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FIXED POINTS OF α_s-β_s-ψ-CONTRACTIVE MAPPINGS IN S-METRIC SPACES

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Abstract. In this paper, we have developed the idea of α-β-ψ-contractive mapping in S-metric space and renamed it α_s-β_s-ψ-contractive mapping. We have proved some results of fixed point present in literature in partially ordered S-metric space using α_s-β_s-admissible and α_s-β_s-ψ-contractive mapping.

1. Introduction and Preliminaries

The theory of fixed point has been applied to different fields of study throughout the last four-five decades. Samet et al. [20] attempted to generalize Banach fixed point theorem to contribute by developing the idea of α-admissible mappings and further the idea of α-ψ-contractive mappings in metric spaces. The study of Samet et al. [20] demonstrate that Banach's fixed point result and other conclusions are natural implications of their results.

The notion of α-admissible mappings is further expanded to S-metric space, S_b-metric space, G-metric space, etc. Zhou et al. [24] expanded the notion of α-admissible mappings to S-metric space for mapping and pair of mappings.

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Further, they also defined various types of contractions of mappings viz. type-A, type-B, etc. [24].

Priyobarta et al. [16] also introduce the notion of α-admissible mappings in the perspective of S-metric spaces and denote it as α_s-admissible mappings. Further, they established many theorems of fixed point regarding various types of contractive mappings due to α_s-admissibility.

Recently, the presence of fixed points, in partially ordered sets has been studied in [1, 2, 3, 4, 6, 7, 8, 11, 12, 14, 15, 17]. In the row of extension and generalization, Asgari et al. [2] considered α-ψ-contractive type mappings with a supplementary condition for partially ordered set and solved a first-order boundary value problem in connection with its lower solution. Further Asgari et al. [3] introduce the notion of α-β-ψ-contractive mappings and proved various results of the fixed point in a partially ordered metric space. For more information reader are suggested to see the papers [5, 9, 10, 13, 18, 19, 22, 23, 25].

In this paper, we have introduced the notion of α-β-ψ-contractive mappings in S-metric space and denote it as α_s-β_s-ψ-contractive mappings and established some theorems of the fixed point in S-metric space equipped with a partial order. The proposed theorems are expansions in the S-metric space of theorems found in the literature, specifically, the results of Ran and Reurings [17], Harjani and Sadrangani [6] and Nieto et al. [12, 13]. Further, we applied the collected results to find the solution to the boundary value issues of the first-order ODE in comparison to its lower solution.

Definition 1.1. If $(U, \leq)$ is a partially ordered set. The mapping $G : U \to U$ is considered as monotonic non-decreasing if

$$l \leq l' \implies G(l) \leq G(l'), \quad \text{for all} \quad l, l' \in U.$$

Definition 1.2. ([20]) We consider Ψ a collection of mappings $\psi : [0, +\infty) \to [0, +\infty)$ such that ψ is non-decreasing and

$$\sum_{0}^{\infty} \psi^n(k) < +\infty, \text{ for all } \ k > 0,$$

where, ψ^n represents n^{th} iteration of ψ.

Lemma 1.3. ([20]) *If a mapping $\psi : [0, +\infty) \to [0, +\infty)$ is non-decreasing such that*

$$\lim_{n\to\infty} \psi^n(k) = 0, \ for \ all \ \ k > 0,$$

then $\psi(k) < k$.

In 2012, Sedghi et al. [21] introduced the concept of S-metric space and defined it as follows;

Definition 1.4. ([21]) Let U be a nonempty set. An S-metric on U is a function $S : U \times U \times U \to [0, \infty)$ that satisfies the following conditions for each $l_1, l_2, l_3, a \in U$:

($\mathcal{S}_1$) $S(l_1, l_2, l_3) \geq 0$,
($\mathcal{S}_2$) $S(l_1, l_2, l_3) = 0$ if and only if $l_1 = l_2 = l_3$,
($\mathcal{S}_3$) $S(l_1, l_2, l_3) \leq S(l_1, l_1, a) + S(l_2, l_2, a) + S(l_3, l_3, a)$.

The pair (U, S) is called an S-metric space.

Example 1.5. ([21]) Let U be a nonempty set and d be an ordinary metric on U. Then $S(l_1, l_2, l_3) = d(l_1, l_3) + d(l_2, l_3)$ is an S-metric on U.

Lemma 1.6. ([21]) *Let (U, S) be an S-metric space. Then for all $l_1, l_2 \in U$, we have*

$$S(l_1, l_1, l_2) = S(l_2, l_2, l_1).$$

Definition 1.7. ([21]) Let (U, S) be an S-metric space,

(i) A sequence $\{l_n\}$ in X converges to l if $S(l_n, l_n, l) \to 0$ as $n \to +\infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that, for all $n \geq n_0$, $S(l_n, l_n, l) < \varepsilon$, and we denote this by $\lim_{n\to+\infty} l_n = l$.

(ii) A sequence $\{l_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in N$ such that $S(l_n, l_n, l_m) < \varepsilon$ for each $n, m \geq n_0$.

(iii) The S-metric space (U, S) is said to be complete if every Cauchy sequence is convergent.

2. Main results

We extended the concept of α-β-ψ-contractive mappings of Asgari and Badehian [3] in partially ordered, complete S-metric space and defined it as follows.

Definition 2.1. Let $(U, \leq, S)$ be a partially ordered, complete S-metric space. The mapping $G : U \to U$ is said to be an α_s-β_s-ψ-contractive mapping of type-A if $\alpha_s, \beta_s : U \times U \times U \to [0, +\infty)$ and $\psi \in \Psi$ are such that

$$\alpha_s(l_1, l_2, l_3) S(G(l_1), G(l_2), G(l_3)) \leq \beta_s(l_1, l_2, l_3) \psi(S(l_1, l_2, l_3), \tag{2.1}$$

for all $l_1, l_2, l_3 \in U$ with $l_1 \geq l_2 \geq l_3$.

