsilon)=f(\upsilon,z(\upsilon),\mathfrak{I}^{\varrho_3;\varsigma}z(\upsilon)), \quad \upsilon\in\mathbb{J}, \\
z(a)=0,\ \int_a^b\varsigma'(t)z(t)dt+\hbar=\sum_{r=1}^{m-2}\xi_r z(\theta_r), \\
z(\upsilon)=\overline{z}(\upsilon), \upsilon\in\mathbb{J}.
\end{cases}
$$

Then, by Lemma 3.1, we get

$$
z(\upsilon)=\Theta_z.
$$

It follows from Theorem 3.4, that

$$
\begin{aligned}
\|z-\overline{z}\| &= \sup_{\upsilon\in\mathbb{J}}|z(\upsilon)-\Theta_{\overline{z}}|\leq\sup_{\upsilon'\in\mho}|z(\upsilon)-\Theta_z|+\sup_{\upsilon'\in\mho}|\Theta_z-\Theta_{\overline{z}}| \\
&\leq \varepsilon\Delta+\kappa\kappa_1\Lambda\|z-\overline{z}\|.
\end{aligned}
$$

Thus

$$
\|z-\overline{z}\|\leq\Upsilon_f\varepsilon,
$$

where

$$
\Upsilon_f=\frac{\Delta}{1-\kappa\kappa_1\Lambda}>0.
$$

Now, by choosing $\varphi_\kappa(\varepsilon)=\Upsilon_f\varepsilon$ such that $\varphi_\kappa(0)=0$, then the problem (4.3) is GUH stability. □

5. Examples

Example 5.1. Consider the implicit-type problem

$$\begin{cases} {}_H\mathfrak{D}^{\frac{4}{3},\frac{1}{3};2^{\upsilon}} z(\upsilon) = \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} \left(\frac{1}{1+|z(\upsilon)|+\left|{}_H\mathfrak{D}^{\frac{4}{3},\frac{1}{3};2^{\upsilon}} z(\upsilon)\right|} \right) + \frac{1}{2}, \ \upsilon \in [0,1], \\ z(0) = 0, \ \int_0^1 \varsigma'(t) z(t) dt + \frac{1}{2} = \frac{3}{5} z(\frac{4}{11}) + \frac{4}{5} z(\frac{5}{11}). \end{cases} \tag{5.1}$$

Here $\varrho_1 = \frac{4}{3}, \varrho_2 = \frac{1}{3}, \xi_1 = \frac{3}{5}$, $\xi_2 = \frac{4}{5}$, $\theta_1 = \frac{4}{11}$, $\theta_2 = \frac{5}{11}$, $\zeta = \frac{14}{9}$, $\varsigma(\upsilon) = 2^{\upsilon}$ and $\hbar = \frac{1}{2}$.

Set

$$f(\upsilon, z, y) = \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} \left(\frac{1}{1+z+y} \right) + \frac{1}{2}.$$

For each $z, z^*, y, y^* \in \mathbb{R}$, we have

$$\begin{aligned} |f(\upsilon, z, y) - f(\upsilon, z^*, y^*)| &= \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} \left| \frac{1}{1+z+y} - \frac{1}{1+z^*+y^*} \right| \\ &\leq \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} \frac{|z-z^*| + |y-y^*|}{(1+z+y)(1+z^*+y^*)} \\ &\leq \frac{1}{9} \left(|z-z^*| + |y-y^*| \right), \text{ for } \upsilon \in [0,1]. \end{aligned}$$

Hence, (H_1) holds with $\kappa = \frac{1}{9}$. Moreover, for each $(\upsilon, z, y) \in \mathbb{J} \times \mathbb{R}^2$, we have

$$\begin{aligned} |f(\upsilon, z, y)| &\leq \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} \left(\frac{1}{1+|z(\upsilon)|+|y(\upsilon)|} \right) + \frac{1}{2} \\ &\leq \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} + \frac{1}{2}. \end{aligned}$$

Consequently, (H_2) holds with $\Theta(\upsilon) = \frac{e^{-\upsilon}}{(e^{\upsilon}+8)} + \frac{1}{2}$.

Next, we can find that $\Omega = -0.11756 \neq 0$ and $\frac{\kappa}{1-\kappa}\Lambda = 0.779 < 1$, the Theorem 3.2 is fulfilled, then the implicit-type problem (5.1) has a unique solution on $[0,1]$. Also, we can find $\frac{\kappa}{1-\kappa}\Lambda^* = 0.674 < 1$, the Theorem 3.3 is fulfilled, then the implicit-type problem (5.1) has at least one solution on $[0,1]$.Additionally, we can find $\Upsilon_1 = 0.104 < 1$, hence the Corollary 3.8 hold.

