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ANALYTIC SOLUTION OF HIGH ORDER FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. The existence of solution of the fractional order differential equations is very important mathematical field. Thus, in this work, we discuss, under some hypothesis, the existence of a positive solution for the nonlinear fourth order fractional boundary value problem which includes the p-Laplacian transform. The proposed method in the article is based on the fixed point theorem. More precisely, Krasnosilsky's theorem on a fixed point and some properties of the Green's function were used to study the existence of a solution for fourth order fractional boundary value problem. The main theoretical result of the paper is explained by example.

1. Introduction

Analytic solutions of high-order fractional boundary value problems constitute a fascinating field of study that combines elements from fractional calculus and boundary value problems. These problems involve differential equations of high order with fractional derivatives, where the order of the derivative is a non-integer value [20]. Fractional calculus provides a powerful tool to describe phenomena with long-range memory and non-local interactions, making it particularly useful in various scientific and engineering applications.

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Boundary value problems, on the other hand, are mathematical models that involve finding solutions to differential equations subject to specified conditions at the boundaries of the domain. They arise in numerous areas, including physics, biology, finance, and engineering, and have been extensively studied using classical calculus techniques.

The combination of high-order differential equations and fractional derivatives in boundary value problems introduces additional challenges and complexities to the solution process. Analytic solutions, which express the solution explicitly in terms of known functions, play a crucial role in gaining insights into the behavior of the system, understanding the underlying dynamics, and developing efficient computational methods.

The study of analytic solutions of high-order fractional boundary value problems encompasses various aspects, such as existence and uniqueness of solutions, qualitative properties of solutions, stability analysis, and numerical approximation techniques. Researchers in this field employ a range of mathematical tools, including Laplace transforms, Fourier transforms, Mellin transforms, series solutions, and special functions, among others, to tackle these challenging problems ([2]-[4]).

The development of analytic solutions for high-order fractional boundary value problems has numerous applications in diverse fields. For instance, in physics, they can be used to model phenomena involving anomalous diffusion, viscoelastic materials, and fractional quantum mechanics. In engineering, these solutions find utility in describing the behavior of complex systems, such as electrical circuits, control systems, and heat transfer processes [18]. Moreover, they can provide valuable insights into biological processes, such as population dynamics, epidemic spread, and tumor growth.

The aim of this article is to study the following nonlinear fourth order fractional boundary value problem. Let $\nu, \gamma \in (1,2], a, b \in (0,1)$ and $p > 1$,

$$\begin{cases} D^{\gamma}(\varphi_p(D^{\nu}w(t))) = g(t)f(w(t)), & 0 < t < 1, \\ w(0) = D^{\nu}w(0) = 0, \\ w(1) = \lambda w(a), \\ D^{\nu}w(1) = \mu D^{\nu}w(b), \end{cases} \tag{1.1}$$

where $\lambda,\ \mu \geq 0,\ 0 < a, b < 1, D^{\nu}(resp.\ D^{\gamma})$ is the standard Riemann-Liouville fractional differential operator of order ν (resp. of order γ), $\varphi_p(x) = |x|^p x$ is the p-Laplacian operator, h is a measurable function on [0,1], and the nonlinear term f is a continuous function.

In recent years, fractional calculus has gained substantial popularity and importance due to its applications as a novel modeling technique in a variety of engineering and scientific fields, such as viscoelasticity [10], thermoelasticity ([1], [14]), system control ([7], [8], [13], [19]), hydrology [9] and fractional dynamics ([5], [16], [17], [21]). Fractional differential equations are the best tool to explain the fractional models. Therefore, the theory of fractional calculus is used in the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes is a key mechanism in the investigation and finding patterns of processes in physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology. Consequently, the exploring of fractional differential equations is gaining more importance and attention.

Over the last decade, there has been significant development in ordinary and partial differential equations involving Riemann Liouville fractional derivatives. This is well illustrated in the monograph [12]. Bai [6] considered the following three-point boundary value problem of the fractional-order differential equation

$$\begin{cases} D^{\nu} w(t) + f(t, w(t)) = 0, \quad 0 < t < 1, \\ w(0) = 0, \\ w(1) = \beta w(\mu), \end{cases} \tag{1.2}$$

where $1 < \nu \leq 2$, $0 < \mu < 1$ are such that $0 < \beta\mu^{\nu-1} < 1$. By using the contraction map principle and fixed-point index theory, the author investigated the existence and uniqueness of positive solutions for the problem (1.2). Xu, Jiang et al. [23] deduced some new properties of Green's function of (1.2). They obtained the existence, uniqueness, and multiplicity of positive solutions to singular problems, depending on some fixed point theorems.