Definition 2.2. Let $(U, \leq, S)$ be a partially ordered, complete S-metric space. The mapping $G : U \to U$ is said to be an α_s-β_s-ψ-contractive mapping of type-B if $\alpha_s, \beta_s : U \times U \times U \to [0, +\infty)$ and $\psi \in \Psi$ are such that

$$\alpha_s(l_1, l_1, l_2)S(G(l_1), G(l_1), G(l_2)) \leq \beta_s(l_1, l_1, l_2)\psi(S(l_1, l_1, l_2)), \tag{2.2}$$

for all $l_1, l_2 \in U$ with $l_1 \geq l_2$.

Example 2.3. A mapping $G : U \to U$ satisfying the Banach contraction principle and $\alpha_s(l_1, l_2, l_3) = \beta_s(l_1, l_2, l_3) = 1$ for all $l_1, l_2, l_3 \in U$ with $\psi(k) = \delta k$ for all $k \geq 0$, where $\delta \in [0, 1)$. Then G is an α_s-β_s-ψ-contractive mapping.

Definition 2.4. Let $G : U \to U$, $\alpha_s, \beta_s : U \times U \times U \to [0, +\infty)$ and $c_{\alpha_s} > 0$, $c_{\beta_s} \geq 0$. G is said to be an α_s-β_s-admissible mapping if for all $l_1,\ l_2,\ l_3 \in U$ with $l_1 \geq l_2 \geq l_3$,

(a) $\alpha_s(l_1, l_2, l_3) \geq c_{\alpha_s} \implies \alpha_s(G(l_1), G(l_2), G(l_3)) \geq c_{\alpha_s}$;
(b) $\beta_s(l_1, l_2, l_3) \leq c_{\beta_s} \implies \beta_s(G(l_1),\ G(l_2),\ G(l_3)) \leq c_{\beta_s}$;
(c) $0 \leq \frac{c_{\beta_s}}{c_{\alpha_s}} \leq 1$.

Example 2.5. Let $U = (0, +\infty)$ and $G : U \to U$ be defined by $G(l) = e^l$, for all $l \in U$. If $\alpha_s, \beta_s : U \times U \times U \to [0, +\infty)$ are such that

$$\alpha_s(l_1, l_2, l_3) = \begin{cases} 3, & if\ l_1 \geq l_2 \geq l_3; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta_s(l_1, l_2, l_3) = \begin{cases} \frac{1}{4}, & if\ l_1 \geq l_2 \geq l_3; \\ 0, & \text{otherwise.} \end{cases}$$

If we take $c_{\alpha_s} = 2$ and $c_{\beta_s} = \frac{1}{2}$, then G is α_s-β_s-admissible.

Theorem 2.6. *Let $(U, \leq, S)$ be a partially ordered, complete S-metric space. Let a non-decreasing mapping $G : U \to U$ be an α_s-β_s-ψ-contractive mapping of type A with;*

(a) *G is α_s-β_s-admissible;*
(b) *there exists $l_0 \in U$ such that $l_0 \leq G(l_0)$;*
(c) *there exists $c_{\alpha_s} > 0$, $c_{\beta_s} \geq 0$ such that $\alpha_s(G(l_0), G(l_0), l_0) \geq c_{\alpha_s}$, $\beta_s(G(l_0), G(l_0), l_0) \leq c_{\beta_s}$;*
(d) *G is continuous.*

Then, $G(l^*) = l^*$ for some $l^* \in U$, that is, G has a fixed point..

Proof. Let there exists $l_0 \in U$ such that $l_0 \leq G(l_0)$. If $G(l_0) = l_0$ then, there is nothing to prove. Suppose $G(l_0) \neq l_0$. Since $l_0 \leq G(l_0)$ and mapping is non-decreasing, by induction we get

$$l_0 \leq G(l_0) \leq G^2(l_0) \leq G^3(l_0) \leq \cdots \leq G^n(l_0) \leq G^{n+1}(l_0) \leq \cdots . \tag{2.3}$$

Due to α_s-β_s-admissibility of G, if $\alpha_s(G(l_0), G(l_0), l_0) \geq c_{\alpha_s}$, then

$$\begin{aligned} &\alpha_s(G^2(l_0), G^2(l_0), G(l_0)) \geq c_{\alpha_s}, \cdots, \\ &\alpha_s(G^{n+1}(l_0), G^{n+1}(l_0), G^n(l_0)) \geq c_{\alpha_s}. \end{aligned} \tag{2.4}$$

And if $\beta_s(G(l_0), G(l_0), l_0) \leq c_{\beta_s}$, then

$$\begin{aligned} &\beta_s(G^2(l_0), G^2(l_0), G(l_0)) \leq c_{\beta_s}, \\ &\beta_s(G^{n+1}(l_0), G^{n+1}(l_0), G^n(l_0)) \leq c_{\beta_s}. \end{aligned} \tag{2.5}$$

From (2.1), (2.3) and (2.5)

$$\begin{aligned} c_{\alpha_s} S(G^2(l_0), G^2(l_0), G(l_0)) &\leq \alpha_s(G(l_0), G(l_0), l_0).S(G^2(l_0), G^2(l_0), G(l_0)) \\ &\leq \beta_s(G(l_0), G(l_0), l_0).\psi(S(G(l_0), G(l_0), l_0)) \\ &\leq c_{\beta_s}\psi(S(G(l_0), G(l_0), l_0)). \end{aligned}$$

Thus,

$$\begin{aligned} S(G^2(l_0), G^2(l_0), G(l_0)) &\leq \frac{c_{\beta_s}}{c_{\alpha_s}}\psi(S(G(l_0), G(l_0), l_0)) \\ &\leq \psi(S(G(l_0), G(l_0), l_0)). \end{aligned}$$

