Nonlinear Functional Analysis and Applications
Vol. 28, No. 2 (2023), pp. 589-599
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2023.28.02.16
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON A SUBCLASS OF K-UNIFORMLY ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS AND THEIR PROPERTIES

Ma'moun I. Y. Alharayzeh[1] **and Habis S. Al-zboon**[2]

[1]Department of Scientific Basic Sciences, Faculty of Engineering Technology,
Al-Balqa Applied University, Amman 11134, Jordan
e-mail: mamoun@bau.edu.jo, mamoun.math@gmail.com

[2]Department of Curriculum and Instruction, College of Education,
Al-Hussein Bin Talal University, Jordan
e-mail: habis.s.alzboon@ahu.edu.jo

Abstract. The object of this study is to introduce a new subclass of univalent analytic functions on the open unit disk. This subclass is created by utilizing univalent analytic functions with negative coefficients. We first explore the specific properties that functions in this subclass must possess before examining their coefficient characterization. By applying this approach, we observe several fascinating features, including coefficient approximations, growth and distortion theorems, extreme points and a demonstration of the radius of starlikeness and convexity for functions belonging to this subclass.

1. Introduction and definition

Let A denote the class of all analytic function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}$$

defined on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S denote the subclass of A consisting of functions that are univalent in D. Further, let $S^*(\mu)$ and $C(\mu)$ be the classes of functions respectively

[0]Received August 30, 2022. Revised March 18, 2023. Accepted March 20, 2023.
[0]2020 Mathematics Subject Classification: 30C45.
[0]Keywords: New class, uniformaly, univalent function.
[0]Corresponding author: Ma'moun I. Y. Alharayzeh(mamoun@bau.edu.jo).

starlike of order μ and convex of order μ, for $0 \le \mu < 1$. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in D, respectively.

Let T be the subclass of S, consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \ge 0. \tag{1.1}$$

Defined on the open unit disk D. A function $f \in T$ is called a function with negative coefficients. The class T was introduced and studied by Silverman [6]. In [6] Silverman investigated, the subclasses of T denoted by $S_T^*(\mu)$, and $C_T(\mu)$, for $0 \le \mu < 1$ that are respectively starlike of order μ and convex of order μ.

Let $\Sigma(\alpha, \beta, \gamma)$ be the subclass of A consisting of functions $f(z)$ which satisfy the inequality:

$$\left| \frac{zf'(z) - f(z)}{\alpha z f'(z) + (1-\gamma) f(z)} \right| < \beta,$$

where $0 \le \alpha \le 1$, $0 < \beta \le 1$ and $0 \le \gamma < 1$ for all $z \in D$. This class of functions was studied by Darus [3].

In this present paper, we introduce new class $M(\alpha, \beta, \gamma, k)$ of functions as the following, mimic to the Aqlan et al. [2] and also in [1],[4],[5].

Definition 1.1. For $0 \le \alpha \le 1$, $0 \le \beta < 1$, $0 \le \gamma < 1$ and $k \ge 0$, we let $M(\alpha, \beta, \gamma, k)$ consist of functions $f \in T$ satisfying the condition

$$\operatorname{Re}\left(\frac{zf'(z) - f(z)}{\alpha z f'(z) + (1-\gamma) f(z)} \right) > k \left| \frac{zf'(z) - f(z)}{\alpha z f'(z) + (1-\gamma) f(z)} - 1 \right| + \beta. \tag{1.2}$$

Our first result, is the coefficient estimate for functions $f \in M(\alpha, \beta, \gamma, k)$, and the others include, the growth and distortion theorem, further we obtain the extreme points. Finally we determined the radius of starlikeness and convexity, for the function belonging to the class $M(\alpha, \beta, \gamma, k)$. First of all, let us look at the coefficient estimates.

