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# ON THE SEQUENCE SPACE $l(p, s)$ AND SOME MATRIX TRANSFORMATIONS

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**Abstract.** Bulut and Cakar [2] have defined the space $l(p, s)$ and determined some classes of matrix transformations $(l(p, s), l_\infty)$ and $(l(p, s), c)$. The object of this paper is to investigate some further clases of infinite matrices, *i.e.*, $(l(p, s), v^\sigma)$ and $(l(p, s), v^\sigma_\infty)$, where $v^\sigma$ is the space of all bounded sequences all of whose $\sigma$- means are equal, $v^\sigma_\infty$ is the space of $\sigma$-bounded sequence.

## 1. Preliminaries, background and Notation

We denote the set of all sequences(real or complex) by $\omega$. Any subspace of $\omega$ is called the sequence space. So the sequence space is the set of scalar sequences(real of complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let $l_\infty$, $c$ and $c_0$, respectively, denotes the space of all bounded sequences, the space of convergent sequences and the sequences converging to zero. Also, by $cs$, $l_1$ and $l(p)$ we denote the spaces of all convergent, absolutely and $p$-absolutely convergent series, respectively.

Let $X, Y$ be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, the matrix $A$ defines

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the $A$-transformation from $X$ *into* $Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$ exists and is in $Y$; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $A \in (X : Y)$ we mean the characterizations of matrices from $X$ to $Y$ *i.e.*, $A : X \to Y$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$ which is called as the $A$-limit of $x$.

Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\phi$ on $l_\infty$ is said to be an invariant mean or a $\sigma$-mean if and only if
(i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$;
(ii) $\phi(e) = 1$, where $e$={1,1,1,.....}; and
(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

Through out this paper, we deal only with mappings $\sigma$ as one to one and are such that $\sigma^m(n) \neq n$, for all positive integers $n$ and $m$, where $\sigma^m(n)$ denotes the $m$th iterate of the mapping $\sigma$ at $n$. If $\sigma$ is the translation mapping $n \to n+1$, a $\sigma$ mean is often called a Banach limit (see, [1, 3-5]). If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown (see, [12]) that

$$v^\sigma = \left\{ x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x \right\},$$

where,

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} T^j x_n,\ T^j x_n = x_{\sigma^j(n)},\ t_{-1,n}(x) = 0.$$

We define $v_\infty^\sigma$ the space of $\sigma-$bounded sequences (see [9]) in the following:

$$v_\infty^\sigma = \{x \in w : \sup_{m,n} |\phi_{m,n}(x)| < \infty\},$$

where,

$$\begin{aligned}\phi_{m,n}(x) &= t_{m,n}(x) - t_{m-1,n}(x)\\ &= \frac{1}{m(m+1)}\sum_{j=1}^{m} j(T^j x_n - T^{j-1}x_n). \end{aligned} \tag{1.1}$$

If $\sigma(n) = n+1$, then $v_\infty^\sigma$ is the set of almost bounded sequences $f_\infty$ (see, [3, 8, 10-14]). The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., (see, [1, 2, 7-9, 13]). In [2], Bulut and Cakar have defined the space

$l(p,s)$ and characterized the classes $(l(p,s), l_\infty)$ and $(l(p,s), c)$. The object of this paper is to characterize the classes of matrices $(l(p,s), v^\sigma)$ and $(l(p,s), v_\infty^\sigma)$, where the space $l(p,s)$ is defined as follows:

$$l(p,s) = \left\{ x : \sum_k k^{-s} |x_k|^{p_k} < \infty, \ s \geq 0 \right\},$$

which is paranormed by

$$g(x) = \left( \sum_k k^{-s} |x_k|^{p_k} \right)^{1/M},$$

where $M = \max(1, \sup_k p_k)$. Further, $l(p,s)$ is a metric space with the metric defined by $d(x,y) = g(x-y)$. The Köthe-Toeplitz dual is the set

$$M(p,s) = \left\{ a = (a_k) : \sum_k k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} |a_k|^{q_k} < \infty, \quad s \geq 0, \right.$$
$$\left. \text{for some integer N} > 1 \right\}.$$