In general,

$$S(G^{n+1}(l_0), G^{n+1}(l_0), G^n(l_0)) \leq \psi^n(S(G(l_0), G(l_0), l_0)).$$

This implies

$$S(G^{n+1}(l_0), G^{n+1}(l_0), G^n(l_0)) \to 0,$$

as $n \to +\infty$. Now it can be proved that $\{G^n(l_0)\}_{n=1}^{\infty}$ is a Cauchy sequence. As $\psi \in \Psi$, so for fixed $\varepsilon > 0$ there exist $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq N(\varepsilon)} \psi^n(S(G(l_0), G(l_0), l_0)) < \varepsilon.$$

For $m, n \in \mathbb{N}$ such that $m > n > N(\varepsilon)$,

$$\begin{aligned} &S(G^n(l_0), G^n(l_0), G^m(l_0)) \\ &\leq 2S(G^n(l_0), G^n(l_0), G^{n+1}(l_0)) + S(G^{n+1}(l_0), G^{n+1}(l_0), G^m(l_0)) \\ &\leq 2\{S(G^n(l_0), G^n(l_0), G^{n+1}(l_0)) + S(G^{n+1}(l_0), G^{n+1}(l_0), G^{n+2}(l_0)) \\ &\quad + \cdots + S(G^{m-1}(l_0), G^{m-1}(l_0), G^m(l_0))\} \end{aligned}$$

$$\begin{aligned}
&\leq 2\{\psi^n S(G(l_0), G(l_0), l_0) + \psi^{n+1} S(G(l_0), G(l_0), l_0) \\
&\quad + \cdots + \psi^{m-1} S(G(l_0), G(l_0), l_0)\} \\
&= 2\sum_{k=n}^{m-1} \psi^k (S(G(l_0), G(l_0), l_0)) \\
&\leq 2 \sum_{n \geq N(\varepsilon)} \psi^n (S(G(l_0), G(l_0), l_0)) \\
&< \varepsilon.
\end{aligned}$$

Since $(U, \leq, S)$ is a complete space, the sequence $\{G^n(l_0)\}_{n=1}^{\infty}$ will converge in it, that is, there exists $l^* \in U$ such that $\lim_{n\to+\infty} G^n(l_0) = l^*$.

Now it can verify that the limit l^* is a fixed point of the function G. Since G is a continuous function, there exists $\delta > 0$ for each $\varepsilon > 0$ such that

$$S(l, l, l^*) < \delta \implies S(G(l), G(l), G(l^*)) < \frac{\varepsilon}{3}, \text{ for } \ l \in U.$$

Suppose $\eta = \min\{\frac{\varepsilon}{3}, \delta\}$, since the sequence $\{G^n(l_0)\}_{n=1}^{\infty}$ converges to l^*, there exist $n_0 \in \mathbb{N}$ such that,

$$S(G^n(l_0), G^n(l_0), l^*) \leq \eta, \text{ for all } \ n \geq n_0, \ n \in \mathbb{N}.$$

Taking $n \geq n_0$, $n \in \mathbb{N}$ we get,

$$\begin{aligned}
&S(G(l^*), G(l^*), l^*) \\
&\leq 2S(G(l^*), G(l^*), G(G^n(l_0))) + S(G^{n+1}(l_0), G^{n+1}(l_0), l^*) \\
&= 2S(G(G^n(l_0)), G(G^n(l_0)), G(l^*)) + S(G^{n+1}(l_0), G^{n+1}(l_0), l^*) \\
&< 2 \times \frac{\varepsilon}{3} + \eta \\
&\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $S(G(l^*), G(l^*), l^*) = 0$ that is $G(l^*) = l^*$. □

Remark 2.7. The hypothesis of continuity of G has been eliminated in the next theorem.

Theorem 2.8. *If $(U, \leq, S)$ is a partially ordered, complete S-metric space. Let a non-decreasing mapping $G : U \to U$ be an α_s-β_s-ψ-contractive mapping of type-A with*

(a) *G be α_s-β_s-admissible;*
(b) *there exists $l_0 \in U$ such that $l_0 \leq G(l_0)$;*
(c) *there exists $c_{\alpha_s} > 0, c_{\beta_s} > 0$ such that $\alpha_s(G(l_0), G(l_0), l_0) \geq c_{\alpha_s}$, $\beta_s(G(l_0), G(l_0), l_0) \leq c_{\beta_s}$;*

(d) *if there is a sequence* $\{l_n\}_{n=1}^{\infty}$ *in* U *such that* $\alpha_s(l_n, l_n, l_{n+1}) \geq c_{\alpha_s}$, $\beta_s(l_n, l_n, l_{n+1}) \leq c_{\beta_s}$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} l_n = l'$ *in* U, *then* $\alpha_s(l_n, l_n, l') \geq c_{\alpha_s}, \beta_s(l_n, l_n, l') \leq c_{\beta_s}$;

(e) *for non-decreasing sequence* $\{l_n\}$ *such that* $l_n \to l'$ *in* U, $l_n \leq l'$ *for all* $n \in \mathbb{N}$.

Then, $G(l^*) = l^*$ *for some* $l^* \in U$.

