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NEW OPERATOR PRESERVING INTEGRAL INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. For a polynomial $P(z)$ of degree n having no zero in $|z| < 1$, it was recently asserted by Shah and Liman [17] that for every $R \geq 1$, $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p ,$$

where B is a B_n-operator, $\sigma(z) = Rz$, $R \geq 1$ and $\Lambda_n := \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}$ with parameters $\lambda_0, \lambda_1, \lambda_2$ in the sense of Rahman [13]. The proof of this result is incorrect. In this paper, we present certain new L_p inequalities for $\mathcal{B}_n$-operators which not only provide a correct proof of the above inequality and other related results but also extend these inequalities for $0 \leq p < 1$ as well.

1. Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in \mathcal{P}_n$, define

$$\|P(z)\|_0 := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right\},$$

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$$\|P(z)\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^p\right\}^{1/p}, \ 1 \le p < \infty,$$
$$\|P(z)\|_\infty := \max_{|z|=1}|P(z)|$$

and denote for any complex function $\psi : \mathbb{C} \to \mathbb{C}$ the composite function of P and ψ, defined by $(P \circ \psi)(z) := P(\psi(z)) \quad (z \in \mathbb{C})$, as $P \circ \psi$.

If $P \in \mathcal{P}_n$, then

$$\left\|P'(z)\right\|_p \le n\,\|P(z)\|_p\,, \quad p \ge 1 \tag{1.1}$$

and

$$\|P(Rz)\|_p \le R^n\,\|P(z)\|_p\,, \ \ R > 1, \ \ p > 0. \tag{1.2}$$

Inequality (1.1) was found out by Zygmund [18] whereas inequality (1.2) is a simple consequence of a result of Hardy [8]. Arestov [2] proved that (1.1) remains true for $0 < p < 1$ as well. For $p = \infty$, the inequality (1.1) is due to Bernstein (for reference, see [11, 15, 16]) whereas the case $p = \infty$ of inequality (1.2) is a simple consequence of the maximum modulus principle (see [11, 12, 15]). Both the inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.1) and (1.2) can be respectively replaced by

$$\left\|P'(z)\right\|_p \le n\frac{\|P(z)\|_p}{\|1+z\|_p}, \quad p \ge 0 \tag{1.3}$$

and

$$\|P(Rz)\|_p \le \frac{\|R^n z+1\|_p}{\|1+z\|_p}\,\|P(z)\|_p\,, \ \ R > 1, \ \ p > 0. \tag{1.4}$$

Inequality (1.3) is due to De-Bruijn [6](see also [3]) for $p \ge 1$. Rahman and Schmeisser [14] extended it for $0 < p < 1$ whereas the inequality (1.4) was proved by Boas and Rahman [5] for $p \ge 1$ and later it was extended for $0 < p < 1$ by Rahman and Schmeisser [14]. For $p = \infty$, the inequality (1.3) was conjectured by Erdös and later verified by Lax [9] whereas inequality (1.4) was proved by Ankeny and Rivlin [1].

As a compact generalization of inequalities (1.1) and (1.2), Aziz and Rather [4] proved that if $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \le 1$, $R \ge 1$, and $p > 0$,

$$\|P(Rz) - \alpha P(z)\|_p \le |R^n - \alpha|\,\|P(z)\|_p\,. \tag{1.5}$$

and if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R \geq 1$, and $p > 0$,

$$\|P(Rz) - \alpha P(z)\|_p \leq \frac{\|(R^n - \alpha)z + (1-\alpha)\|_p}{\|1+z\|_p} \|P(z)\|_p \,. \tag{1.6}$$

Inequality (1.6) is the corresponding compact generalization of inequalities (1.3) and (1.4).

Rahman [13] (see also Rahman and Schmeisser [15, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that maps $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!} \tag{1.7}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \;\; C(n,r) = n!/r!(n-r)!,$$

lie in the half plane

$$|z| \leq |z - n/2|$$

and proved that if $P \in \mathcal{P}_n$, then

$$|B[P \circ \sigma](z)| \leq R^n \, |\Lambda_n| \, \|P(z)\|_\infty \quad \text{for} \;\; |z| = 1. \tag{1.8}$$

And if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then

$$|B[P \circ \sigma](z)| \leq \frac{1}{2} \left\{R^n \, |\Lambda_n| + |\lambda_0|\right\} \|P(z)\|_\infty \quad \text{for} \;\; |z| = 1, \tag{1.9}$$

(see [13, Inequality (5.2) and (5.3)] where $\sigma(z) = Rz$, $R \geq 1$ and

$$\Lambda_n := \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{1.10}$$

As an extension of inequality (1.8) to L_p-norm, recently W. M. Shah and A. Liman [17, Theorem 1] proved that if $P \in \mathcal{P}_n$, then for every $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq R^n \, |\Lambda_n| \, \|P(z)\|_p \,, \tag{1.11}$$

where $B \in \mathcal{B}_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (1.10).

While seeking the desired extension of inequality (1.9) to L_p-norm, they [17, Theorem 2] have made an incomplete attempt by claiming to have proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n |\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p \,, \tag{1.12}$$

where $B \in \mathcal{B}_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (1.10).

Further, it has been claimed in [17] to have proved the inequality (1.12) for self-inversive polynomials as well.

The proof of inequality (1.12) and other related results including the Lemma 4 in [17] given by Shah and Liman is not correct. The reason being that the authors in [17] deduce line 10 from line 7 on page 84, line 19 on page 85 from Lemma 3 [17] and line 16 from line 14 on page 86 by using the fact that if $P^*(z) := z^n\overline{P(1/\overline{z})}$, then for $\sigma(z) = Rz$, $R \geq 1$ and $|z| = 1$,

$$|B[P^* \circ \sigma](z)| = |B[(P^* \circ \sigma)^*](z)|,$$

which is not true, in general, for every $R \geq 1$ and $|z| = 1$. To see this, let

$$P(z) = a_n z^n + \cdots + a_k z^k + \cdots + a_1 z + a_0$$

be an arbitrary polynomial of degree n, then

$$P^\star(z) =: z^n\overline{P(1/\overline{z})} = \bar{a_0}z^n + \bar{a_1}z^{n-1} + \cdots + \bar{a_k}z^{n-k} + \cdots + \bar{a_n}.$$

Now with $\mu_1 := \lambda_1 n/2$ and $\mu_2 := \lambda_2 n^2/8$, we have

$$B[P^\star \circ \rho](z) = \sum_{k=0}^{n} \left(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1)\right) \bar{a_k} z^{n-k} R^{n-k},$$

and in particular for $|z| = 1$, we get

$$B[P^\star \circ \rho](z) = R^n z^n \sum_{k=0}^{n} \left(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1)\right) \overline{a_k \left(\frac{z}{R}\right)^k},$$

whence

$$|B[P^\star \circ \rho](z)| = R^n \left| \sum_{k=0}^{n} \overline{(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1))} a_k \left(\frac{z}{R}\right)^k \right|.$$