2. Coefficient estimates

In this section, we shall obtain the coefficient estimates for the function f belongs to the class $M(\alpha, \beta, \gamma, k)$). Our first result is the following:

Theorem 2.1. *A function f given by* (1.1) *is in the class $M(\alpha, \beta, \gamma, k)$ if and only if*

$$\sum_{n=2}^{\infty} \tau_n a_n \le \tau_1, \tag{2.1}$$

where

$$\tau_n = |(2+k-\beta)(1+\alpha n-\gamma)+(k+1)(1-n)| \tag{2.2}$$

such that $0 \le \alpha \le 1$, $0 \le \beta < 1, 0 \le \gamma < 1$ *and* $k \ge 0$.

Proof. We have $f \in M(\alpha, \beta, \gamma, k)$ if and only if the condition (1.2) is satisfied. Let

$$w = \frac{zf'(z) - f(z)}{\alpha z f'(z) + (1-\gamma) f(z)},$$

upon the fact that,

$Re(w) \ge k\,|w-1| + \beta$ if and only if $(k+1)\,|w-1| \le 1-\beta$.

Now

$$(k+1)\,|w-1| = (k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)\,a_n z^n}{(1+\alpha-\gamma)\,z - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^n} - 1\right| \le 1-\beta,$$

is equivalent to

$$(k+1)\left|\frac{\sum_{n=2}^{\infty}(1-n)\,a_n z^{n-1}}{(1+\alpha-\gamma) - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}} - 1\right| \le 1-\beta.$$

After reduction to a common denominator, we have

$$(k+1)\left|\frac{\sum_{n=2}^{\infty}(1+\alpha n-\gamma+1-n)\,a_n z^{n-1} - (1+\alpha-\gamma)}{(1+\alpha-\gamma) - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}}\right| \le 1-\beta. \tag{2.3}$$

The above inequality reduces to

$$(k+1)\,\frac{\sum_{n=2}^{\infty}|1+\alpha n-\gamma+1-n|\,a_n - (1+\alpha-\gamma)}{1+\alpha-\gamma - \sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n} \le 1-\beta. \tag{2.4}$$

Then

$$(k+1)\left[\sum_{n=2}^{\infty}|1+\alpha n-\gamma+1-n|\,a_n - (1+\alpha-\gamma)\right]$$
$$\le (1+\alpha-\gamma)(1-\beta) - (1-\beta)\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n.$$

Thus

$$\left[(k+1)\sum_{n=2}^{\infty}|1+\alpha n-\gamma+1-n|+(1-\beta)\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\right]a_n$$
$$\leq(1+\alpha-\gamma)(1-\beta)+(1+\alpha-\gamma)(1+k),$$

which yields to (2.1).

Conversely, suppose that (2.1) holds and we have to show (2.3) holds. Here the inequality (2.1) is equivalent to (2.4). So it suffices to show that,

$$\left|\frac{\sum_{n=2}^{\infty}(1+\alpha n-\gamma+1-n)\,a_n z^{n-1}-(1+\alpha-\gamma)}{1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}}\right| \tag{2.5}$$
$$\leq\frac{\sum_{n=2}^{\infty}|1+\alpha n-\gamma+1-n|\,a_n-(1+\alpha-\gamma)}{1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n}.$$

Since,

$$\left|1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}\right|\geq|1+\alpha-\gamma|-\left|\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}\right|,$$

we have

$$\left|1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n z^{n-1}\right|\geq 1+\alpha-\gamma-\sum_{n=2}^{\infty}(\alpha n+1-\gamma)\,a_n,$$

where $|z|<1$. And hence we obtain (2.5). □

Theorem 2.2. *Let the function f given by* (1.1) *be in the class $M(\alpha,\beta,\gamma,k)$. Then*

$$a_n\leq\frac{\tau_1}{\tau_n},\qquad n=2,3,4,..., \tag{2.6}$$

where τ_n is given by (2.2). *Equality holds for the functions given by,*

$$f(z)=z-\frac{\tau_1\,z^n}{\tau_n}. \tag{2.7}$$

Proof. Since $f\in M(\alpha,\beta,\gamma,k)$, Theorem 2.1 holds.