Proof. Proceeding as in the Theorem 2.6, since the sequence $\{G^n(l_0)\}$ is a Cauchy sequence, there exists an element $l \in U$ such that $\lim_{n\to+\infty} G^n(l_0) = l$. This limit is a fixed point of G which can be proved as follows:

Since $\{G^n(l_0)\}_{n=1}^{\infty}$ converges to l, therefore, for some $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$S(G^n(l_0), G^n(l_0), l) < \frac{\varepsilon}{3}, \text{ for all } n \geq n_0.$$

Since, the sequence $\{G^n(l_0)\}$ is a non-decreasing sequence, on taking account (e), we have

$$G^n(l_0) \leq l. \tag{2.6}$$

Using (2.1), (2.5), (2.6) and (d), we get

$$\begin{aligned}
c_{\alpha_s} S(l, l, G(l)) &\leq c_{\alpha_s} S(G(G^n(l_0)), G(G^n(l_0)), G(l)) \\
&\quad + 2c_{\alpha_s} S(G^{n+1}(l_0), G^{n+1}(l_0), l) \\
&\leq \alpha_s(G^n(l_0), G^n(l_0), l) S(G(G^n(l_0)), G(G^n(l_0)), G(l)) \\
&\quad + 2c_{\alpha_s} S(G^{n+1}(l_0), G^{n+1}(l_0), l) \\
&\leq \beta_s(G^n(l_0), G^n(l_0), l) \psi(S(G^n(l_0), G^n(l_0), l)) \\
&\quad + 2c_{\alpha_s} S(G^{n+1}(l_0), G^{n+1}(l_0), l) \\
&\leq c_{\beta_s} \psi(S(G^n(l_0), G^n(l_0), l)) + 2c_{\alpha_s} S(G^{n+1}(l_0), G^{n+1}(l_0), l),
\end{aligned}$$

therefore,

$$\begin{aligned}
S(l, l, G(l)) &< \frac{c_{\beta_s}}{c_{\alpha_s}} \psi(S(G^n(l_0), G^n(l_0), l)) + 2S(G^{n+1}(l_0), G^{n+1}(l_0), l) \\
&< \frac{\varepsilon}{3} + 2\frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Hence, $S(l, l, G(l)) = 0$, that is $G(l) = l$. □

Example 2.9. Let $(\mathbb{R}, \leq)$ and S metric defined on it by $S(p, q, r) = |p - q| + |q - r|$, for all $p, q, r \in \mathbb{R}$. Then $(\mathbb{R}, S)$ is a complete S-metric space. The function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{G}(r) = \begin{cases} \frac{r}{15}, & \text{if } r \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and the mappings $\alpha_s, \beta_s : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ given by

$$\alpha_s(p,q,r) = \begin{cases} 2, & \text{if } p,q,r \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_s(p,q,r) = \begin{cases} \frac{1}{3}, & \text{if } p,q,r \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(k) = \frac{k}{2}$ for $k > 0$. Clearly function $\mathcal{G}$ is continuous, non-decreasing and α_s-β_s-ψ-contractive of type A. Let $c_{\alpha_s} = \frac{3}{2}$ and $c_{\beta_s} = \frac{1}{2}$. Then $\mathcal{G}$ is α_s-β_s-admissible. For $p,q,r \in [0,+\infty)$ with $p \geq q \geq r$, we have

$$\alpha_s(p,q,r) \geq c_{\alpha_s} \implies \alpha_s(\mathcal{G}(p), \mathcal{G}(q), \mathcal{G}(r)) = \alpha_s(\frac{p}{15}, \frac{q}{15}, \frac{r}{15}) \geq c_{\alpha_s},$$

also

$$\beta_s(p,q,r) \leq c_{\beta_s} \implies \beta_s(\mathcal{G}(p), \mathcal{G}(q), \mathcal{G}(r)) = \beta_s(\frac{p}{15}, \frac{q}{15}, \frac{r}{15}) \leq c_{\beta_s}.$$

Also, there exists $r_0 \in U$ such that

$$\alpha_s(\mathcal{G}(r_0), \mathcal{G}(r_0), r_0) \geq c_{\alpha_s}$$

and

$$\beta_s(\mathcal{G}(r_0), \mathcal{G}(r_0), r_0) \leq c_{\beta_s}.$$

Since $0 \leq \mathcal{G}(0) = 0$, $r_0 \leq \mathcal{G}(r_0)$. Hence each postulates (a)-(d) of Theorem 2.6 holds. Therefore, $G(l^*) = l^*$ for some $l^* \in U$. Here $0 \in U$ is a point such that $G(0) = 0$.

Remark 2.10. In the next example mapping is discontinuous and follows Theorem 2.8.

Example 2.11. Let $(\mathbb{R}, \leq)$ and S-metric defined on it is

$$S(p,q,r) = |p-q| + |q-r| + |r-p|$$

for all $p,q,r \in \mathbb{R}$. Then $(\mathbb{R}, S)$ is a complete S-metric space. Define $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ and $\alpha_s, \beta_s : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ by

$$\mathcal{G}(r) = \begin{cases} 2r - \frac{1}{2}, & \text{if } r \geq \frac{1}{2}; \\ \frac{r}{10}, & \text{if } 0 \leq r < \frac{1}{2}; \\ 0, & \text{if } r < 0 \end{cases}$$

and

$$\alpha_s(p,q,r) = \begin{cases} 1, & \text{if } p,q,r \in [0, \frac{1}{2}]; \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta_s(p,q,r) = \begin{cases} \frac{1}{3}, & \text{if } p,q,r \in [0, \frac{1}{2}]; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that, the mapping $\mathcal{G}$ is discontinuous and non-decreasing. Let $\psi(k) = \frac{k}{3}$, for all $k > 0$. Obviously, if $p, q, r \in \mathbb{R} - [0, \frac{1}{2}]$, then the mapping $\mathcal{G}$ is α_s-β_s-ψ-contractive of type-A. Let $p, q, r \in [0, \frac{1}{2}]$ with $p \geq q \geq r$, $c_{\alpha_s} = \frac{1}{2}$ and $c_{\beta_s} = \frac{1}{3}$. Then $\alpha_s(p, q, r) \geq c_{\alpha_s}$ and $\beta_s(p, q, r) \leq c_{\beta_s}$. Hence, we have

$$\begin{aligned}\alpha_s(p,q,r)S(\mathcal{G}p,\mathcal{G}q,\mathcal{G}r) &= |\mathcal{G}p - \mathcal{G}q| + |\mathcal{G}q - \mathcal{G}r| + |\mathcal{G}r - \mathcal{G}p| \\ &= |\frac{p}{10} - \frac{q}{10}| + |\frac{q}{10} - \frac{r}{10}| + |\frac{r}{10} - \frac{p}{10}| \\ &= \frac{1}{10}(|p-q| + |q-r| + |r-p|)\end{aligned}$$

and

$$\begin{aligned}\beta_s(p,q,r)\psi(S(p,q,r)) &= \frac{1}{3} \times \frac{1}{3}S(p,q,r) \\ &= \frac{1}{9}(|p-q| + |q-r| + |r-p|).\end{aligned}$$