Nowadays, the number of scientific papers devoted to the study of boundary value problems of nonlinear fractional differential equations progressively increases. Many authors pay attention to the existence results for boundary value problems of nonlinear fractional differential equations using some fixed-point theorems, such as the Krasnoselskii fixed-point theorem [11], the Schauder fixed point theorem, and the Leggett-Williams fixed point theorem [22]. The investigating nonlinear fractional differential equations involving the p-Laplacian operator [15] is not fully researched and needs new creative approaches and ideas.

This article is focused on the conversion of equation (1.1) to an equivalent integral form. Next, we get the main conclusion asserting the existence of nontrivial non-negative solutions of the problem (1.1), by using some properties of the associated Green function and the Guo-Krasnoselskii fixed point theorem. Finally, an example is given to demonstrate the effectiveness of the obtained result.

2. Definitions and preliminaries of fractional calculus

This section presents necessary definitions and theorems that are necessary to obtain the main result of the article.

Definition 2.1. ([16]) Let n be a positive integer and $n-1<\nu\leq n$.

(i) The Riemann-Liouville integral of order ν for the function f is defined by:

$$(I_x^{\nu}f)(x)=\frac{1}{\Gamma(\nu)}\int_0^x\frac{f(\xi)}{(x-\xi)^{1-\nu}}d\xi,\quad x>0,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

(ii) The Riemann-Liouville's derivatives of order ν for the function f is denoted by $D^{\nu}f$, and is defined by

$$D_x^{\nu}f(x)=\frac{1}{\Gamma(n-\nu)}(\frac{d}{dx})^n\int_0^x\frac{f(\xi)}{(x-\xi)^{1-n+\nu}}d\xi,\quad x>0.$$

Theorem 2.2. ([8]) *Let n be a positive integer and $n-1<\nu\leq n$. If u and D^{ν} are continuous and integrable on $(0,1)$, then*

$$I_{0+}^{\nu}D_{0+}^{\nu}w(t)=w(t)+d_0+d_1t+d_2t^2+...+d_{n-1}t^{n-1}$$

for some $d_i\in\mathbb{R},\ i=0,1,2,\ldots,n-1$.

Theorem 2.3. ([6]) *Let $h\in C[0,1]$ and assume that $1-\lambda a^{\nu-1}>0$. Then the following problem*

$$\begin{cases}D^{\nu}w(t)+g(t)=0,\ 0<t<1\\ w(0)=0,\qquad w(1)=\lambda w(a)\end{cases}$$

has a unique solution which is given by

$$w(t)=\int_0^1F(t,s)g(s)ds,$$

where

$$F(t,s)=\begin{cases}\frac{t^{\nu-1}(1-s)^{\nu-1}}{\Gamma(\nu)(1-\lambda a^{\nu-1})}, & \text{if } 0\leq max(t,a)\leq s\leq 1,\\ \frac{t^{\nu-1}(1-s)^{\nu-1}-(1-\lambda a^{\nu-1})(t-s)^{\nu-1}}{\Gamma(\nu)(1-\lambda a^{\nu-1})}, & \text{if } 0\leq a\leq s\leq t\leq 1,\\ \frac{t^{\nu-1}(1-s)^{\nu-1}-\lambda t^{\nu-1}(a-s)^{\nu-1}}{\Gamma(\nu)(1-\lambda a^{\nu-1})}, & \text{if } 0\leq t\leq s\leq a\leq 1,\\ \frac{t^{\nu-1}(1-s)^{\nu-1}-\lambda t^{\nu-1}(a-s)^{\nu-1}}{\Gamma(\nu)(1-\lambda a^{\nu-1})}-\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}, & \text{if } 0\leq s\leq min(t,a)\leq 1.\end{cases}\tag{2.1}$$