But

$$|B[(P^\star \circ \rho)^\star](z)| = R^n \left| \sum_{k=0}^{n} \left(\lambda_0 + \mu_1 k + \mu_2 k(k-1)\right) a_k \left(\frac{z}{R}\right)^k \right|,$$

so the asserted identity does not hold in general for every $R \geq 1$ and $|z| = 1$ as e.g. the immediate counterexample of $P(z) := z^n$ demonstrates in view of $P^\star(z) = 1$, $|B[P^\star \circ \rho](z)| = |\lambda_0|$ and

$$|B[(P^\star \circ \rho)^\star](z)| = |\lambda_0 + \lambda_1(n^2/2) + \lambda_2 n^3(n-1)/8|, \quad |z| = 1.$$

The main aim of this paper is to present correct proofs of the results mentioned in [17] by investigating the dependence of

$$\|B[P\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P](z)\|_p$$

on $\|P(z)\|_p$ for arbitrary real or complex numbers α, β with $|\alpha|\le 1$, $|\beta|\le 1$, $R>1$, $0\le p<\infty$, $\sigma(z):=Rz$,

$$\phi_n\left(R,\alpha,\beta\right):=\beta\left\{\left(\frac{R+1}{2}\right)^n-|\alpha|\right\}-\alpha, \tag{1.13}$$

and establish certain generalized L_p-mean extensions of the inequalities (1.8) and (1.9) for $0\le p<\infty$.

2. Lemmas

For the proofs of our main results, we need the following lemmas. The first Lemma is easy to prove.

Lemma 2.1. *If $P\in P_n$ and $P(z)$ has all its zeros in $|z|\le 1$, then for every $R\ge 1$ and $|z|=1$,*

$$|P(Rz)|\ge\left(\frac{R+1}{2}\right)^n|P(z)|\,.$$

The following Lemma follows from Corollary 18.3 of [10, p. 65].

Lemma 2.2. *If all the zeros of polynomial $P\in\mathcal{P}_n$ lie in $|z|\le 1$, then all the zeros of the polynomial $B[P](z)$ also lie in $|z|\le 1$.*

Lemma 2.3. *If $F\in P_n$ has all its zeros in $|z|\le 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)|\le|F(z)|\ \text{ for }\ |z|=1,$$

then for arbitrary real or complex numbers α,β with $|\alpha|\le 1$, $|\beta|\le 1$, $R\ge 1$, and $|z|\ge 1$,

$$\begin{aligned}&|B[P\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P](z)|\\&\le|B[P^*\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P^*](z)|\end{aligned}\tag{2.1}$$

where $P^*(z):=z^n\overline{P(1/\overline{z})}$, $B\in\mathcal{B}_n$, $\sigma(z):=Rz$, Λ_n and $\phi_n\left(R,\alpha,\beta\right)$ are defined by (1.10) *and* (1.13) *respectively.*

Proof. Since the polynomial $F(z)$ of degree n has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = 1, \tag{2.2}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (2.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \text{ for } |z| > 1 .$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leq m \leq n$, so that we can write

$$F(z) = F_1(z)F_2(z),$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = 1$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < 1$. This implies with the help of inequality (2.2) that

$$P(z) = P_1(z)F_1(z),$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (2.2), we get

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = 1,$$

where $F_2(z) \neq 0$ for $|z| = 1$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < 1$. Hence the polynomial

$$f(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z),$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $f(z)$, we obtain for every $R > 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|f(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{2}\right)^{n-1} |H(e^{i\theta})| \\
&= \left(\frac{R+1}{2}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|e^{i\theta} - te^{i\delta}|} |(e^{i\theta} - te^{i\delta})H(e^{i\theta})| \\
&\geq \left(\frac{R+1}{2}\right)^{n-1} \left(\frac{R+t}{1+t}\right) |f(e^{i\theta})|.
\end{aligned}$$

This implies for $R > 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{1+t}{R+t}\right)|f(Re^{i\theta})| \geq \left(\frac{R+1}{2}\right)^{n-1}|f(e^{i\theta})|. \tag{2.3}$$

Since $R > 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{2}{1+R} > \frac{1+t}{R+t}$, from inequality (2.3), we obtain $R > 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta}| > \left(\frac{R+1}{2}\right)^{n}|f(e^{i\theta})|. \tag{2.4}$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{2}\right)^{n}|f(z)|$$

for $|z| = 1$ and $R > 1$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > 1$, we have

$$\begin{aligned}|f(Rz) - \alpha f(z)| &\geq |f(Rz)| - |\alpha||f(z)| \\ &> \left\{\left(\frac{R+1}{2}\right)^{n} - |\alpha|\right\}|f(z)|, \quad |z| = 1.\end{aligned} \tag{2.5}$$

Also, inequality (2.4) can be written in the form

$$|f(e^{i\theta})| < \left(\frac{2}{R+1}\right)^{n}|f(Re^{i\theta})| \tag{2.6}$$

for every $R > 1$ and $0 \leq \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{2}{R+1}\right)^{n} < 1$, from inequality (2.6), we obtain for $0 \leq \theta < 2\pi$ and $R > 1$,

$$|f(e^{i\theta}| < |f(Re^{i\theta})|.$$

Equivalently,

$$|f(z)| < |f(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $f(Rz) - \alpha f(z)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (2.5) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$, all the zeros of the polynomial

$$\begin{aligned}T(z) &= f(Rz) - \alpha f(z) + \beta\left\{\left(\frac{R+1}{2}\right)^{n} - |\alpha|\right\}f(z) \\ &= f(Rz) + \phi_n(R,\alpha,\beta)f(z) \\ &= \big(P(Rz) - \lambda F(Rz)\big) + \phi_n\left(R,\alpha,\beta\right)\big(P(z) - \lambda F(z)\big) \\ &= \big(P(Rz) + \phi_n\left(R,\alpha,\beta\right)P(z)\big) - \lambda\big(F(Rz) + \phi_n\left(R,\alpha,\beta\right)F(z)\big)\end{aligned}$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. Using Lemma 2.2 and the fact that B is a linear operator, we conclude that all the zeros of polynomial

$$\begin{aligned} W(z) &= B[T](z) \\ &= (B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[F](z)) \end{aligned}$$

also lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$\begin{aligned} &|B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P](z)| \\ &\leq |B[F \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[F](z)| \end{aligned} \tag{2.7}$$

for $|z| \geq 1$ and $R > 1$. If inequality (2.7) is not true, then exist a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} &|B[P \circ \sigma](z_0) + \phi_n(R, \alpha, \beta) B[P](z_0)| \\ &> |B[F \circ \sigma](z_0) + \phi_n(R, \alpha, \beta) B[F](z_0)|. \end{aligned}$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi_n(R, \alpha, \beta)F(z)$ lie in $|z| < 1$. Hence by Lemma 2.2, all the zeros of $B[F \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[F](z)$ also lie in $|z| < 1$, which shows that

$$B[F \circ \sigma](z_0) + \phi_n(R, \alpha, \beta) B[F](z_0) \neq 0.$$

We take

$$\lambda = \frac{B[P \circ \sigma](z_0) + \phi_n(R, \alpha, \beta)B[P](z_0)}{B[F \circ \sigma](z_0) + \phi_n(R, \alpha, \beta)B[F](z_0)},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $W(z_0) = 0$. This contradicts the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus (2.7) holds and this completes the proof of Lemma 2.3. □

Lemma 2.4. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R > 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P](z)| \\ &\leq |B[P^* \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P^*](z)| \end{aligned} \tag{2.8}$$

where $P^*(z) := z^n\overline{P(1/\overline{z})}$, $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ and $\phi_n(R, \alpha, \beta)$ is defined by (1.13).