Example 5.2. Consider the integrodifferential-type problem:

$$\begin{cases} {}_H\mathfrak{D}^{\frac{5}{3},\frac{1}{2};\upsilon} z(\upsilon) = \frac{1}{(10+\upsilon)} \left[\frac{|z(\upsilon)|}{1+|z(\upsilon)|} + \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} z(\upsilon) \right) \right] + \frac{\upsilon}{2}, \ \upsilon \in [0,1], \\ z(0) = 0, \ \int_0^1 \varsigma'(t) z(t) dt + \frac{1}{30} = \frac{2}{5} z(\frac{4}{5}). \end{cases} \tag{5.2}$$

Here $\xi_1 = \frac{2}{5}$, $\theta_1 = \frac{4}{5}$, $\varrho_1 = \frac{5}{3}$, $\varrho_2 = \frac{1}{2}$, $\varrho_3 = \frac{3}{2}$, $\zeta = \frac{11}{6}$ and $\hbar = \frac{1}{30}$ with $\varsigma(\upsilon) = \upsilon$. For each $z, z^*, y, y^* \in \mathbb{R}$, we have

$$\begin{aligned}
&\left| f(\upsilon, z, \mathfrak{I}^{\frac{3}{2};\varsigma} z) - f(\upsilon, y, \mathfrak{I}^{\frac{3}{2};\varsigma} y) \right| \\
&= \frac{1}{(10+\upsilon)} \left| \frac{|z|}{1+|z|} + \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} z\right) - \frac{|y|}{1+|y|} - \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} y\right) \right| \\
&\leq \frac{1}{(10+\upsilon)} \left[\frac{|z-y|}{(1+|z|)(1+|y|)} + \left| \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} z\right) - \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} y\right) \right| \right] \\
&\leq \frac{1}{(10+\upsilon)} \left[\ |z-y| + \left| \mathfrak{I}^{\frac{3}{2};\upsilon} z - \mathfrak{I}^{\frac{3}{2};\upsilon} y \right| \right].
\end{aligned}$$

For $\upsilon \in [0,1]$, we have

$$\left| f(\upsilon, z, \mathfrak{I}^{\frac{3}{2};\upsilon} z) - f(\upsilon, y, \mathfrak{I}^{\frac{3}{2};\upsilon} y) \right| \leq \frac{1}{10} \left[\ |z-y| + \left| \mathfrak{I}^{\frac{3}{2};\upsilon} z - \mathfrak{I}^{\frac{3}{2};\upsilon} y \right| \right].$$

Hence, (H_1) holds with $\kappa = \frac{1}{10}$. Moreover, for each $(\upsilon, z, y) \in \mathbb{J} \times \mathbb{R}^2$, we have

$$\begin{aligned}
|f(\upsilon, z, y)| &\leq \frac{1}{(10+\upsilon)} \left[\frac{|z(\upsilon)|}{1+|z(\upsilon)|} + \sin\left(\mathfrak{I}^{\frac{3}{2};\upsilon} z(\upsilon)\right) \right] + \frac{\upsilon}{2} \\
&\leq \frac{2}{(10+\upsilon)} + \frac{\upsilon}{2}.
\end{aligned}$$

Consequently, (H_2) holds with $\Theta(\upsilon) = \frac{2}{(10+\upsilon)} + \frac{\upsilon}{2}$.

Next, we can find that $\Omega = 0.213$, $\kappa\kappa_1\Lambda = 0.379 < 1$, $\kappa\kappa_1\Lambda^\star = 0.337 < 1$ and $\Upsilon_2 = 0.110 < 1$. Therefore, by the applying of Theorems 3.3, 3.5, and Corollary 3.9, the problem (5.2) has a solution which is unique.

6. Conclusions

In this work, we have proven the existence and uniqueness of solution for nonlinear implicit-type FDEs and integrodifferential-type FDEs with nonlocal integral-multipoint boundary conditions in the frame of ς-Hilfer FD. The analysis of the main results is based on the employment of the fixed point theorems of Banach and Krasnoselskii. The obtained results extend many fundamental results existing in the current literature for other types of FDs. We made some observations and special cases related to the function ς which generates many other FDs.

Further, it will be of interest to investigate the current problem in this work for the Mittag-Leffler power low [11] and for fractal fractional operators [10].

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