Theorem 2.4. *Let $h \in C[0,1]$ and assume that $1-\mu^{p-1}b^{\beta-1}>0$. Then the following problem*

$$\begin{cases} D^{\gamma}(\varphi_p(D^{\nu}w(t))) = g(t), & 0<t<1 \\ w(0)=D^{\nu}w(0)=0,\ w(1)=\lambda w(a), & D^{\nu}w(1)=\mu D^{\nu}w(b) \end{cases} \tag{2.2}$$

has a unique solution which is given by:

$$w(t)=\int_0^1 F(t,s)\varphi_q(\int_0^1 G(s,v)g(v)ds)dv,$$

where

$$G(t,s)=\begin{cases} \frac{t^{\gamma-1}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}, & \text{if } 0\le max(t,b)\le s\le 1, \\ \frac{t^{\gamma-1}(1-s)^{\gamma-1}-(1-\mu^{p-1}b^{\nu-1})(t-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}, & \text{if } 0\le b\le s\le t\le 1, \\ \frac{t^{\gamma-1}(1-s)^{\gamma-1}-\mu t^{\gamma-1}(b-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}, & \text{if } 0\le t\le s\le b\le 1, \\ \frac{t^{\gamma-1}(1-s)^{\gamma-1}-\mu t^{\gamma-1}(b-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}-\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}, & \text{if } 0\le s\le min(t,b)\le 1. \end{cases} \tag{2.3}$$

Proof. Let $t\in(0,1)$, and assume that $D^{\gamma}(\varphi_p(D^{\nu}w(t)))=g(t)$. Then from Theorem 2.2, we have

$$\varphi_p(D^{\nu}w(t)) = I^{\gamma}g(t)+d_1t^{\gamma-1}+d_2t^{\gamma-2},\quad d_1,d_2\in\mathbb{R}. \tag{2.4}$$

$D^{\nu}w(0)=0$, implies that $d_2=0$. From the boundary condition $D^{\nu}w(1)=\mu D^{\nu}w(b)$, we obtain

$$d_1=-\int_0^1\frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}g(s)ds+\int_0^b\frac{\mu^{p-1}(b-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}g(s)ds.$$

Substituting d_1 *and* d_2 in the equation (2.4), we find

$$\begin{aligned}\varphi_p(D^{\nu}w(t)) &= \int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}g(s)ds-\int_0^1\frac{t^{\gamma-1}(1-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}g(s)ds \\ &\quad+\int_0^b\frac{\mu^{p-1}t^{\gamma-2}(b-s)^{\gamma-1}}{\Gamma(\gamma)(1-\mu^{p-1}b^{\gamma-1})}g(s)ds \\ &=-\int_0^1 G(t,s)g(s)ds,\end{aligned}$$

where $G(t,s)$ is given by equation (2.3). So, boundary value problem (2.2) is equivalent to

$$\begin{cases} D^{\nu}w(t)+\varphi(\int_0^1 G(t,s)g(s)ds)=0, & 0<t<1, \\ w(0)=0, \\ w(1)=\lambda w(a). \end{cases}$$

Consequently, from Lemma 2.3, the unique solution of problem (2.2) is given by

$$w(t) = \int_0^1 F(t,s)\varphi_q\left(\int_0^1 G(t,s)g(s)ds\right)ds.$$

This completes the proof. □

The following theorems represent important properties of the Green functions G and F.

Theorem 2.5. ([22]) *The functions G and F defined respectively by equations* (2.1) *and* (2.3) *are nonnegative continuous on* $[0,1]\times[0,1]$*, moreover, for all* $t,s\in[0,1]$*, we have*

$$G(t,s) \le G(s,s) \tag{2.5}$$

and

$$F(t,s) \le F(s,s). \tag{2.6}$$

Theorem 2.6. *There exist positive continuous functions η_1 and η_2 such that for all $s\in[0,1]$, we have*

$$\min_{a\le t\le 1} F(t,s) \ge \eta_1(s)F(s,s), \tag{2.7}$$

$$\min_{b\le t\le 1} G(t,s) \ge \eta_2(s)G(s,s). \tag{2.8}$$

Proof. For simplification, let us use the following notation

$$F(t,s) = \begin{cases} f_1(t,s), & if \quad 0\le s\le min(t,a)\le 1,\\ f_2(t,s), & if \quad 0\le a\le s\le t\le 1,\\ f_3(t,s), & if \quad 0\le t\le s\le a\le 1,\\ f_4(t,s), & if \quad 0\le max(t,a)\le s\le 1.\end{cases}$$