Now

$$\sum_{n=2}^{\infty}\tau_n\,a_n\quad\leq\quad\tau_1,$$

we have,

$$a_n \leq \frac{\tau_1}{\tau_n}.$$

Clearly the function given by (2.7) satisfies (2.6) and therefore f given by (2.7) is in $M(\alpha, \beta, \gamma, k)$ for this function, the result is clearly sharp. □

3. Growth and distortion theorems for the subclass $M(\alpha, \beta, \gamma, k)$

In this section, growth and distortion theorem will be considered and the covering property for function in the class $M(\alpha, \beta, \gamma, k)$ will also be given.

Theorem 3.1. *Let the function f given by* (1.1) *be in the class $M(\alpha, \beta, \gamma, k)$. Then for $0 < |z| = r < 1$, we have*

$$r - \frac{\tau_1}{\tau_2} r^2 \leq |f(z)| \leq r + \frac{\tau_1}{\tau_2} r^2, \tag{3.1}$$

where τ_1 and τ_2 can be found by (2.2)*. Equality holds for the function,*

$$f(z) = z - \frac{\tau_1 z^2}{\tau_2}, \qquad (z = \pm r, \ \pm ir).$$

Proof. We only prove the right hand side inequality in (3.1), since the other inequality can be justified using similar arguments.

Since $f \in M(\alpha, \beta, \gamma, k)$ by Theorem 2.1, we have,

$$\sum_{n=2}^{\infty} \tau_n \, a_n \leq \tau_1.$$

Now

$$\begin{aligned} \tau_2 \sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} \tau_2 \, a_n \\ &\leq \sum_{n=2}^{\infty} \tau_n a_n \\ &\leq \tau_1. \end{aligned}$$

And therefore

$$\sum_{n=2}^{\infty} a_n \leq \frac{\tau_1}{\tau_2}. \tag{3.2}$$

Since

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

we have,

$$\begin{aligned} |f(z)| &= \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \\ &\leq |z| \; + |z|^2 \sum_{n=2}^{\infty} a_n \, |z|^{n-2} \\ &\leq r + r^2 \sum_{n=2}^{\infty} a_n . \end{aligned}$$

By aid of inequality (3.2), yields the right hand side inequality of (3.1). □

Theorem 3.2. *If the function f given by* (1.1) *is in the class $M(\alpha, \beta, \gamma, k)$ for* $0 < |z| = r < 1$, *then we have*

$$1 - \frac{2\,\tau_1\, r}{\tau_2} \leq \left| f'(z) \right| \leq 1 + \frac{2\,\tau_1\, r}{\tau_2}, \tag{3.3}$$

where τ_1 and τ_2 can be found by (2.2). *Equality holds for the function f given by*

$$f(z) = z - \frac{\tau_1\, z^2}{\tau_2}.$$

Proof. Since $f \in M(\alpha, \beta, \gamma, k)$ by Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \tau_n \, a_n \leq \; \tau_1.$$

Now,

$$\begin{aligned} \tau_2 \sum_{n=2}^{\infty} n a_n &\leq 2 \sum_{n=2}^{\infty} \tau_n \, a_n \\ &\leq 2\, \tau_1. \end{aligned}$$

Hence

$$\sum_{n=2}^{\infty} n a_n \leq \frac{2\,\tau_1}{\tau_2}. \tag{3.4}$$

Since

$$f'(z) = 1 - \sum_{n=2}^{\infty} n \, a_n \, z^{n-1},$$

we have

$$1 - |z| \sum_{n=2}^{\infty} n a_n \; |z|^{n-2} \leq \left| f'(z) \right| \; \leq \; 1 + \; |z| \sum_{n=2}^{\infty} n a_n \, |z|^{n-2},$$

where, $|z| < 1$. By using the inequality (3.4), we get Theorem 3.2 this completes the proof. □

