Nonlinear Functional Analysis and Applications
Vol. 28, No. 3 (2023), pp. 613-622
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.03.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

FINITE ELEMENT METHOD FOR SOLVING BOUNDARY CONTROL PROBLEM GOVERNED BY ELLIPTIC VARIATIONAL INEQUALITIES WITH AN INFINITE NUMBER OF VARIABLES

Ghada Ebrahim Mostafa

Department of Mathematics, Faculty of Science,
Al-Azhar University, Nasr City, Cairo, Egypt
e-mail: Ghadaali@azhar.edu.eg

Abstract. In this paper, finite element method is applied to solve boundary control problem governed by elliptic variational inequality with an infinite number of variables. First, we introduce some important features of the finite element method, boundary control problem governed by elliptic variational inequalities with an infinite number of variables in the case of the control and observation are on the boundary is introduced. We prove the existence of the solution by using the augmented Lagrangian multipliers method. A triangular type finite element method is used.

1. Introduction

Finite element and boundary element methods are major numerical tools for different types of boundary value problems and for studying partial differential equations modeling real-world problems. Functional analysis plays a vital role in reducing the problem in a form amenable to compute analysis. It is a basic tool for error estimation between solutions of continuous and discrete problems and convergence of solutions of the latter to the original problem. The finite element method is a general technique to build finite-dimensional

[0]Received July 13, 2022. Revised April 23, 2023. Accepted June 1, 2023.
[0]2020 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Optimal control, elliptic variational inequalities, augmented Lagrangian multipliers, operator with an infinite number of variables, finite element method (FEM).
[0]Corresponding author: Gh. E. Mostafa(Ghadaali@azhar.edu.eg).

spaces of a Hilbert space of some classes of functions, such as Sobolev spaces of different orders, and their subspaces, in order to apply the Ritz and Galerkin methods to a variational problem. The technique is based on ideas like (i) Division of the domain in which the problem is posed in a set of simple subdomains, called elements-often these elements are triangles, quadrilaterals and tetrahedra. (ii) A space H of functions defined on Ω is then approximated by appropriate functions defined on each subdomain with suitable matching conditions at interfaces ([13]).

Some important features of the finite element method are:

(1) arbitrary geometries,

(2) unstructured meshes,

(3) robustness,

(4) sound mathematical foundation.

Arbitrary geometries means that, in principle, the method can be applied to domains of arbitrary shapes with different boundary conditions. By unstructured meshes, we mean that, in principle, one can place finite elements anywhere from the complex cross-sections of biological tissues to the exterior of aircraft, to internal flows in turbo machinery, without the use of a globally fixed coordinate frame. Robustness means that the scheme developed for assemblage after local approximation over individual elements is stable in appropriate norms and insensitive to singularities or distortions of the meshes (This property is not available in classical different methods) ([13]).

The finite element method has been applied in every conceivable area of engineering, such as structural analysis, semiconductor devices, meteorology, flow through porous media, heat conduction, wave propagation, electromagnetism, environmental studies, biomechanics ([13]).

A set of inequalities defining a control of a system governed by self-adjoint elliptic operators with an infinite number of variables are presented in Gali et al. ([8]). In Gali et al. ([8, 9]) the optimal control problem for system described by elliptic operators with an infinite number of variables have been discussed. El-Zahaby ([5]) presented the necessary conditions for control problems governed by elliptic variational inequalities with an infinite number of variables. Necessary conditions for optimality in distributed control problem governed by parabolic variational inequalities with an infinite number of variables are established by El-Zahaby et al. ([7]). Boundary control problem with nonlinear state equation with an infinite number of variables is established by El-Zahaby and Mostafa ([6]).

In this paper we shall use the theory of Barbu ([2, 3]) to introduce boundary control problem governed by elliptic equation with nonlinear boundary value condition in the case of infinite number of variables and will apply finite element method.