Therefore,

$$\frac{1}{10}(|p-q| + |q-r| + |r-p|) \leq \frac{1}{9}(|p-q| + |q-r| + |r-p|).$$

In other words,

$$\alpha_s(p,q,r)S(\mathcal{G}p,\mathcal{G}q,\mathcal{G}r) \leq \beta_s(p,q,r)\psi(S(p,q,r)),$$

for all $p, q, r \in \mathbb{R}$. Therefore, the mapping $\mathcal{G}$ is an α_s-β_s-ψ-contractive mapping of type-A. Moreover, there exists $r_0 \in \mathbb{R}$ such that $\alpha_s(\mathcal{G}r_0, \mathcal{G}r_0, r_0) \geq c_{\alpha_s}$ and $\beta_s(\mathcal{G}r_0, \mathcal{G}r_0, r_0) \leq c_{\beta_s}$. Let $r_0 = 0$. Then

$$\alpha_s(\mathcal{G}r_0,\mathcal{G}r_0,r_0) = \alpha_s(\mathcal{G}(0),\mathcal{G}(0),0) = \alpha_s(0,0,0) = 1 \geq \frac{1}{2}$$

and

$$\beta_s(\mathcal{G}r_0,\mathcal{G}r_0,r_0) = \beta_s(\mathcal{G}(0),\mathcal{G}(0),0) = \beta_s(0,0,0) = \frac{1}{3} \leq c_{\beta_s} = \frac{1}{3}.$$

Since $0 = r_0 \leq 0 = \mathcal{G}r_0$, that is, $r_0 \leq \mathcal{G}r_0$, $\mathcal{G}$ is α_s-β_s-admissible. Now, if the sequence $\{r_n\}$ is non-decreasing in $\mathbb{R}$ such that $\alpha_s(r_n, r_n, r_{n+1}) \geq c_{\alpha_s}$ and $\beta_s(r_n, r_n, r_{n+1}) \leq c_{\beta_s}$ for all $n \in \mathbb{N}$ and $r_n \to r$, then by definition of α_s and β_s, $r_n \in [0, \frac{1}{2})$, that is, $r \in [0, \frac{1}{2})$. In addition, $\{r_n\}$ is non-decreasing hence $r_n \leq r$. Hence, all the hypotheses of Theorem 2.8 are satisfied, therefore $\mathcal{G}$ has a fixed point. 0 and $\frac{1}{2}$ are fixed points for $\mathcal{G}$.

Remark 2.12. It is clear that the fixed point of G may not be unique(see above Example 2.11). The following theorems are obtained by applying additional conditions to the hypotheses of Theorem 2.6 and 2.8 to obtain a unique fixed point.

Theorem 2.13. *Considering all the hypotheses of Theorems* 2.6 *or* 2.8*, there exists* $p \in U$ *for all* $l_1, l_2, \in U$ *with* $l_1 \geq p,\ l_2 \geq p$ *such that*

$$\begin{cases} \alpha_s(l_1, l_1, p) \geq c_{\alpha_s} \ \ \textit{and} \ \ \ \beta_s(l_1, l_1, p) \leq c_{\beta_s} \\ \alpha_s(l_2, l_2, p) \geq c_{\alpha_s} \ \ \textit{and} \ \ \beta_s(l_2, l_2, p) \leq c_{\beta_s} \end{cases} \tag{2.7}$$

provides unique fixed point of G.

Proof. Suppose l' and l'' are two fixed points of G, that is, $G(l') = l'$ and $G(l'') = l''$. Then there exists $p \in U$ for l' and l'' such that (2.7) holds. Now by the first part of (2.7), we have

$$\alpha_s(l', l', p) \geq c_{\alpha_s} \ \text{ and } \ \ \beta_s(l', l', p) \leq c_{\beta_s}, \ l' \geq p. \tag{2.8}$$

Since G is α_s-β_s-admissible, we get

$$\alpha_s(G(l'), G(l'), G(p)) \geq c_{\alpha_s} \text{ and } \beta_s(G(l'), G(l'), G(p)) \leq c_{\beta_s},$$
$$G(l') \geq G(p).$$

Therefore, $\alpha_s(l', l', G(p)) \geq c_{\alpha_s}$ and $\beta_s(l', l', G(p)) \leq c_{\beta_s}$, $l' \geq G(p)$. Continuing this process, we have

$$\alpha_s(l', l', G^n(p)) \geq c_{\alpha_s} \text{ and } \beta_s(l', l', G^n(p)) \leq c_{\beta_s}, l' \geq G^n(p), \text{ for all } n \in \mathbb{N}. \tag{2.9}$$

Using the α_s-β_s-ψ-contractivity of G, we have

$$\begin{aligned} c_{\alpha_s} S(l', l', G^n(p)) &= c_{\alpha_s} S(G(l'), G(l'), G(G^{n-1}(p))) \\ &\leq \alpha_s(l', l', G^{n-1}(p)) S(G(l'), G(l'), G(G^{n-1}(p))) \\ &\leq \beta_s(l', l', G^{n-1}(p)) \psi(S(l', l', G^{n-1}(p))) \\ &\leq c_{\beta_s} \psi(S(l', l', G^{n-1}(p))). \end{aligned}$$