Proof. By hypothesis the polynomial $P(z)$ of degree n does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $P^*(z) = z^n\overline{P(1/\overline{z})}$ of degree n lie in $|z| \leq 1$. Applying Lemma 2.3 with $F(z)$ replaced by $P^*(z)$, it follows that

$$\begin{aligned} &|B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P](z)| \\ &\leq |B[P^* \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P^*](z)| \end{aligned}$$

for $|z| \geq 1, |\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$. This proves the Lemma 2.4. □

Next we describe a result of Arestov [2].

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \mathbb{C}^{n+1}$ *and* $P(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$C_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:
(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \leq 1\}$,
(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \geq 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.5. ([2, Th. 2]) *Let* $\phi(x) = \psi(\log x)$ *where* ψ *is a convex non decreasing function on* $\mathbb{R}$. *Then for all* $P \in \mathcal{P}_n$ *and each admissible operator* Λ_γ,

$$\int_0^{2\pi} \phi\left(|C_\gamma P(e^{i\theta})|\right) d\theta \leq \int_0^{2\pi} \phi\left(c(\gamma, n)|P(e^{i\theta})|\right) d\theta,$$

where $c(\gamma, n) = max\,(|\gamma_0|, |\gamma_n|)$.

In particular Lemma 2.5 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$ and $\phi : x \to \log x$ as well. Therefore, we have for $0 \leq p < \infty$,

$$\left\{\int_0^{2\pi} \phi\left(|C_\gamma P(e^{i\theta})|^p\right) d\theta\right\}^{1/p} \leq c(\gamma, n)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{2.9}$$

From Lemma 2.5, we deduce the following result.

Lemma 2.6. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for each* $p > 0$, $R > 1$ *and* η *real,* $0 \leq \eta < 2\pi$,

$$\begin{aligned}\int_0^{2\pi} \Big|&\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta) B[P](e^{i\theta})\big)e^{i\eta} \\ &+ \big(B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, \bar{\alpha}, \bar{\beta}) B[P^*]^*(e^{i\theta})\big)\Big|^p d\theta \\ \leq \Big|&(R^n + \phi_n(R, \alpha, \beta))\Lambda_n e^{i\eta} + (1 + \phi_n(R, \bar{\alpha}, \bar{\beta}))\bar{\lambda_0}\Big|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta,\end{aligned}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $B[P^* \circ \sigma]^*(z) := (B[P^* \circ \sigma](z))^*$, Λ_n *and* $\phi_n(R, \alpha, \beta)$ *are defined by* (1.10) *and* (1.13) *respectively.*

Proof. Since $P(z)$ does not vanish in $|z|<1$ and $P^*(z)=z^n\overline{P(1/\bar{z})}$, by Lemma 2.3, we have for $R>1$,

$$\begin{aligned}&|B[P\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P](z)\\&\leq|B[P^*\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P^*](z)|\end{aligned}\tag{2.10}$$

Also, since

$$P^*(Rz)+\phi_n\left(R,\alpha,\beta\right)P^*(z)=R^nz^n\overline{P(1/R\bar{z})}+\phi_n\left(R,\alpha,\beta\right)z^n\overline{P(1/\bar{z})},$$

therefore,

$$\begin{aligned}&B[P^*\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[P^*](z)\\&=\lambda_0\big(R^nz^n\overline{P(1/R\bar{z})}+\phi_n\left(R,\alpha,\beta\right)z^n\overline{P(1/\bar{z})}\big)+\lambda_1\left(\frac{nz}{2}\right)\Big(nR^nz^{n-1}\overline{P(1/R\bar{z})}\\&\quad-R^{n-1}z^{n-2}\overline{P'(1/R\bar{z})}+\phi_n\left(R,\alpha,\beta\right)\big(nz^{n-1}\overline{P(1/\bar{z})}-z^{n-2}\overline{P'(1/\bar{z})}\big)\Big)\\&\quad+\frac{\lambda_2}{2!}\left(\frac{nz}{2}\right)^2\Big(n(n-1)R^nz^{n-2}\overline{P(1/R\bar{z})}-2(n-1)R^{n-1}z^{n-3}\overline{P'(1/R\bar{z})}\\&\quad+R^{n-2}z^{n-4}\overline{P''(1/R\bar{z})}+\phi_n\left(R,\alpha,\beta\right)\big(n(n-1)z^{n-2}\overline{P(1/\bar{z})}\\&\quad-2(n-1)z^{n-3}\overline{P'(1/\bar{z})}+z^{n-4}\overline{P''(1/\bar{z})}\big)\Big)\end{aligned}$$

and hence,

$$\begin{aligned}&B[P^*\circ\sigma]^*(z)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[P^*]^*(z)\\&=\big(B[P^*\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P^*](z)\big)^*\\&=\left(\bar{\lambda_0}+\bar{\lambda_1}\frac{n^2}{2}+\bar{\lambda_2}\frac{n^3(n-1)}{8}\right)\left(R^nP(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)P(z)\right)\\&\qquad-\left(\bar{\lambda_1}\frac{n}{2}+\bar{\lambda_2}\frac{n^2(n-1)}{4}\right)\Big(R^{n-1}zP'(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)zP'(z)\Big)\\&\qquad+\bar{\lambda_2}\frac{n^2}{8}\Big(R^{n-2}z^2P''(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)z^2P''(z)\Big).\end{aligned}\tag{2.11}$$

Also, for $|z|=1$

$$\begin{aligned}&|B[P^*\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P^*](z)|\\&=|B[P^*\circ\sigma]^*(z)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[P^*]^*(z)|.\end{aligned}$$

Using this in (2.10), we get for $|z|=1$ and $R>r\geq 1$,

$$\begin{aligned}&|B[P\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[P](z)|\\&\leq|B[P^*\circ\sigma]^*(z)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[P^*]^*(z)|.\end{aligned}$$