This paper is organized as follows: In Section 2, some functional spaces with an infinite number of variables will be introduced. In Section 3, the main result is introduced where the boundary control problems governed by elliptic variational inequalities with an infinite number of variables, the finite element discretization of the state equation and optimal control problem, the existence of the solution is proved by using the augmented Lagrangian multipliers method. A triangular type finite element method is used.

2. Preliminaries

We consider the space $L_2(R^\infty, dp(x))$ which is constructed over $R^\infty = R^1 \times R^1 \times \cdots$, with the measure $dp(x) = p_1(x_1) \otimes p_2(x_2) \cdots$ $(R^\infty \ni x = (x_k)_{k=1}^\infty$, $x_k \in R^1)$ where $(p_k(t))_{k=1}^\infty$ is a fixed weight such that $0 < p_k(t) \in C^\infty(R^1)$, $\int\limits_{R^1} p_k(t)dt = 1$, that is, the space of functions which are measurable and such that

$$\|u\|_{L_2(R^\infty,dp(x))} = \left(\int\limits_{R^\infty} |u|^2 dp(x)\right)^{\frac{1}{2}} < \infty. \tag{2.1}$$

We shall often set $L_2(R^\infty, dp(x)) = L_2(R^\infty)$. It is a classical result that $L_2(R^\infty)$ is a Hilbert space for the scalar product

$$(u, v)_{L_2(R^\infty)} = \int\limits_{R^\infty} u(x)v(x)dp(x)$$

associated to the above norm (2.1) ([12]).

We introduce the scalar product

$$(u, v)_{W^l(R^\infty)} = \sum_{|\alpha|\le l} (D^\alpha u, D^\alpha v)_{L^2(R^\infty)}, \tag{2.2}$$

where D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\ldots},\ |\alpha| = \sum_{i=1}^\infty \alpha_i,$$

and the differentiation is in the sense of generalized functions, and after the completion, we obtain the Sobolev spaces $W^l(R^\infty)$; $l = 1, 2, \cdots$.

In short, Sobolev spaces $W^l(R^\infty)$ of order l on R are defined by

$$W^l(R^\infty) = \{\varphi : D^\alpha\varphi \in L_2(R^\infty),\ \forall\alpha, |\alpha| \leq l\}.$$

This space forms a Hilbert space denoted with the scalar product (2.2) ([4]). As in the l case of bounded region, the spaces $W^l(R^\infty)$ form a sequence of positive spaces. We can construct negative spaces $W^{-l}(R^\infty)$ with respect to the zero space $W^0(R^\infty) = L_2(R^\infty)$ and then we are equipped with the following ([8]):

$$W^l(R^\infty) \subseteq W^0(R^\infty) = L_2(R^\infty) \subseteq W^{-1}(R^\infty),$$

$$\|u\|_{W^l(R^\infty)} \geq \|u\|_{L_2(R^\infty)} \geq \|u\|_{W^{-l}(R^\infty)},$$

analogous to the above chain we have a chain of the form:

$$W_0^l(R^\infty) \subseteq L_2(R^\infty) \subseteq W_0^{-1}(R^\infty),$$

$$\|u\|_{W_0^l(R^\infty)} \geq \|u\|_{L_2(R^\infty)} \geq \|u\|_{W_0^{-l}(R^\infty)},$$

where

$$W_0^l(R^\infty) = \left\{\varphi : \varphi \in W^l(R^\infty), \left.\frac{\partial^k\varphi}{\partial n^k}\right|_\Gamma = 0,\ 0 \leq k \leq l-1\right\},$$

where $\frac{\partial^k\varphi}{\partial n^k}$ is the normal k-order derivative on Γ oriented to the exterior of R^∞ and $W_0^{-l}(R^\infty)$ is its dual, the norm on $W_0^l(R^\infty)$ is given by

$$\begin{aligned}\|u\|_{W_0^l(R^\infty, dp(x))} &= \left(\sum_{|\alpha|\leq l}(D^\alpha u, D^\alpha u)_{L^2(R^\infty)}\right)^{\frac{1}{2}} \\ &= \left(\sum_{|\alpha|\leq l}\|D^\alpha\|^2_{L^2(R^\infty)}\right)^{\frac{1}{2}}. \qquad (2.3)\end{aligned}$$