Therefore

$$\begin{aligned} S(l', l', G^n(p)) &\leq \frac{c_{\beta_s}}{c_{\alpha_s}} \psi(S(l', l', G^{n-1}(p))) \\ &\leq \psi(S(l', l', G^{n-1}(p))) \\ &\leq \psi(\psi(S(l', l', G^{n-2}(p)))) \\ &\ \ \vdots \\ &\leq \psi^n(S(l', l', p)). \end{aligned}$$

Which implies,

$$S(l', l', G^n(p)) \leq \psi^n(S(l', l', p)) \ \text{ for all } n \in \mathbb{N},$$

this implies $G^n(p) \to l'$ as $n \to +\infty$. Similarly, for the second part of (2.7), $G^n(p) \to l''$. Therefore $l' = l''$ proves uniqueness of fixed point of G. □

Note: Similarly, we can easily prove the following theorems (2.14), (2.15) and (2.16) obtained by replacing the inequality $l_0 \leq G(l_0)$ by $l_0 \geq G(l_0)$ in the assumption (b) of the theorems (2.6), (2.8) and (2.13) respectively.

Theorem 2.14. *Let $(U, \leq, S)$ be a partially ordered, complete S-metric space and $G : U \to U$ be a non-decreasing, α_s-β_s-ψ-contractive mapping of type-A satisfying;*

(a) *G is α_s-β_s-admissible;*
(b) *there exists $l_0 \in U$ such that $l_0 \geq G(l_0)$;*
(c) *there exists $c_{\alpha_s} > 0, c_{\beta_s} \geq 0$ such that $\alpha_s(l_0, l_0, G(l_0)) \geq c_{\alpha_s}$, $\beta_s(l_0, l_0, G(l_0)) \leq c_{\beta_s}$;*
(d) *G is continuous.*

Then, there exists a fixed point of G.

Theorem 2.15. *Let $(U, \leq, S)$ be a partially ordered, complete S-metric space. If a non-decreasing mapping $G : U \to U$ is α_s-β_s-ψ-contractive mapping of type-A with;*

(a) *G is α_s-β_s-admissible;*
(b) *there exists $l_0 \in U$ such that $l_0 \geq G(l_0)$;*
(c) *there exists $c_{\alpha_s} > 0, c_{\beta_s} \geq 0$ such that $\alpha_s(l_0, l_0, G(l_0)) \geq c_{\alpha_s}$, $\beta_s(l_0, l_0, G(l_0)) \leq c_{\beta_s}$;*
(d) *if $\{l_n\}_{n=1}^{\infty}$ is a sequence in U and $\lim_{n\to\infty} l_n = l$, if $\alpha_s(l_{n+1}, l_{n+1}, l_n) \geq c_{\alpha_s}$, $\beta_s(l_{n+1}, l_{n+1}, l_n) \leq c_{\beta_s}$ for all $n \in N$ implies $\alpha_s(l, l, l_n) \geq c_{\alpha_s}$, $\beta_s(l, l, l_n) \leq c_{\beta_s}$;*
(e) *if there exists a non-increasing sequence $\{l_n\}$ in U such that $l_n \to l$ then $l \leq l_n$ for all $n \in N$.*

Then, there exists a fixed point of G.

Theorem 2.16. *Considering all the postulates of the Theorems 2.14 or 2.15, if there exists $p \in U$ for all $l_1, l_2 \in U$ such that $l_1 \geq p$, $l_2 \geq p$ and*

$$\begin{cases} \alpha_s(l_1, l_1, p) \geq c_{\alpha_s} \text{ and } \beta_s(l_1, l_1, p) \leq c_{\beta_s}, \\ \alpha_s(l_2, l_2, p) \geq c_{\alpha_s} \text{ and } \beta_s(l_2, l_2, p) \leq c_{\beta_s}. \end{cases} \tag{2.10}$$

Then, there exists a unique fixed point of G.

3. Applications to ordinary differential equations

Here, we have proved the uniqueness of a solution of the following first-order boundary value problem with continuous $T : J \times R \to R$ and α_s-β_s-ψ-contractive mapping of type-A considering existence of a lower solution.

$$\begin{cases} x'(j) = T(j, x(j)), \quad j \in J = [0, M]; \\ x(0) = x(M), \end{cases} \tag{3.1}$$

where $M \geq 0$ and function $T : J \times R \to R$ is continuous.

Nieto and Rod.-Lopez [12] solved the differential equation (3.1) in the relation of its lower solution as:

Theorem 3.1. ([12]) *The problem* (3.1) *with continuous* $T : J \times R \to R$ *and some* $\lambda > 0$, $\mu > 0$ *with* $\mu < \lambda$ *such that, for all* $l_1, l_2 \in R$ *with* $l_1 \leq l_2$,

$$\mu(l_2 - l_1) \geq T(j, l_2) + \lambda l_2 - T(j, l_1) - \lambda l_1 \geq 0,$$

then, the existence of a lower solution for (3.1)*, provides the existence of a unique solution of* (3.1)*.*

Also, Sadarangani and Harjani [7] have proved the theorem:

Theorem 3.2. ([7]) *The problem* (3.1) *with continuous* $T : J \times R \to R$ *and suppose that there exists* $\lambda > 0$ *such that for all* $l_1, l_2 \in R$ *with* $l_1 \leq l_2$,

$$\lambda\psi(l_2 - l_1) \geq T(j, l_2) + \lambda l_2 - T(j, l_1) \geq 0,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ *given by* $\psi(k)$=k-$\phi(k)$ *for* $\phi : [0, +\infty) \to [0, +\infty)$ *continuous, increasing with* $\phi(k)$=0 *only for* $k = 0$ *and* $\lim_{k\to+\infty} \phi(k)$=$+\infty$ *for all* $k \in (0, +\infty)$*. If* (3.1) *has a lower solution exists, then it is unique solution.*

Now we solve problem (3.1) using the above theorems.