Since all the zeros of $P^*(z)$ lie in $|z|\leq 1$, as before, all the zeros of $P^*(Rz)+\phi_n(R,\alpha,\beta)P^*(z)$ lie in $|z|<1$ for all real or complex numbers α,β with

$|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > 1$. Hence by Lemma 2.2, all the zeros of $B[P^* \circ \sigma](z) + \phi_n(R,\alpha,\beta)B[P^*](z)$ lie in $|z| < 1$, therefore, all the zeros of $B[P^* \circ \sigma]^*(z) + \phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z)$ lie in $|z| > 1$. Hence by the maximum modulus principle,

$$\begin{aligned} &|B[P \circ \sigma](z) + \phi_n\,(R,\alpha,\beta)\,B[P^*](z)| \\ &< |B[P^* \circ \sigma]^*(z) + \phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[P^*]^*(z)| \quad \text{for} \quad |z| < 1. \end{aligned} \tag{2.12}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned} C_\gamma P(z) =& \big(B[P \circ \sigma](z) + \phi_n(R,\alpha,\beta)B[P](z)\big)e^{i\eta} \\ &+ \big(B[P^* \circ \sigma]^*(z) + \phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z)\big) \\ =& \left\{(R^n + \phi_n(R,\alpha,\beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}\right\} a_n z^n \\ &+ \cdots + \left\{(R^n + \phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\Lambda_n} + e^{i\eta}(1 + \phi_n(R,\alpha,\beta))\lambda_0\right\} a_0 \end{aligned}$$

does not vanish in $|z| < 1$. Therefore, C_γ is an admissible operator. Applying (2.9) of Lemma 2.5, the desired result follows immediately for each $p > 0$. □

From Lemma 2.6, we deduce the following more general result.

Lemma 2.7. *If $P \in P_n$, then for every $p > 0$, $R > 1$ and η real, $0 \leq \eta < 2\pi$,*

$$\begin{aligned} &\int_0^{2\pi} |\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta} \\ &\qquad + \big(B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})\big)|^p d\theta \\ &\leq |(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1 + \phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta, \end{aligned}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, $B[P^* \circ \sigma]^*(z) := (B[P^* \circ \sigma](z))^*$, Λ_n and $\phi_n\,(R,\alpha,\beta)$ are defined by (1.10) *and* (1.13) *respectively.*

Proof. If all the zeros of $P(z)$ lie in $|z| \geq 1$, then the result follows by Lemma 2.6. Henceforth, we assume that $P(z)$ has at least one zero in $|z| < 1$ so that we can write

$$P(z) = P_1(z)P_2(z) = a\prod_{j=1}^{k}(z - z_j)\prod_{j=k+1}^{n}(z - z_j), \;\; 0 \leq k \leq n-1, \;\; a \neq 0$$

where all the zeros of $P_1(z)$ lie in $|z| \geq 1$ and all the zeros of $P_2(z)$ lie in $|z| < 1$. First we assume that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $P_1(z)$ lie in $|z| > 1$. Let $P_2^*(z) = z^{n-k}\overline{P_2(1/\bar{z})}$, then all the zeros of $P_2^*(z)$

lie in $|z|>1$ and $|P_2^*(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$f(z)=P_1(z)P_2^*(z)=a\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\bar{z}_j),$$

then all the zeros of $f(z)$ lie in $|z|>1$ and for $|z|=1$,

$$|f(z)|=|P_1(z)|\,|P_2^*(z)|=|P_1(z)|\,|P_2(z)|=|P(z)|\,. \tag{2.13}$$

Therefore, it follows by Rouche's theorem that the polynomial $g(z)=P(z)+\mu f(z)$ does not vanish in $|z|\le 1$ for every μ with $|\mu|>1$, so that all the zeros of $g(z)$ lie in $|z|\ge\delta$ for some $\delta>1$ and hence all the zeros of $T(z)=g(\delta z)$ lie in $|z|\ge 1$. Applying (2.12) and (2.11) to the polynomial $T(z)$, we get for $R>1$ and $|z|<1$,

$$\begin{aligned}
&|B[T\circ\sigma](z)+\phi_n(R,\alpha,\beta)\,B[T](z)|\\
&<|B[T^*\circ\sigma]^*(z)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[T^*]^*(z)|\\
&=\Bigg|\left(\bar{\lambda}_0+\bar{\lambda}_1\frac{n^2}{2}+\bar{\lambda}_2\frac{n^3(n-1)}{8}\right)\left(R^nT(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)T(z)\right)\\
&\qquad-\left(\bar{\lambda}_1\frac{n}{2}+\bar{\lambda}_2\frac{n^2(n-1)}{4}\right)\left(R^{n-1}zT'(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)zT'(z)\right)\\
&\qquad+\bar{\lambda}_2\frac{n^2}{8}\left(R^{n-2}z^2T''(z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)z^2T''(z)\right)\Bigg|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|B[T\circ\sigma](z)+\phi_n(R,\alpha,\beta)\,B[T](z)|\\
&=\Bigg|\left(\bar{\lambda}_0+\bar{\lambda}_1\frac{n^2}{2}+\bar{\lambda}_2\frac{n^3(n-1)}{8}\right)\left(R^ng(\delta z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)g(\delta z)\right)\\
&\qquad-\left(\bar{\lambda}_1\frac{n}{2}+\bar{\lambda}_2\frac{n^2(n-1)}{4}\right)\left(R^{n-1}\delta zg'(\delta z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)\delta zg'(\delta z/r)\right)\\
&\qquad+\bar{\lambda}_2\frac{n^2}{8}\left(R^{n-2}\delta^2z^2g''(\delta z/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)\delta^2z^2g''(\delta z)\right)\Bigg|
\end{aligned}$$

for $|z|<1$. If $z=e^{i\theta}/\delta, 0\le\theta<2\pi$, then $|z|=(1/\delta)<1$ as $\delta>1$ and we get

$$\begin{aligned}
&|B[T\circ\sigma](e^{i\theta}/\delta)+\phi_n(R,\alpha,\beta)B[T](e^{i\theta}/\delta)|\\
&=\Bigg|\left(\bar{\lambda_0}+\bar{\lambda_1}\frac{n^2}{2}+\bar{\lambda_2}\frac{n^3(n-1)}{8}\right)\left(R^n g(e^{i\theta}/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)g(e^{i\theta})\right)\\
&\quad-\left(\bar{\lambda_1}\frac{n}{2}+\bar{\lambda_2}\frac{n^2(n-1)}{4}\right)\left(R^{n-1}e^{i\theta}g'(e^{i\theta}/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)e^{i\theta}g'(e^{i\theta})\right)\\
&\quad+\bar{\lambda_2}\frac{n^2}{8}\left(R^{n-2}e^{2i\theta}g''(e^{i\theta}/R)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)e^{2i\theta}g''(e^{i\theta})\right)\Bigg|\\
&=|B[g^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[g^*]^*(e^{i\theta})|.
\end{aligned}$$