3. Main results

Definition 3.1. Let A be a second order self-adjoint elliptic partial deferential operator with an infinite number of variables that maps $W^1(R^\infty)$ onto $W^{-1}(R^\infty)$ and takes the form:

$$\begin{aligned}Ay(x) &= -\sum_{k=1}^{\infty}\frac{1}{\sqrt{p_k(x_k)}}\frac{\partial^2}{\partial x_k^2}\sqrt{p_k(x_k)}y(x) + q(x)y(x) \\ &= -\sum_{k=1}^{\infty}(D_k^2 y(x) + q(x)y(x), \qquad (3.1)\end{aligned}$$

where

$$(D_k y)(x) = \frac{1}{\sqrt{p_k(x_k)}} \frac{\partial}{\partial x_k} \sqrt{p_k(x_k)} y(x)$$

and $q(x)$ is a real valued function from $L_2(R^\infty, dp(x))$ such that

$$q(x) \geq v, \ 1 \geq v > 0.$$

We consider a chain of the form:

$$W_0^1(R^\infty) \subseteq L_2(R^\infty) \subseteq W_0^{-1}(R^\infty), \tag{3.2}$$

where $l = 1$.

Every continuous bilinear form $\pi(y, w)$ can be written in the form:

$$\pi(y, w) = (Ay, w), \quad y, w \in W_0^1(R^\infty), \tag{3.3}$$

where A is bounded operator.

In our consideration, we have an operator of the form (3.1) with and maps $y, w \in W_0^1(R^\infty)$ and maps $W_0^1(R^\infty)$ into $W_0^{-1}(R^\infty)$, then

$$\begin{aligned}
\pi(y, w) = & -\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k)}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k)} y(x), w)_{L^2(R^\infty)} \\
& + (q(x)y(x), w(x))_{L^2(R^\infty)} \\
= & \sum_{k=1}^{\infty} (\frac{1}{\sqrt{p_k(x_k)}} \frac{\partial}{\partial x_k} \sqrt{p_k(x_k)} y(x), \frac{1}{\sqrt{(p_k(x_k)}} \frac{\partial}{\partial x_k} \sqrt{p_k(x_k)} w(x))_{L^2(R^\infty)} \\
& + (q(x)y(x), w(x))_{L^2(R^\infty)} \\
= & \sum_{k=1}^{\infty} (D_k y(x), D_k w(x))_{L^2(R^\infty)} + (q(x)y(x), w(x))_{L^2(R^\infty)} \\
= & \sum_{k=1}^{\infty} \int_{R^\infty} (D_k(y(x)), (D_k(w(x)) dp(x) + \int_{R^\infty} q(x)y(x)w(x) dp(x).
\end{aligned}$$

To set our problem, we need to prove the following lemma ([8]), which enable us to formulate our problem.

Lemma 3.2. *The continuous bilinear form $\pi(y, w)$ in (3.3) is coercive, that means*

$$\pi(y, y) \geq v\|y\|^2, \quad v > 0, \ y \in W^1(R^\infty). \tag{3.4}$$

Proof. It is well known that the ellipticity of A is sufficient for the coerciveness of $\pi(y,w)$ on $W_0^1(R^\infty)$ (see [12]). In fact

$$\begin{aligned}
\pi(y,y) &= \sum_{k=1}^{\infty}\int_{R^\infty}(D_k(y(x))(D_k(y(x))dp(x)+\int_{R^\infty} q(x)y(x)y(x)dp(x)\\
&\geq \sum_{k=1}^{\infty}(D_k y(x), D_k y(x))_{L_2(R^\infty)} + v(y(x),y(x))_{L_2(R^\infty)}\\
&= \sum_{k=1}^{\infty}\|D_k y\|^2_{L_2(R^\infty)} + v\|y\|^2_{L_2(R^\infty)}\\
&= \sum_{k=1}^{\infty}\|D_k y\|^2_{L_2(R^\infty)} + v\|D_k y\|^2_{L_2(R^\infty)} - v\|D_k y\|^2_{L_2(R^\infty)}\\
&\quad + v\|y\|^2_{L_2(R^\infty)}\\
&= v||y\|^2_{W_0^1(R^\infty)} + (1-v)\|D_k y\|^2_{L_2(R^\infty)}.
\end{aligned}$$