Theorem 3.3. *If the function f given by* (1.1) *is in the class $M(\alpha,\beta,\gamma,k)$ then f is starlike of order δ, where*

$$\delta = 1 - \frac{\tau_1}{-\tau_1 + \tau_2},$$

where τ_1 and τ_2 can be found by (2.2)*. The result is sharp with*

$$f(z) = z - \frac{\tau_1 z^2}{\tau_2}.$$

Proof. It is sufficient to show that (2.1) implies

$$\sum_{n=2}^{\infty} a_n(n-\delta) \leq 1-\delta. \tag{3.5}$$

That is,

$$\frac{n-\delta}{1-\delta} \leq \frac{\tau_n}{\tau_1}, \qquad n \geq 2. \tag{3.6}$$

The above inequality is equivalent to

$$\delta \leq 1 - \frac{\tau_1\,(n-1)}{-\tau_1+\tau_n} = \psi(n), \quad \text{where}, \quad n \geq 2.$$

And $\psi(n) \geq \psi(2)$, (3.6) holds true for any $0 \leq \alpha \leq 1,\ 0 \leq \beta < 1,\ 0 \leq \gamma < 1$ and $k \geq 0$. This completes the proof of Theorem 3.3. □

4. Extreme points of the class $M(\alpha,\beta,\gamma,k)$

The extreme points of the class $M(\alpha,\beta,\gamma,k)$ is given by the following theorem.

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_n(z) = z - \frac{\tau_1 z^n}{\tau_n}, \qquad n = 2,3,4,\ldots,$$

where τ_n is given by (2.2).

Then $f \in M(\alpha,\beta,\gamma,k)$ if and only if it can be represented in the form

$$f(z) = \sum_{n=1}^{\infty} y_n f_n(z), \tag{4.1}$$

where $y_n \geq 0$ and $\sum\limits_{n=1}^{\infty} y_n = 1$.

Proof. Suppose f can be expressed as in (4.1). Our goal is to show that $f \in M(\alpha, \beta, \gamma, k)$. By (4.1) we have

$$f(z) = \sum_{n=1}^{\infty} y_n \left\{ z - \frac{\tau_1 \, z^n}{\tau_n} \right\}.$$

Then

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \frac{\tau_1 \, y_n z^n}{\tau_n}.$$

So that

$$a_n = \frac{\tau_1 \, y_n}{\tau_n}, \qquad n \geq 2.$$

Now, we have

$$\sum_{n=2}^{\infty} y_n = \; 1 - y_1 \; \leq \; 1.$$

Setting

$$\sum_{n=2}^{\infty} y_n \frac{\tau_1}{\tau_n} \quad \times \quad \frac{\tau_n}{\tau_1} = \sum_{n=2}^{\infty} y_n = 1 - y_1 \leq 1.$$

It follows from Theorem 2.1 that the function $f \in M(\alpha, \beta, \gamma, k)$.

Conversely, it suffices to show that

$$a_n = \frac{\tau_1 \, y_n}{\tau_n}.$$

Now we have $f \in M(\alpha, \beta, \gamma, k)$ then by previous Theorem 2.2

$$a_n \leq \frac{\tau_1}{\tau_n}, \qquad n \geq 2.$$

That is,

$$\frac{\tau_n \, a_n}{\tau_1} \leq 1,$$

but $y_n \; \leq \; 1$.
Setting,

$$y_n = \frac{\tau_n \, a_n}{\tau_1}, \qquad n \geq 2.$$

Which yields to the desired result. This completes the proof. □

Corollary 4.2. *The extreme point of the class* $M(\alpha,\beta,\gamma,k)$ *are the function*

$$f_1(z)=z,$$

and

$$f_n(z)=z-\frac{\tau_1\, z^n}{\tau_n}, \qquad n=2,3,4,\dots\,,$$

where τ_n *is given by* (2.2).

Finally, in this paper we consider the radius of starlikeness and convexity.

5. Radius of starlikeness and convexity

The radius of starlikeness and convexity for the class $M(\alpha,\beta,\gamma,k)$ is given by the following theorems.