Equivalently for $|z|=1$,

$$\begin{aligned}
&|B[g\circ\sigma](z)]+\phi_n\left(R,\alpha,\beta\right)B[g](z)|\\
&\quad<|B[g^*\circ\sigma]^*(z)+\phi_n\left(R,\bar{\alpha},\bar{\beta}\right)B[g^*]^*(z)|.
\end{aligned}$$

Since all the zeros of $g(z)$ lie in $|z|\geq 1$, all the zeros of $g^*(z)=z^n\overline{g(1/\bar{z})}$ lie in $|z|\leq 1$ and hence as before, all the zeros of $g^*(Rz)+\phi_n\left(R,\alpha,\beta\right)g^*(z)$ lie in $|z|<1$. By Lemma 2.2, all the zeros of $B[g^*\circ\sigma](z)+\phi_n\left(R,\alpha,\beta\right)B[g^*](z)$ lie in $|z|<1$ and therefore, all the zeros of $B[g^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[g^*]^*(z)$ lie in $|z|>1$. Thus

$$B[g^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[g^*]^*(z)\neq 0 \quad\text{for}\quad |z|\leq 1.$$

An application of Rouche's theorem shows that the polynomial

$$\begin{aligned}
M(z)=&\left(B[g\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[g](z)\right)e^{i\eta}\\
&+B[g^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[g^*]^*(z)
\end{aligned}\tag{2.14}$$

does not vanish in $|z|\leq 1$. Replacing $g(z)$ by $P(z)+\mu f(z)$ and noting that B is a linear operator, it follows that the polynomial

$$\begin{aligned}
M(z)=&\left(B[P\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[P](z)\right)e^{i\eta}\\
&+\left(B[P^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z)\right)\\
&+\mu\big(\left(B[f\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[f](z)\right)e^{i\eta}\\
&+(B[f^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z))\big)
\end{aligned}\tag{2.15}$$

does not vanish in $|z|\leq 1$ for every μ with $|\mu|>1$. We claim

$$\begin{aligned}
&|\left(B[P\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[P](z)\right)e^{i\eta}\\
&\qquad+B[P^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z)|\\
&\leq|\left(B[f\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[f](z)\right)e^{i\eta}\\
&\qquad+B[f^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z)|
\end{aligned}\tag{2.16}$$

for $|z| \leq 1$. If inequality (2.16) is not true, then there a point $z = z_0$ with $|z_0| \leq 1$ such that

$$\begin{aligned}&|\big(B[P\circ\sigma](z_0)+\phi_n(R,\alpha,\beta)B[P](z_0)\big)e^{i\eta}\\&\qquad+B[P^*\circ\sigma]^*(z_0)+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z_0)|\\&>|\big(B[f\circ\sigma](z_0)+\phi_n(R,\alpha,\beta)B[f](z_0)\big)e^{i\eta}\\&\qquad+B[f^*\circ\sigma]^*(z_0)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z_0)|.\end{aligned}$$

Since $f(z)$ does not vanish in $|z| \leq 1$, proceeding similarly as in the proof of (2.14), it follows that the polynomial

$$\begin{aligned}&\big(B[f\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[f](z)\big)e^{i\eta}\\&\qquad+B[f^*\circ\sigma]^*(z)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z)\end{aligned}$$

does not vanish in $|z| \leq 1$. Hence

$$\begin{aligned}&\big(B[f\circ\sigma](z_0)+\phi_n(R,\alpha,\beta)B[f](z_0)\big)e^{i\eta}\\&\qquad+B[f^*\circ\sigma]^*(z_0)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z_0)\neq 0.\end{aligned}$$

We take

$$\mu=-\frac{(B[P\circ\sigma](z_0)+\phi_n(R,\alpha,\beta)B[P](z_0))e^{i\eta}+B[P^*\circ\sigma]^*(z_0)+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z_0)}{(B[f\circ\sigma](z_0)+\phi_n(R,\alpha,\beta)B[f](z_0))e^{i\eta}+B[f^*\circ\sigma]^*(z_0)+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(z_0)}$$

so that μ is well-defined real or complex number with $|\mu| > 1$ and with this choice of μ, from (2.15) , we get $M(z_0) = 0$. This clearly is a contradiction to the fact that $M(z)$ does not vanish in $|z| \leq 1$. Thus (2.16) holds, which in particular gives for each $p > 0$ and η real,

$$\begin{aligned}&\int_0^{2\pi}\Big|\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta}\\&\qquad+B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})\Big|^p d\theta\\&\leq\int_0^{2\pi}\Big|\big(B[f\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[f](e^{i\theta})\big)e^{i\eta}\\&\qquad+B[f^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[f^*]^*(e^{i\theta})\Big|d\theta.\end{aligned}\tag{2.17}$$

Using Lemma 2.7 and (2.13), we get for each $p > 0$,

$$\begin{aligned}\int_0^{2\pi} &\Big|\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta} \\ &\quad + B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})\Big|^p d\theta \\ &\le |(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1+\phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi}\Big|f(e^{i\theta})\Big|^p d\theta \\ &= |(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1+\phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi}\Big|P(e^{i\theta})\Big|^p d\theta. \quad (2.18)\end{aligned}$$

Now if $P_1(z)$ has a zero on $|z| = 1$, then applying (2.18) to the polynomial $Q(z) = P_1(tz)P_2(z)$ where $t < 1$, we get for each $p > 0$, $R > 1$ and η real,

$$\begin{aligned}\int_0^{2\pi} &|\big(B[Q\circ\sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[Q](e^{i\theta})\big)e^{i\eta} \\ &\quad + \big(B[Q^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,\bar{\alpha},\bar{\beta})B[Q^*]^*(e^{i\theta})\big)|^p d\theta \\ &\le |(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1+\phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi}\Big|Q(e^{i\theta})\Big|^p d\theta. \quad (2.19)\end{aligned}$$

Letting $t \to 1$ in (2.19) and using continuity, the desired result follows immediately and this proves Lemma 2.7. □

Lemma 2.8. *If $P \in P_n$ and $P^*(z) = z^n\overline{P(1/\bar{z})}$, then for every $p > 0$, α, $\beta \in \mathbb{C}$ with $|\alpha| \le 1$, $|\beta| \le 1$ and $R > 1$,*

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} &\Big|\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta} \\ &\quad + \big(B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})\big)\Big|^p d\theta \\ &\le \int_0^{2\pi}\Big|(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1+\phi_n(R,\alpha,\beta))\lambda_0\Big|^p d\eta \\ &\quad \times \int_0^{2\pi}\Big|P(e^{i\theta})\Big|^p d\theta \qquad (2.20)\end{aligned}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, Λ_n and $\phi_n(R,\alpha,\beta)$ are defined by (1.10) *and* (1.13) *respectively. The result is best possible and the extremal polynomial is* $P(z) = \beta z^n$, $\beta \ne 0$.