So, if $a\le t\le 1$, then

$$F(t,s) = \begin{cases} f_1(t,s), & if \quad 0\le s\le a\le t\le 1,\\ f_2(t,s), & if \quad 0\le a\le s\le t\le 1,\\ f_4(t,s), & if \quad 0\le a\le t\le s\le 1\end{cases}$$

and

$$\begin{aligned}\min_{a\le t\le 1} F(t,s) &= \begin{cases} \min_{a\le t\le 1} f_1(t,s), & if\ 0\le s\le a,\\ \min_{a\le t\le 1} f_2(t,s), & if\ a\le s\le 1,\end{cases}\\ &= \begin{cases} f_1(1,s), & if\ 0\le s\le a,\\ \min_{a\le t\le 1}(f_2(1,s), f_4(a,s)), & if\ a\le s\le 1.\end{cases}\end{aligned}$$

Let

$$\eta_1(s) = \begin{cases} \frac{f_1(1,s)}{F(s,s)}, & if\ 0 \leq s \leq a, \\ \min_{a \leq t \leq 1}(f_2(1,s), f_4(a,s), & if\ a \leq s \leq 1. \end{cases}$$

Then, it is easy to see that for all $s \in [0,1]$ the following inequality is obtained

$$\min_{a \leq t \leq 1} F(t,s) \geq \eta_1(s) F(s,s). \tag{2.9}$$

The inequality (2.9) can be proved similarly. □

The following theorem helps us to proof of the main results of this work.

Theorem 2.7. ([11]) *Let $(E, ||.||)$ be a Banach space, and $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \Omega_1 \subset \Omega_2$, and let*

$$T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P,$$

be a completely continuous operator such that either

(i) $||Tw|| \leq ||w||$ *for* $w \in P \cap \partial\Omega_1$ *and* $||Tw|| \geq ||w||$ *for* $w \in P \cap \partial\Omega_2$ *or*

(ii) $||Tw|| \geq ||w||$ *for* $w \in P \cap \partial\Omega_1$ *and* $||Tw|| \leq ||w||$ *for* $w \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Existent of the non-negative solution

Now, let $\rho = \max(a, b)$. Consider the following hypotheses:

(H_1) The function $h : [0,1] \longrightarrow [0, \infty)$ is a nontrivial Lebesgue integrable function.

(H_2) The function f is nonnegative continuous such that there exist $r_2 > r_1 > 0$ satisfying

(i) $f(x) \geq \theta_1 \varphi_P(r_1)$ for all $x \in [0, r_1]$,

(ii) $f(x) \leq \theta_2 \varphi_P(r_2)$ for all $x \in [0, r_2]$,

where

$$\theta_1 = \left(\int_\rho^1 \eta_1(s) F(s,s) ds \right)^{1-p} \left(\int_\rho^1 \eta_2(\tau) G(\tau,\tau) g(\tau) d\tau \right)^{-1}$$

and

$$\theta_2 = \left(\int_0^1 F(s,s) ds \right)^{1-p} \left(\int_0^1 G(\tau,\tau) g(\tau) d\tau \right)^{-1}.$$

Taking into account the above theoretical material, the following theorem is the main result of this paper.

Theorem 3.1. *Assume that hypothesis* $(H_1) - (H_2)$ *are satisfied. If*

$$1 - \lambda a^{\nu-1} > 0 \quad and \quad 1 - \mu^{p-1} b^{\gamma-1} > 0,$$

then the fractional boundary value problem (1.1) *has a nontrivial nonnegative solution* u *such that* $r_1 \leq ||w|| \leq r_2$.

Proof. Let $E = C([0,1], \mathbb{R})$ be a set of continuous real functions on [0,1], in which the following norm is valid

$$||w|| = \max_{0 \leq t \leq 1} |w(t)|.$$

Then $(E, ||w||)$ is a real Banach space.

Let $P = \{u \in E, \ w(t) \geq 0 \text{ for all } \in [0,1]\}$. Then P is a cone in E. Additionally, for $\eta > 0$, let $B_\rho := \{x \in E : ||x|| < \eta\}$. Let's define the operator $T : P \longrightarrow E$ by

$$Tw(t) = \int_0^1 F(t,s) \left(\int_0^1 G(t,\tau) g(\tau) f(w(\tau)) d\tau \right)^{\frac{1}{p-1}} ds.$$

From Lemma 2.4, u is a nonnegative solution for problem (1.1), if and only if it is a fixed point of the operator T. Moreover the functions f, h, K and G are nonnegative. Consequently, the operator T maps P into itself.