Remark 3.3. For some $\lambda > 0$, problem (3.1) can be expressed as

$$\begin{cases} x'(j) + \lambda x(j) = T(j, x(j)) + \lambda x(j), \quad j \in J = [0, M]; \\ x(0) = x(M). \end{cases}$$

The corresponding integral equation to this differential equation is given by

$$x(j) = \int_0^M G(j,t)[T(t, x(t)) + \lambda x(t)]dt,$$

where

$$G(j,t) = \begin{cases} \frac{e^{\lambda(M+t-j)}}{e^{\lambda M}-1}; & 0 \leq t < j \leq M; \\ \frac{e^{\lambda(t-j)}}{e^{\lambda M}-1}; & 0 \leq j < t \leq M. \end{cases}$$

$G(j,t)$ is known as the Green function in differential equation theory.

Theorem 3.4. *Consider the given problem* (3.1) *with continuous* $T : J \times R \to R$ *holding the following conditions:*

(a) *for all* $l_1, l_2 \in R$ *with* $l_2 \geq l_1$*, and* $\psi \in \Psi$ *there exists* $\lambda > 0$ *such that*

$$\lambda\psi(l_2 - l_1) \geq T(j, l_2) + \lambda l_2 - T(j, l_1) - \lambda l_1 \geq 0;$$

(b) *for all* $j \in I$ *and* $a, b \in R$ *there exists* $\xi : R^3 \to R$ *such that if* $\xi(a, a, b) \geq 0$ *implies*

$$\xi\Big(\int_0^M G(t,j)[T(t,x(t))+\lambda x(t)]dt, \int_0^M G(t,j)[T(t,x(t))+\lambda x(t)]dt, \gamma(j)\Big) \geq 0,$$

where $\gamma \in C(J, R)$ *is lower solution of* (3.1)*;*

(c) *for all* $x, y \in C(J, R)$ *and* $j \in J$, $\xi(x(j), x(j), y(j)) \geq 0$ *implies*

$$\xi\Big(\int_0^M G(j,t)[T(t,x(t)) + \lambda x(t)]ds, \int_0^M G(j,t)[T(t,x(t)) + \lambda x(t)]ds,$$
$$\int_0^M G(j,t)[T(t,y(t)) + \lambda x(t)]ds\Big) \geq 0;$$

(d) *if* $z_n \to z \in C(J, R)$ *and* $\xi(z_n, z_n, z_{n+1}) \geq 0$ *implies* $\xi(z_n, z_n, z) \geq 0$ *for all* $n \in N$.

Then, there exists a unique solution if a lower solution exists.

Proof. Let $U = C(J, R)$ and define $\mathcal{A} : U \to U$ by

$$[\mathcal{A}(x)](j) = \int_0^M G(j,t)[T(t,x(t)) + \lambda x(t)]dt, \ j \in J.$$

Note that solution of (3.1) is a fixed point of $\mathcal{A}$. U is a partially ordered set with order relation.

$$x \leq y \quad \Leftrightarrow \quad x(j) \leq y(j) \ \ for \ all \ j \in J, \ where \ x, y \in U.$$

If we define

$$S(x, x, y) = \sup 2|x(j) - y(j)| \ \ for \ x, y \in U, \ j \in J.$$

Then (U, S) is a complete S-metric space. Let us take a sequence $\{x_n\} \subseteq U$, which is monotonic, non-decreasing and converges to $x^* \in U$. Then for each $j \in J$,

$$x_1(j) \leq x_2(j) \leq x_3(j) \leq \cdots \leq x_n(j) \leq \cdots .$$

Since the sequence $\{x_n(j)\}$ converges to $x^*(j)$ implies that $x_n(j) \leq x^*(j)$ for all $n \in N$ and $j \in J$. Therefore, $x_n \leq x^*$ for all $n \in N$. $\mathcal{A}$ is non-decreasing, for all $y \leq x$ where $x, y \in U$, we have

$$T(j, y) + \lambda y \leq T(j, x) + \lambda x,$$

also $G(j, t) \geq 0$ for all $(j, t) \in J \times J$, therefore

$$\begin{aligned}[\mathcal{A}x](t) &= \int_0^M G(j,t)[T(t,x(t)) + \lambda x(t)]dt \\ &\geq \int_0^M G(j,t)[T(t,y(t)) + \lambda y(t)]dt = [\mathcal{A}y](j).\end{aligned}$$

In addition, for $x \geq y$ using (a) and by the definition of $G(j,t)$, we have

$$\begin{aligned} S(\mathcal{A}x, \mathcal{A}x, \mathcal{A}y) &= \sup 2|\mathcal{A}x(j) - \mathcal{A}y(j)|, \ j \in J \\ &\leq \sup_{j\in J} \int_0^M 2G(j,t)|T(t,x(t)) + \lambda x(t) - T(t,y(t)) - \lambda y(t)|dt \\ &\leq \sup_{j\in J} \int_0^M 2G(j,t)|\lambda\psi(x(t) - y(t))|dt \\ &\leq \sup_{j\in J} \int_0^M G(j,t)\lambda\psi(2|x(t) - y(t)|)dt \\ &\leq \lambda\psi(S(x,x,y)) \sup_{j\in J} \int_0^M G(j,t)dt \\ &= \lambda\psi(S(x,x,y)) \sup_{j\in J} \frac{1}{e^{\lambda M} - 1}\left(\frac{1}{\lambda}e^{\lambda(M+t-j)}|_0^j + \frac{1}{\lambda}e^{\lambda(t-j)}|_j^M\right) \\ &= \lambda\psi(S(x,x,y)) \times \frac{1}{\lambda} \\ &= \psi(S(x,x,y)), \end{aligned}$$

it implies that

$$S(\mathcal{A}x, \mathcal{A}x, \mathcal{A}y) \leq \psi(S(x,x,y)).$$

Define $\alpha_s : U \times U \times U \to [0, +\infty)$ by

$$\alpha_s(x,x,y) = \begin{cases} 1, & \text{if } \xi(x(j), x(j), y(j)) \geq 0, \ j \in J; \\ 0, & \text{otherwise} \end{cases}$$

and $\beta_s : U \times U \times U \to [0, +\infty)$ by

$$\beta_s(x,x,y) = \begin{cases} 1, & \text{if } \xi(x(j), x(j), y(j)) \geq 0, \ j \in J; \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y \in U$ with $x \geq y$. Then

$$\alpha_s(x,x,y)S(\mathcal{A}x, \mathcal{A}x, \mathcal{A}y) \leq \beta_s(x,x,y)\psi(S(x,x,y)).$$