Proof. Since $B[P^*\circ\sigma]^*(z) + \phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(z)$ is the conjugate polynomial of $B[P^*\circ\sigma](z) + \phi_n(R,\alpha,\beta)B[P^*](z)$,

$$\begin{aligned}&|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})|\\&=|B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})|,\ 0\le\theta<2\pi\end{aligned}$$

and therefore for each $p>0$, $R>1$ and $0\le\theta<2\pi$, we have

$$\begin{aligned}&\int_0^{2\pi}|\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta}\\&\qquad+\big(B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})\big)|^p d\eta\\&=\int_0^{2\pi}||B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})|e^{i\eta}\\&\qquad+|B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})||^p d\eta\\&=\int_0^{2\pi}||B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})|e^{i\eta}\\&\qquad+|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})||^p d\eta.\end{aligned}\tag{2.21}$$

Integrating both sides of (2.21) with respect to θ from 0 to 2π and using Lemma 2.7, we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}|\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta}\\&\qquad+\big(B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})\big)|^p d\eta d\theta\\&=\int_0^{2\pi}\int_0^{2\pi}||B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})|e^{i\eta}\\&\qquad+|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})||^p d\eta d\theta\\&=\int_0^{2\pi}\left(\int_0^{2\pi}\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta}\right.\\&\qquad\left.+\big(B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,\bar{\alpha},\bar{\beta})B[P^*]^*(e^{i\theta})\big)|^p d\theta\right)d\eta\\&\le\int_0^{2\pi}|(R^n+\phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta}+(1+\phi_n(R,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p d\eta\\&\qquad\times\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\\&\le\int_0^{2\pi}|(R^n+\phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta}+(1+\phi_n(R,\alpha,\beta))\lambda_0|^p d\eta\\&\qquad\times\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.\end{aligned}$$

This completes the proof of Lemma 2.8. $\square$

3. Main Results

We first present the following result which is a compact generalization of the inequalities (1.1),(1.2), (1.5) and (1.8) and extends inequality (1.11) for $0 \le p < 1$ as well.

Theorem 3.1. *If $P \in \mathcal{P}_n$, then for arbitrary real or complex numbers α, β with $|\alpha| \le 1$, $|\beta| \le 1$, $R > 1$ and $0 \le p < \infty$,*

$$\begin{aligned} &\|B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta) B[P](z)\|_p \\ &\le |R^n + \phi_n(R, \alpha, \beta)| \, |\Lambda_n| \, \|P(z)\|_p, \end{aligned} \tag{3.1}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, Λ_n and $\phi_n(R, \alpha, \beta)$ are defined by (1.10) *and* (1.13) *respectively. The result is best possible and equality in* (3.1) *holds for $P(z) = az^n, a \neq 0$.*

Proof. By hypothesis $P \in \mathcal{P}_n$, we can write

$$P(z) = P_1(z)P_2(z) = c\prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (z - z_j), \;\; k \ge 1, \, c \neq 0,$$

where all the zeros of $P_1(z)$ lie in $|z| \le 1$ and all the zeros of $P_2(z)$ lie in $|z| > 1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z| < 1$. Let $P_2^*(z) = z^{n-k}\overline{P_2(1/\bar{z})}$, then all the zeros of $P_2^*(z)$ lie in $|z| < 1$ and $|P_2^*(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$F(z) = P_1(z)P_2^*(z) = c\prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n} (1 - z\bar{z}_j),$$

then all the zeros of $F(z)$ lie in $|z| < 1$ and for $|z| = 1$,

$$|F(z)| = |P_1(z)| \, |P_2^*(z)| = |P_1(z)| \, |P_2(z)| = |P(z)|. \tag{3.2}$$

Observe that $P(z)/F(z) \to 1/\prod_{j=k+1}^{n}(-\bar{z}_j)$ when $z \to \infty$, so it is regular even at ∞ and thus from (3.2) and by the maximum modulus principle, it follows that

$$|P(z)| \le |F(z)| \;\text{ for }\; |z| \ge 1.$$

Since $F(z) \neq 0$ for $|z| \ge 1$, a direct application of Rouche's theorem shows that the polynomial $H(z) = P(z) + \lambda F(z)$ has all its zeros in $|z| < 1$ for every λ with $|\lambda| > 1$. Therefore, for all real or complex numbers α, β with $|\alpha| \le 1, |\beta| \le 1$ and $R > 1$, it follows that all the zeros of $h(z) = H(Rz) + \phi_n(R, \alpha, \beta)H(z)$ lie in $|z|<1$. Applying Lemma 2.2 to the polynomial $h(z)$ and noting that B is a linear operator, it follows that all the zeros of

$$\begin{aligned}B[h](z) &= B[H\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[H](z)\\ &= B[P\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[P](z)\\ &\quad+\lambda(B[F\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[F](z))\end{aligned}$$

lie in $|z|<1$ for every λ with $|\lambda|>1$. This implies

$$|B[P\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[P](z)|\le|B[F\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[F](z)|$$

for $|z|\ge 1$, which, in particular, gives for each $p>0$, $R>1$ and $0\le\theta<2\pi$,

$$\begin{aligned}&\int_0^{2\pi}|B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})|^p d\theta\\ &\le\int_0^{2\pi}|B[F\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[F](e^{i\theta})|^p d\theta. \qquad (3.3)\end{aligned}$$

Again, since all the zeros of $F(z)$ lie in $|z|<1$, it follows, as before, that all the zeros of $B[F(Rz)]+\phi_n(R,\alpha,\beta)F(z)$ also lie in $|z|<1$. Therefore, the operator C_γ defined by

$$\begin{aligned}C_\gamma F(z) &= B[F\circ\sigma](z)+\phi_n(R,\alpha,\beta)B[F](z)\\ &= (R^n+\phi_n(R,\alpha,\beta))\Big(\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\Big)b_nz^n+\cdots+\lambda_0b_0\end{aligned}$$

is admissible. Hence by (2.9) of Lemma 2.5, for each $p>0$, we have

$$\begin{aligned}&\int_0^{2\pi}|B[F\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[F](e^{i\theta})|^p d\theta\\ &\le|R^n+\phi_n(R,\alpha,\beta)|\Big|\lambda_0+\lambda_1\frac{n^2}{2}+\lambda_2\frac{n^3(n-1)}{8}\Big|^p\int_0^{2\pi}|F(e^{i\theta})|^p d\theta. \qquad (3.4)\end{aligned}$$

Combining inequalities (3.3) and (3.4) and noting that $|F(e^{i\theta})|=|P(e^{i\theta})|$, we obtain for each $p>0$ and $R>1$,

$$\begin{aligned}&\int_0^{2\pi}|B[P\circ\sigma](e^{i\theta})+\phi_n(R,\alpha,\beta)B[P](e^{i\theta})|^p d\theta\\ &\le|R^n+\phi_n(R,\alpha,\beta)||\Lambda_n|\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \qquad (3.5)\end{aligned}$$

In case $P_1(z)$ has a zero on $|z|=1$, then the inequality (3.5) follows by continuity. To obtain this result for $p=0$, we simply make $p\to 0+$. □

A variety of interesting results can be deduced from Theorem 3.1 as special cases. Here we mention a few of these.