## 2. Some matrix transformations

We note that, if $Ax$ is defined, then it follows from (1.1) that, for all integers $n, m \geq 0$

$$\phi_{m,n}(Ax) = \sum_k \alpha(n,k,m) x_k,$$

where

$$\alpha(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} j\{a(\sigma^j(n),k) - a(\sigma^{j-1}(n),k)\}.$$

**Theorem 2.1.** *Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every $k$, then $A \in (l(p,s), v_\infty^\sigma)$ if and only if there exists an integer $N > 1$ such that*

$$\sup_{m,n} \sum_k |\alpha(n,k,m)|^{q_k} k^{s(q_k-1)} N^{-q_k} < \infty. \tag{2.1}$$

*Proof.* **Necessity:** Let $A \in (l(p,s), v_\infty^\sigma)$ and that $x \in l(p,s)$. Put

$$q_n(x) = \sup_m |\phi_{mn}(Ax)|.$$

For $n > 0$, $q_n$ is continuous semi-norm on $l(p,s)$ and $(q_n)$ is pointwise bounded on $l(p,s)$. Suppose that (2.1) is not true. Then there exists $x \in l(p,s)$ with

$$\sup_n q_n(x) = \infty.$$

By the principle of condensation of singularities (see, [15]), the set

$$\left\{x \in l(p,s) : \sup_n q_n(x) = \infty\right\}$$

is of second category in $l(p,s)$ and hence nonempty, *i.e.*, there is $x \in l(p,s)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that $q_n$ is pointwise bounded on $l(p,s)$. Now, by Banach-Steinhauss theorem, there is constant $M$ such that

$$q_n(x) \leq Mg(x). \tag{2.2}$$

Applying (2.2) to the sequence $x = (x_k)$ defined as in [2, p. 42] by replacing $a_{nk}$ by $a(n,k,m)$, we then obtain the necessity of (2.1).

**Sufficiency:** Let (2.1) holds and $x \in l(p,s)$. Using the following inequality

$$|ab| \leq C(|a|^q C^{-q} + |b|^p)$$

for $C > 0$ and $a, b$ two complex numbers ($p > 1$ and $p^{-1} + q^{-1} = 1$) (see [7, 15]),we have

$$\begin{aligned} |\phi_{m,n}(Ax)| &= \left|\sum_k \alpha(n,k,m)x_k\right| \\ &\leq \sum_k |\alpha(n,k,m)x_k| \\ &\leq \sum_k N[|\alpha(n,k,m)|^{q_k} k^{s(q_k-1)} N^{-q_k} + |x_k|^{p_k} k^{-s}]. \end{aligned}$$

Taking the supremum over $m, n$ and using (2.2) we get $Ax \in v_\infty^\sigma$ for $x \in l(p,s)$, *i.e.*, $A \in (l(p,s), v_\infty^\sigma)$. This completes the proof of the Theorem 2.1. □

**Theorem 2.2.** *Let* $1 < p_k \leq \sup_k p_k = H < \infty$ *for every* $k$, *then* $A \in (l(p,s), v^\sigma)$ *if and only if there exists an integer* $N > 1$ *such that*

(i) $\sup_{m,n} \sum_k |t(n,k,m)|^{q_k} k^{s(q_k-1)} N^{-q_k} < \infty$,

(ii) $\lim_m t(n,k,m) = a_k$ *uniformly in* $n$, *for every* $k$.

*Proof.* **Necessity:** Let $A \in (l(p,s), v^\sigma)$ and that $x \in l(p,s)$. Write $q_n(x) = \sup_m |t_{mn}(Ax)|$. It is easy to see that for $n \geq 0$, $q_n$ is continuous semi-norm on $l(p,s)$ and $q_n$ is pointwise bounded on $l(p,s)$. Suppose that (i) is not true. Then there exists $x \in l(p,s)$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities [15], the set

$$\left\{ x \in l(p,s) : \sup_n q_n(x) = \infty \right\}$$

is of second category in $l(p,s)$ and hence non empty, *i.e.*, there is $x \in l(p,s)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that $(q_n)$ is pointwise bounded on $l(p,s)$. Now by Banach-Steinhauss theorem, there is constant $M$ such that

$$q_n(x) \leq M g(x). \tag{2.3}$$

Now define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} (sgn\ t(n,k,m)) k^{\frac{s}{p_k}} N^{\frac{-q_k}{p_k}}, & 1 \leq k \leq k_0, \\ 0, & k > k_0. \end{cases}$$

Then it is easy to see that $x \in l(p,s)$. Applying this sequence to (2.3), we get the condition (i). Since $e_k \in l(p,s)$, condition (ii) follows immediately by taking $x = e_k$.