On the other hand, by using the Arzela-Ascoli theorem, we can prove that $T : P \longrightarrow P$ is completely continuous. Thereafter, let $\rho = \max(a,b)$ and $u \in P \cap \partial B_{r_1}$. Then from Lemma 2.6 and hypothesis (H_2), we get

$$\begin{aligned} Tw(t) &= \int_0^1 F(t,s) \left(\int_0^1 G(t,\tau) g(\tau) f(w(\tau)) d\tau \right)^{\frac{1}{p-1}} ds \\ &\geq \int_\rho^1 \eta_1(s) F(s,s) \left(\int_\rho^1 \eta_2(\tau) G(\tau,\tau) g(\tau) f(w(\tau)) d\tau \right)^{\frac{1}{p-1}} ds \\ &\geq r_1 \theta_1^{\frac{1}{p-1}} \int_\rho^1 \eta_1 F(t,s) \left(\int_\rho^1 \eta_2 G(t,\tau) g(\tau) f(w(\tau)) d\tau \right)^{\frac{1}{p-1}} ds \\ &= ||w||. \end{aligned} \tag{3.1}$$

If $u \in B_{r_2}$, then from Lemma 2.6 and hypothesis (H_1), we get

$$\begin{aligned} Tw(t) &= \int_0^1 F(t,s)\left(\int_0^1 G(t,\tau)g(\tau)f(w(\tau))d\tau\right)^{\frac{1}{p-1}} ds \\ &\leq \int_0^1 F(s,s)\left(\int_0^1 G(\tau,\tau)g(\tau)f(w(\tau))d\tau\right)^{\frac{1}{p-1}} ds \\ &\leq r_2\theta_2^{\frac{1}{p-1}} \int_0^1 F(s,s)\left(\int_0^1 G(\tau,\tau)g(\tau)f(w(\tau))d\tau\right)^{\frac{1}{p-1}} ds \\ &= ||w||. \end{aligned} \tag{3.2}$$

Therefore, Lemma 2.7 implies that the operator T has a fixed point u in $P \cap (\overline{B_{r_2}} \setminus B_{r_1})$, that is, u satisfies $r_1 \leq ||w|| \leq r_2$. This completes the proof of Main Theorem of this section. Eventually, we give an example to illustrate the validity of the main result. □

Example 3.2. In this example, problem (2.1) is considered in the special case. Precisely, we consider the following fractional boundary value problem

$$\begin{cases} D^{1\frac{1}{2}}(\varphi_{\frac{9}{5}}(D^{\frac{4}{3}}w(t))) = \sqrt{t}\ln(w(t)+15), \quad 0<t<1, \\ w(0) = D^{\frac{4}{3}}w(0) = 0, \\ w(1) = w(\frac{1}{8}), \\ D^{\frac{4}{3}}w(1) = D^{\frac{4}{3}}w(\frac{1}{8}). \end{cases} \tag{3.3}$$

Firstly, we have

$$1-\lambda a^{\nu-1} = 1-(\frac{1}{3^{\frac{1}{3}}}) = \frac{1}{2} > 0$$

and

$$1-\mu^{p-1}b^{\gamma-1} = 1-(\frac{1}{2\sqrt{2}}) > 0.$$

On the other hand, $g(t) = t^{1/2}$, which is nontrivial nonnegative Lebesgue integrable function on $[0,1]$. So, hypothesis (H_1) is satisfied. It's easy to show that

$$\eta_1 = \begin{cases} \dfrac{\frac{(1-s)^{\frac{1}{3}}}{2} - (\frac{1}{8}-s)^{\frac{1}{3}}}{s^{\frac{1}{3}}(1-s)^{\frac{1}{3}} - s^{\frac{1}{3}}(\frac{1}{8}-s)^{\frac{1}{3}}}, & if\ 0 \leq s \leq \frac{1}{8}, \\ \dfrac{1}{2s^{\frac{1}{3}}}, & if\ \frac{1}{8} \leq s \leq 1 \end{cases}$$

and

$$\eta_2 = \begin{cases} \dfrac{\frac{\sqrt{1-s}}{2\sqrt{2}} - \sqrt{\frac{1}{8}-s}}{\sqrt{s}\sqrt{1-s} - \sqrt{s}\sqrt{\frac{1}{8}-s}}, & if\ 0 < s \leq \frac{1}{8}, \\ \dfrac{1}{2\sqrt{2}\sqrt{s}} & if\ \frac{1}{8} \leq s \leq 1. \end{cases}$$

With the help of simple calculations we obtain $\theta_1 \approx 5.8961$ and $\theta_2 \approx 0.1411$.