The following result follows from Theorem 3.1 by taking $\beta = 0$.

Corollary 3.2. *If $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \sigma](z) - \alpha B[P](z) \right\|_p \leq |R^n - \alpha| \, |\Lambda_n| \, \| P(z) \|_p \,, \tag{3.6}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ and Λ_n is defined by (1.10). *The result is best possible and equality in* (3.6) *holds for $P(z) = az^n, a \neq 0$.*

Setting $\alpha = 0$ in Corollary 3.2, we get the following sharp result.

Corollary 3.3. *If $P \in \mathcal{P}_n$, then for $R > 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \sigma](z) \right\|_p \leq |R^n| \, |\Lambda_n| \, \| P(z) \|_p \,, \tag{3.7}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ and Λ_n is defined by (1.10). *The result is best possible and equality in* (3.7) *holds for $P(z) = az^n, a \neq 0$.*

Remark 3.4. Corollary 3.3 not only includes inequality (1.11) as a special case but also extends it for $0 \leq p < 1$ as well. Further inequality (1.8) follows from Corollary 3.3 by letting $p \to \infty$ in (3.7).

The case $B[P](z) = P(z)$ of Theorem 3.1 yields the following interesting result which is a compact generalization of inequalities (1.1), (1.2) and (1.5).

Corollary 3.5. *If $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > 1$, and $p > 0$,*

$$\left\| P(Rz) + \phi_n(R, \alpha, \beta) \, P(z) \right\|_p \leq |R^n + \phi_n(R, \alpha, \beta)| \, \| P(z) \|_p \,, \tag{3.8}$$

where $\phi_n(R, \alpha, \beta)$ is defined by (1.13). *The result is best possible and equality in* (3.8) *holds for $P(z) = az^n, a \neq 0$.*

Remark 3.6. If we divide the two sides of (3.8) by $R - 1$ with $\alpha = 1$ and then let $R \to 1$, we get for $P \in \mathcal{P}_n$, $|\beta| \leq 1$ and $0 \leq p < \infty$,

$$\left\| zP'(z) + \frac{n\beta}{2} P(z) \right\|_p \leq n \left| 1 + \frac{\beta}{2} \right| \| P(z) \|_p \,. \tag{3.9}$$

The result is best possible and equality in (3.9) holds for $P(z) = az^n, a \neq 0$.

Taking $\alpha = 0$ in (3.1), we obtain:

Corollary 3.7. *If $P \in \mathcal{P}_n$, then for every real or complex number β with $|\beta| \leq 1$, $R > 1$ and $0 \leq p < \infty$,*

$$\left\|B[P(Rz)] + \beta\left(\frac{R+1}{2}\right)^n B[P(z)]\right\|_p \le \left|R^n + \beta\left(\frac{R+1}{2}\right)^n\right| |\Lambda_n| \, \|P(z)\|_p , \tag{3.10}$$

where $B \in \mathcal{B}_n$ and $\phi_n(R,\alpha,\beta)$ is defined by (1.13)*.The result is best possible and equality in* (3.10) *holds for $P(z) = \lambda z^n, \lambda \neq 0$.*

Theorem 3.1 can be sharpened if we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$. In this direction, we next present the following result which in particular includes a generalized L_p mean extension of the inequality (1.9) for $0 \le p < \infty$ and among other things yields a correct proof of inequality (1.12) for each $p \ge 0$ as a special case.

Theorem 3.8. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for then for arbitrary real or complex numbers α, β with $|\alpha| \le 1$, $|\beta| \le 1$, $R > 1$ and $0 \le p < \infty$,*

$$\|B[P \circ \sigma](z) + \phi_n(R,\alpha,\beta)B[P](z)\|_p \le \frac{\|(R^n + \phi_n(R,\alpha,\beta))\,\Lambda_n z + (1 + \phi_n(R,\alpha,\beta))\,\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p , \tag{3.11}$$

where $B \in B_n$, $\sigma(z) := Rz$, Λ_n and $\phi_n(R,\alpha,\beta)$ are defined by (1.10) *and* (1.13) *respectively. The result is best possible and equality in* (3.11) *holds for $P(z) = az^n + b, |a| = |b| \neq 0$.*

Proof. By hypothesis $P \in P_n$ does not vanish in $|z| < 1$, $\sigma(z) = Rz$, therefore, if $P^*(z) = z^n\overline{P(1/\bar{z})}$, then by Lemma 2.3, we have for $0 \le \theta < 2\pi$,

$$|B[P \circ \sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P](e^{i\theta})| \le |B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})|. \tag{3.12}$$

Also, by Lemma 2.8, for each $p > 0$ and η real and $R > 1$,

$$\int_0^{2\pi}\int_0^{2\pi} |\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P](e^{i\theta})\big)e^{i\eta} + \big(B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R,\alpha,\beta)B[P^*](e^{i\theta})\big)|^p d\theta d\eta$$
$$\le \int_0^{2\pi} |(R^n + \phi_n(R,\alpha,\beta))\Lambda_n e^{i\eta} + (1 + \phi_n(R,\alpha,\beta))\lambda_0|^p d\eta \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Now it can be easily verified that for every real number α and $r \ge 1$,

$$\left|r + e^{i\alpha}\right| \ge \left|1 + e^{i\alpha}\right|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi} \left|r + e^{i\alpha}\right|^p d\alpha \geq \int_0^{2\pi} \left|1 + e^{i\alpha}\right|^p d\alpha. \tag{3.13}$$

If $B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \bar{\alpha}, \bar{\beta})B[P](e^{i\theta}) \neq 0$, we take

$$r = \frac{|B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P^*](e^{i\theta})|}{|B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})|},$$

then by (3.12), $r \geq 1$ and from (3.13), we get

$$\begin{aligned}
&\int_0^{2\pi} |\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})\big)e^{i\eta} \\
&\qquad + \big(B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P^*](e^{i\theta})\big)|^p d\eta \\
&= |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})|^p \\
&\qquad \times \int_0^{2\pi} \left|e^{i\eta} + \frac{B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P^*](e^{i\theta})}{B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})}\right|^p d\eta \\
&= |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})|^p \\
&\qquad \times \int_0^{2\pi} \left|e^{i\eta} + \left|\frac{B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P^*](e^{i\theta})}{B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})}\right|\right|^p d\eta \\
&\geq |B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})|^p \int_0^{2\pi} |1 + e^{i\eta}|^p d\eta.
\end{aligned}$$

For $B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta}) = 0$, this inequality is trivially true. Using this in (2.20), we conclude that for each $p > 0$,

$$\begin{aligned}
&\int_0^{2\pi} \left|B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})\right|^p d\theta \int_0^{2\pi} \left|1 + e^{i\eta}\right|^p d\eta \\
&\leq \int_0^{2\pi} |(R^n + \phi_n(R, \alpha, \beta))\Lambda_n e^{i\eta} \\
&\qquad + (1 + \phi_n(R, \alpha, \beta))\lambda_0|^p d\eta \int_0^{2\pi} |P(e^{i\theta})|^p d\theta,
\end{aligned}$$

from which theorem 3.8 follows for $p > 0$. To establish this result for $p = 0$, we simply let $p \to 0+$. This completes the proof of Theorem 3.8. □

For $\beta = 0$, inequality (3.11) reduces to the following result.