Theorem 5.1. *If the function* f *given by* (1.1) *is in the class* $M(\alpha,\beta,\gamma,k)$, *then* f *is starlike of order* δ $(0\le\delta<1)$, *in the disk* $|z|<R$, *where*

$$R=\inf\left[\frac{\tau_n}{\tau_1}\left(\frac{1-\delta}{n-\delta}\right)\right]^{\frac{1}{n-1}}, \quad n=2,3,4,\dots, \tag{5.1}$$

and τ_n *is given by* (2.2).

Proof. Here (5.1) implies

$$\tau_1\,(n-\delta)\,|z|^{n-1} \quad\le\quad \tau_n\,(1-\delta)\,.$$

It suffices to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\delta,$$

for $|z|<R$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n\,|z|^{n-1}}. \tag{5.2}$$

By aid of (2.6), we have

$$\left|\frac{zf'}{f}-1\right| \le \frac{\sum\limits_{n=2}^{\infty}\frac{(n-1)\tau_1\,|z|^{n-1}}{\tau_n}}{1-\sum\limits_{n=2}^{\infty}\frac{\tau_1\,|z|^{n-1}}{\tau_n}} \le 1-\delta. \tag{5.3}$$

Hence (5.3) holds true if

$$\sum_{n=2}^{\infty} \frac{(n-1)\,\tau_1 \ |z|^{n-1}}{\tau_n} \quad \leq \quad \left[1 - \sum_{n=2}^{\infty} \frac{\tau_1 \ |z|^{n-1}}{\tau_n}\right](1-\delta)\,,$$

and it follows that

$$|z|^{n-1} \leq \left[\frac{\tau_n}{\tau_1}\left(\frac{1-\delta}{n-\delta}\right)\right], \qquad n \geq 2$$

as required. □

Theorem 5.2. *If the function f given by* (1.1) *is in the class $M(\alpha,\beta,\gamma\,,k)$, then f is convex of order λ $(\,0 \leq \lambda < 1\,)$, in the disk $|z| < w$, where*

$$w = \inf\left[\frac{\tau_n}{\tau_1}\left(\frac{1-\lambda}{n\,(n-\lambda)}\right)\right]^{\frac{1}{n-1}}, \qquad n = 2, 3, 4, \dots\,,$$

and τ_n is given by (2.2).

Proof. By using the same technique in the proof of Theorem 5.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\lambda, \qquad \text{for} \qquad |z| \leq w,$$

with the aid of (2.6). Thus we have the assertion of Theorem 5.2. □

6. Conclusion

This article aims at finding a new subclass of analytic functions in the open unit disk, that have been characterized using negative analytic function. Further to study certain sufficient requirements for the functions belonging to this class, one of the main requirements needed to satisfy coefficient characterization. This approach, for example, can provide several many fascinating features.

Acknowledgements: The author would like to express deepest thanks to the reviewers for their insightful comments on their paper. The author thank Dr. Abdullah Algunmeeyn for his time and invaluable contributions.

References

[1] M.H. Al-Abbadi and M. Darus. *On subclass of analytic univalent functions associated with negative coefficients*, Int. J. Math. Math. Sci., (2010), 1-11.

[2] E. Aqlan, J.M. Jahangiri and S.R. Kulkarni, *Classes of K-uniformaly convex and starlike functions*, Tamk. J. Math., **35**(3) (2004), 261-266.

[3] M. Darus, *Some subclasses of analytic functions*, J. Inst. Math. & Comp. Sci., **16**(3) (2003), 121-126.

[4] F. Ghanim and M. Darus, *A new subclass of uniformly starlike and convex functions with negative coefficients*, Int. Math. Forum., **4**(23) (2009), 1105-1117.

[5] T.N. Shanmugam, S. Sivasubramanian and M. Darus, *On a subclass of k-uniformly convex functions with negative coefficients*, Int. Math. Forum., **1**(34) (2006), 1677-1689.

[6] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51** (1975), 109-116.