Then $\pi(y,y)\geq v\|y\|^2_{W_0^1(R^\infty)}$, which gives the required. □

Let $U = L_2(R^\infty)$ be the space of controls. For the control u, the state $y(u)\in W^1(R^\infty)$ is given by the solution of

$$\begin{aligned}
&y + Ay = f && \text{in } R^\infty,\\
&\frac{\partial y}{\partial v_A} + \beta(y) \ni u && \text{in } \Gamma_1, \qquad (3.5)\\
&y = 0 && \text{in } \Gamma_2,
\end{aligned}$$

where, $\frac{\partial y}{\partial v_A}$ is the outward normal derivative corresponding to A, $f\in L_2(\Omega)$, Γ_2 is a smooth and open nonempty subset of Γ, Γ_1 is the interior of $\Gamma-\Gamma_2$ and β is a maximal monotone graph in $R\times R$ such that $0\in D(\beta)$.

The optimal control problem (P) can be set in the following form: Minimize the function

$$g_1(y) + \int_{\Gamma_1} g_0(z, y(z))dz + r(u), \qquad (3.6)$$

on all $y\in W^1(R^\infty)$, $z\in\Gamma_1$ and $u\in L_2(R^\infty)$ subject to the system (3.5), where, $g_1 : L_2(R^\infty)\to R^+$ and $r(u)$ satisfy the following conditions:

(i) The function $g_1(y)$ is Lipschitz on bounded subsets of $W^1(R^\infty)$ and there exists a real number C and $k_0\in L_2(R^\infty)$ such that

$$g_1(y)\geq (k_0,y)+C, \quad \forall y\in L_2(R^\infty),$$

that is, $g_1(y)$ is bounded from below by $(k_0,y)+C$.

(ii) The function r is convex lower semicontinuous and for some constants $c_1 > 0$, $c_2 \in R$

$$r(u) \geq c_1\|u\|_U + c_2, \quad \forall u \in U,$$

where, $g_0 : \Gamma_1 \times R \to R^+$ is measurable in z, differentiable in $W^1(\Omega)$ satisfies the conditions:

$$g_0(z, 0) = 0, \quad |\nabla g_0(z, y(z))| \leq c(1 + |y|) \quad \text{a.}e.\ z \in \Gamma_1, \quad \forall y \in R.$$

We define $g_2(y) = g_1(y) + \int\limits_{\Gamma_1} g_0(z, y(z))dz$.

The pervious conditions of g_1 and g_0 implies that the function $g_2(y)$ satisfies condition (i) and is Frechet differentiable in $W^1(R^\infty)$.

Theorem 3.3. *If the assumptions* (i), (ii) *and the coerciveness condition* (3.4) *satisfy, then the problem* (P) *has at least one optimal pair* (y^*, u^*).

Proof. Let $d = \inf\{g_1(y) + \int\limits_{\Gamma_1} g_0(z, y(z))dz + r(u),\ u \in U\}$. Then, by assumptions (i), (ii) we see that $-\infty < d < \infty$.