**Sufficiency:** Let (i) and (ii) hold and $x \in l(p,s)$. For $j \geq 1$

$$\sum_{k=1}^{j} |t(n,k,m)|^{q_k} k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} \leq \sup_m \sum_k |t(n,k,m)|^{q_k} k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} < \infty$$

for every $n$. Therefore,

$$\begin{aligned} \sum_k |\alpha_k|^{q_k} k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} &= \lim_j \lim_m \sum_{k=1}^{j} |t(n,k,m)|^{q_k} k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} \\ &\leq \sup_m \sum_k |t(n,k,m)|^{q_k} k^{s(q_k-1)} N^{\frac{-q_k}{p_k}} \\ &< \infty. \end{aligned}$$

Consequently the series $\sum_k t(n,k,m)x_k$ and $\sum_k \alpha_k x_k$ converges for every $n, m$ and for every $x \in l(p,s)$. Now for $\varepsilon > 0$ and $x \in l(p,s)$. Choose $k_0 \in N$ such that

$$\sum_{k \geq k_0+1} |x_k|^{p_k} k^{-s} < 1.$$

By condition (ii), there exists $m_0$ such that

$$\left| \sum_{k=1}^{k_0} [t(n,k,m) - \alpha_k] \right| < \infty$$

for every $m > m_0$. By condition (i), it follows that

$$\left| \sum_{k \geq k_0+1} [t(n,k,m) - \alpha_k] \right|$$

is arbitrarily small. Therefore

$$\lim_m \sum_k t(n,k,m)x_k = \sum_k \alpha_k x_k \text{ uniformly in } n.$$

Hence $A \in (l(p,s), v^\sigma)$. □

**Corollary 2.3.** *If we choose the mapping $\sigma$ as the translation mapping, we get the results obtained by Mursaleen* [8].

## References

[1] S. Banach, Theorie des operations linearies, Warszawa, 1932.
[2] E. Bullet and O. Cakar, *The sequence space and related matrix transformations*, Comm. Fac. Sci. Univ. Aukara, **28** (1979), 33–44.
[3] A.H. Ganie and N.A. Sheikh, *A note on almost convergent sequences and some matrix transformations*, Int. J. Mod. Math. Sci., **4** (2012), 126–132.
[4] A.H. Ganie and N.A. Sheikh, *On the sequence space of nonabsolute type and matrix transformations* (accepted for publication in Jour. Egyptain Math. Society, ID-JOEMS-D-12-00105).
[5] T. Jalal and A.H. Ganie, *Almost convergence and some matrix transformations*, Int. J. Math (Shakhar new series), **1(1)** (2009), 133–138.
[6] G.G. Lorentz, *A contribution to the theory of divergent sequence*, Acta Math., **80** (1948), 167–190.
[7] I.J. Maddox, *Continuous and Kothe-Toeplitz dual of certain sequence space*, Proc. Camb. Phil. Soc., **65** (1969), 431–435.
[8] Mursaleen, *Infinite matrices and almost convergent sequences*, SEA Bul. Math., **19(1)** (1995), 45–48.
[9] M. Mursaleen, *Some matrix transformtions on sequence spaces of invariant means*, Hacet. J. Math. Stat., **38** (2009), 259–264.
[10] S. Nanda, *On some sequence spaces*, Math. student, **48(4)** (1980), 348–452.
[11] G.M. Petersen. Regular matrix transformations. McGraw-Hill Publishing Co. Ltd., London-New York-Toronto, (1966).
[12] P. Schaefer, *Infinite matricies and invariant means*, Proc. Amer. Math. Soc., **36** (1972), 104–110.
[13] N A. Sheikh, and A.H. Ganie, *Matrix transformations into new sequence spaces related to invariant means*, Chamchuri J. Math., **4** (2012), 71–77.
[14] N.A. Sheikh and A.H. Ganie, *On the $\lambda$-convergent sequence and almost convergence*, Thai J. of Math. (2013), in press.
[15] K. Yosida, Functional Analysis, Springer-Verlag, Berlin Heidelberg, New York, (1996).