Corollary 3.9. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > 1$ and $0 \leq p < \infty$,*

$$\|B[P \circ \sigma](z) - \alpha B[P](z)\|_p \leq \frac{\|(R^n - \alpha)\Lambda_n z + (1 - \alpha)\lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p, \tag{3.14}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, *and* Λ_n *is defined by* (1.10). *The result is best possible and equality in* (3.14) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$.

For $\alpha = 0$, Corollary 3.9 yields the following interesting result.

Corollary 3.10. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for* $R > 1$ *and* $0 \leq p < \infty$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{\|R^n \Lambda_n z + \lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p \,, \tag{3.15}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (1.10). *The result is best possible and equality in* (3.15) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 3.11. If we choose $\alpha = \lambda_0 = \lambda_2 = 0$ in (3.15), we get for $R > 1$ and $0 \leq p < \infty$

$$\left\|P'(Rz)\right\|_p \leq \frac{nR^{n-1}}{\|1 + z\|_p} \|P(z)\|_p \tag{3.16}$$

which in particular yields inequality (1.3).

By the triangle inequality, the following result immediately follows from Corollary 3.10.

Corollary 3.12. *If* $P \in \mathcal{P}_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for* $0 \leq p < \infty$ *and* $R > 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1 + z\|_p} \|P(z)\|_p \,, \tag{3.17}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, Λ_n *is defined by* (1.10).

Remark 3.13. Corollary 3.12 not only validates the inequality (1.12) for $p \geq 1$ but also extends it for $0 \leq p < 1$ as well.

A polynomial $P \in \mathcal{P}_n$ is said be self-inversive if $P(z) = uP^*(z)$ where $|u| = 1$ and $P^*(z)$ is the conjugate polynomial of $P(z)$, that is, $P^*(z) = z^n \overline{P(1/\overline{z})}$. Finally in this paper, we establish the following result for self-inversive polynomials which includes a correct proof of another result of Shah and Liman [17, Theorem 3] as a special case.

Theorem 3.14. *If* $P \in \mathcal{P}_n$ *is a self-inversive polynomial, then for arbitrary real or complex numbers* α, β *with* $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ *and* $0 \leq p < \infty$,

$$\begin{aligned} &\|B[P \circ \sigma](z) + \phi_n(R, \alpha, \beta)B[P](z)\|_p \\ &\leq \frac{\|(R^n + \phi_n(R, \alpha, \beta))\,\Lambda_n z + (1 + \phi_n(R, \alpha, \beta))\,\lambda_0\|_p}{\|1 + z\|_p} \|P(z)\|_p \,, \end{aligned} \tag{3.18}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$, Λ_n *and* $\phi_n(R, \alpha, \beta)$ *are defined by* (1.10) *and* (1.13) *respectively. The result is best possible and equality in* (3.18) *holds for* $P(z) = z^n + 1$.

Proof. Since $P \in P_n$ is self-inversive polynomial, we have for some u with u=1, $P^*(z) = uP(z)$ for all $z \in \mathbb{C}$ where $P^*(z) = z^n\overline{P(1/\bar{z})}$. This gives for $0 \le \theta < 2\pi$,

$$\begin{aligned}&|B[P \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P](e^{i\theta})|\\ &\le |B[P^* \circ \sigma](e^{i\theta}) + \phi_n(R, \alpha, \beta)B[P^*](e^{i\theta})|.\end{aligned}$$

Using this in (2.20) and proceeding similarly as in the proof of Theorem 3.8, we get the desired result for each $p > 0$. To extension to $p = 0$ is obtains by letting $p \to 0+$. □

The following result is an immediate consequence of Theorem 3.14.

Corollary 3.15. *If* $P \in \mathcal{P}_n$ *is a self-inversive polynomial, then for* $0 \le p < \infty$ *and* $R > 1$,

$$\begin{aligned}&\left\|B[P \circ \sigma](z) - \alpha B[P](z)\right\|_p\\ &\le \frac{\|(R^n - \alpha r^n)\Lambda_n z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p}\,\|P(z)\|_p\,,\end{aligned} \tag{3.19}$$

where $B \in \mathcal{B}_n$ *and* $\sigma(z) := Rz$, *and* Λ_n *is defined by* (1.10). *The result is sharp and equality in* (3.19) *holds for* $P(z) = z^n + 1$.

For $\alpha = 0$, Corollary 3.15 reduces to the following interesting result.

Corollary 3.16. *If* $P \in \mathcal{P}_n$ *is a self-inversive polynomial, then for* $0 \le p < \infty$ *and* $R > 1$,

$$\|B[P \circ \sigma](z)\|_p \le \frac{\|R^n\Lambda_n z + \lambda_0\|_p}{\|1+z\|_p}\,\|P(z)\|_p\,, \tag{3.20}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (1.10). *The result is best possible and equality in* (3.20) *holds for* $P(z) = z^n + 1$.

By the triangle inequality, the following result follows immediately from Corollary 3.16.

Corollary 3.17. *If* $P \in \mathcal{P}_n$ *is a self-inversive polynomial, then for* $0 \le p < \infty$ *and* $R > 1$,

$$\|B[P \circ \sigma](z)\|_p \le \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1+z\|_p}\,\|P(z)\|_p\,, \tag{3.21}$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (1.10). *The result is sharp and equality in* (3.21) *holds for* $P(z) = z^n + 1$.

Remark 3.18. Corollary 3.17 establishes a correct proof of a result due to Shah and Liman [17, Theorem 3] for $p \geq 1$ and also extends it for $0 \leq p < 1$ as well.

Lastly letting $p \to \infty$ and setting $\alpha = \beta = 0$ in (3.18), we obtain the following result.

Corollary 3.19. *If* $P \in \mathcal{P}_n$ *is a self-inversive polynomial, then for* $|z| = 1$ *and* $R > 1$,

$$|B[P \circ \sigma](z)| \leq \frac{1}{2} \{R^n |\Lambda_n| + |\lambda_0|\} \|P(z)\|_{\infty},$$

where $B \in \mathcal{B}_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (1.10). *The result is sharp.*

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