Choose $r_1 = 1/3$ and $r_2 = 80$. Then the following inequalities are valid

$$f(x) \geq \theta_1 \varphi_p(r_1) \ for \ \ all \ x \in [0, r_1]$$

and

$$f(x) \leq \theta_2 \varphi_p(r_2) \ for \ \ all \ x \in [0, r_2].$$

Hence, hypothesis (H_2) is satisfied. Finally, Theorem 3.1 implies that problem (3.3) admits a nontrivial nonnegative solution.

Example 3.3. Now let us consider the problem (2.1) in the following special case. We consider the following fractional boundary value problem

$$\begin{cases} D^{1\frac{1}{3}}(\varphi_{\frac{8}{5}}(D^{\frac{5}{3}}w(t))) = sin(w(t)+11), \qquad 0 < t < 1, \\ w(0) = D^{\frac{5}{3}}w(0) = 0, \\ w(1) = w(\frac{1}{7}), \\ D^{\frac{5}{3}}w(1) = D^{\frac{5}{3}}w(\frac{1}{7}). \end{cases} \tag{3.4}$$

Firstly, we have

$$1 - \lambda a^{\nu-1} = 1 - (\frac{1}{3^{\frac{2}{3}}}) = 0.999 > 0$$

and

$$1 - \mu^{p-1} b^{\gamma-1} > 0.$$

On the other hand, $g(t) = 1$, which is nontrivial nonnegative Lebesgue integrable function on $[0, 1]$. So, hypothesis (H_1) is satisfied. It's easy to show that

$$\eta_1 = \begin{cases} \dfrac{\frac{(1-s)^{\frac{2}{3}}}{2} - (\frac{1}{7}-s)^{\frac{2}{3}}}{s^{\frac{2}{3}}(1-s)^{\frac{2}{3}} - s^{\frac{2}{3}}(\frac{1}{7}-s)^{\frac{2}{3}}}, & if \ 0 \leq s \leq \frac{1}{7}, \\ \dfrac{1}{2s^{\frac{2}{3}}}, & if \ \frac{1}{7} \leq s \leq 1 \end{cases}$$

and

$$\eta_2 = \begin{cases} \dfrac{\frac{\sqrt{1-s}}{2\sqrt{2}} - \sqrt{\frac{1}{7}-s}}{\sqrt{s}\sqrt{1-s} - \sqrt{s}\sqrt{\frac{1}{7}-s}}, & if \ 0 < s \leq \frac{1}{7}, \\ \dfrac{1}{2\sqrt{2}\sqrt{s}} & if \ \frac{1}{7} \leq s \leq 1. \end{cases}$$

With the help of simple calculations we obtain $\theta_1 \approx 4.5267$ and $\theta_2 \approx 0.2236$.

Choose $r_1 = 2/3$ and $r_2 = 90$. Then the following inequalities are valid

$$f(x) \geq \theta_1 \varphi_p(r_1) \ for \ \ all \ x \in [0, r_1]$$

and

$$f(x) \leq \theta_2 \varphi_p(r_2) \ for \ \ all \ x \in [0, r_2].$$

Hence, hypothesis (H_2) is satisfied. Finally, Theorem 3.1 implies that problem (3.3) admits a nontrivial nonnegative solution.

4. Conclusion

Our effort in this paper contributes to a growing literature on studying the existence of the solutions of a fractional order differential equations. Under specific hypothesis, we studied the existence of a positive solution for a nonlinear fourth order fractional boundary value problem with p-Laplacian transform. Our method in this article is based on the fixed point theorem and the properties of the Green's function. These theorems provide powerful tools to demonstrate the existence of fixed points of certain mappings or operators, which can be directly linked to the existence of solutions for boundary value problem.

We proposed two examples to show the theoretical result of our work. By utilizing fixed point theorems, we can establish the existence of solutions and lay the groundwork for further analysis and numerical computations of boundary value problem.

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