Hence mapping $\mathcal{A}$ is α_s-β_s-ψ-contractive of type-A. Let $c_{\alpha_s} = c_{\beta_s} = 1$. From (c) for all $x, y \in U$ with $x \geq y$, we get for $\alpha_s(x,x,y) \geq 1 = c_{\alpha_s}$, we have $\xi(x(j), x(j), y(j)) \geq 0$. Then

$$\xi(\mathcal{A}x(j), \mathcal{A}x(j), \mathcal{A}y(j)) \geq 0.$$

It implies that

$$\alpha_s(\mathcal{A}x, \mathcal{A}x, \mathcal{A}y) \geq 1 = c_{\alpha_s}.$$

And also, for $\beta_s(x,x,y) \leq 1 = c_{\beta_s}$, we have $\xi(x(j), x(j), y(j)) \geq 0$. Then

$$\xi(\mathcal{A}x(j), \mathcal{A}x(j), \mathcal{A}y(j)) \geq 0.$$

It implies that

$$\beta_s(\mathcal{A}x, \mathcal{A}x, \mathcal{A}y) \leq 1 = c_{\beta_s},$$

this means that $\mathcal{A}$ is α_s-β_s-admissible. If γ is a lower solution of (3.1), from (b),

$$\xi((\mathcal{A}\gamma)(j), (\mathcal{A}\gamma)(j), \gamma(j)) \geq 0 \quad \Longrightarrow \quad \begin{cases} \alpha_s(\mathcal{A}\gamma, \mathcal{A}\gamma, \gamma) \geq c_{\alpha_s}; \\ \beta_s(\mathcal{A}\gamma, \mathcal{A}\gamma, \gamma) \leq c_{\beta_s}. \end{cases}$$

Now, we prove that $\mathcal{A}\gamma \geq \gamma$. Since γ is lower solution of the considered problem (3.1), therefore

$$\begin{cases} \gamma'(j) \leq h(j, \gamma(j)), \ j \in J = [0, M]; \\ \gamma(0) \leq \gamma(M), \end{cases}$$

for all $j \in J$ and $\lambda > 0$. Hence

$$\gamma'(j) + \lambda\gamma(j) \leq h(j, \gamma(j)) + \lambda\gamma(j),$$

on multiplying by $e^{\lambda j}$, we have

$$(\gamma(j)e^{\lambda j})' \leq (h(j, \gamma(j)) + \lambda\gamma(j))e^{\lambda j}.$$

By integrating from 0 to j, we have

$$\gamma(j)e^{\lambda j} \leq \gamma(0) + \int_0^j [h(t, \gamma(t)) + \lambda\gamma(t)]e^{\lambda t} dt. \tag{3.2}$$

This implies that

$$\gamma(0)e^{\lambda M} \leq \gamma(M)e^{\lambda M} \leq \gamma(0) + \int_0^M [h(t, \gamma(t)) + \lambda\gamma(t)]e^{\lambda t} dt,$$

$$\gamma(0) \leq \int_0^M \frac{e^{\lambda t}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt. \tag{3.3}$$

From (3.2) and (3.3)

$$\begin{aligned}\gamma(j)^{\lambda j} &\leq \int_0^M \frac{e^{\lambda t}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt + \int_0^j [h(t, \gamma(t)) + \lambda\gamma(t)]e^{\lambda t} dt \\ &\leq \int_0^j \frac{e^{\lambda(M+t)}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt + \int_j^M \frac{e^{\lambda t}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt,\end{aligned}$$

and dividing by $e^{\lambda j}$, we obtain

$$\gamma(j) \leq \int_0^j \frac{e^{\lambda(M+t-j)}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt + \int_j^M \frac{e^{\lambda(t-j)}}{e^{\lambda M} - 1}[h(t, \gamma(t)) + \lambda\gamma(t)]dt.$$

Hence, by the definition of green function $G(j, t)$, we have

$$\gamma(j) \leq \int_0^M G(j, t)[h(t, \gamma(t)) + \lambda\gamma(t)]dt = [A\gamma](j)$$

for all $j \in J$, which implies that $\mathcal{A}\gamma \geq \gamma$.

Finally, from (d) if $l_n \to l \in U$, for all n, we have

$$\xi(l_n, l_n, l_{n+1}) \geq 0 \implies \xi(l_n, l_n, l) \geq 0,$$

therefore

$$\alpha_s(l_n, l_n, l_{n+1}) \geq c_{\alpha_s} \implies \alpha_s(l_n, l_n, l) \geq c_{\alpha_s},$$

$$\beta_s(l_n, l_n, l_{n+1}) \leq c_{\beta_s} \implies \beta_s(l_n, l_n, l) \leq c_{\beta_s}.$$

Thus each postulates (a)-(e) of Theorem 2.8 hold. Therefore, $\mathcal{A}$ has a fixed point that is given differential equation (3.1) has a solution. The solution's uniqueness can be verified using Theorem 2.15. □

Theorem 3.5. *If lower solution of the differential equation* (3.1) *replaced by upper solution, Theorem* 3.4 *still holds.*

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