Now let $\{u_n\} \subset U = L_2(\Omega)$ be such that

$$d \leq g_1(y_n) + \int\limits_{\Gamma_1} g_0(z, y_n)dz + r(u_n),$$

where $y_n = y(u_n)$. Then, by assumption (ii) it follows that the sequence $\{u_n\}$ is weakly compact in $U = L_2(\Omega)$ and so by Barbu ([2]) we may infer that a subsequence $\{u_{n_i}\}$ with

$$u_{n_i} \to u^* \text{ weakly in } U = L_2(\Omega) \text{ and } y_{n_i} \to y^* \text{ strongly in } W^1(\Omega).$$

Since $r(u)$ is a weakly lower semicontinuous on $L_2(\Omega)$ and $g_1(y) + \int\limits_{\Gamma_1} g_0(z, y)dz$ is continuous on $W^1(\Omega)$, this yields

$$g_1(y^*) + \int\limits_{\Gamma_1} g_0(z, y^*) + r(u^*) = d,$$

that is, u^* is an optimal control of problem (P). □

We will approximate (3.6) with a finite element method introduced in ([1, 10]).

Assume that Ω is a polygonal domain. Consider a triangulation $\Im_\eta$ of Ω in the following sense: $\Im_\eta$ is a finite set of triangles T such that:

$$T \subset \bar{\Omega}\ \forall T \in \Im_\eta, \qquad U_{T \in \Im_\eta} T = \bar{\Omega},$$

$$T_1^0 \cap T_2^0 = \phi, \qquad \forall\ T_1, T_2 \in \Im_\eta \text{ and } T_1 \neq T_2,$$

where T_i^0 denotes the inner part of the corresponding triangle. Furthermore, for all $T_1, T_2 \in \Im_\eta$ and $T_1 \neq T_2$, exactly one of the following conditions must hold:

(i) $T_1 \cap T_2 = \phi$.
(ii) T_1 and T_2 have only one common vertex.
(iii) T_1 and T_2 have only a whole common edge.

η is the length of the largest edge of the triangles in the triangulation. Define $\wp_\zeta$ as a space of polynomials in T_1 and T_2 of degree less than or equal to ζ, and

$$\sum_\eta = \{\wp \in \bar{\Omega}, \wp \text{ is a vertex of } T \in \Im_\eta\}.$$

The space $W_0^1(R^\infty)$ is approximated by the family of subspaces $(V_\eta^\zeta)_\eta$ with $\zeta = 1$ or $\zeta = 2$, where

$$V_\eta^\zeta = \{v_\eta \in C^0(\bar{\Omega}),\ v_\eta|_{\partial\Omega} = 0 \quad \text{and} \quad v_\eta|_T \in \wp_\zeta,\ \forall T \in \Im_\eta\}.$$

It is obvious that the V_η^ζ are finite dimensional. Then the space $K = W_0^1(R^\infty)$ is approximated by

$$K_\eta^\zeta = \left\{v_\eta \in V_\eta^\zeta, v_\eta(\wp) \geq \psi(\wp),\ \forall \wp \in \sum_\eta^\zeta\right\}.$$

Notice that K_η^ζ are closed convex nonempty subsets of V_η^ζ.

The algorithm of the solution is given by the following steps:

Step 0: Choose an initial iterate μ_h and $h \geq 0$.
Step 1: Solve the linear problem

$$((1+\mu_h)Ay_h, A(v-y_h)) + \int_{R^\infty} |D_k v + v|\, dp(x) - \int_{R^\infty} |(D_k + I)y_h|\, dp(x)$$
$$- \int_{R^\infty} f \cdot (u - y_h) dp(x) = (\mu_h, A(v - y_h)).$$

Step 2: Update $\mu_h^{\alpha+1} = \max\{0, \mu_h + \alpha((D_k + I)(u_h^\alpha))\}$ on each cell.
Step 3: Replace α by $\alpha + 1$ and go back to Step 1.

So we can prove the following theorem:

Theorem 3.4. *Let (y^*, u^*) be an optimal pair of problem (P) and* (3.5)*, where β is locally Lipschitz monotonically increasing function. Then y_h is a solution of* (3.6).

Proof. From what have been introduced, the solution y is approximated by y_h which is the solution of

$$(A(y_h), u - y_h) + \int\limits_{R^\infty} |D_k u + u| dp(x) - \int\limits_{R^\infty} |D_k y_h + y_h| \geq \int\limits_{R^\infty} f \cdot (u - y_h) dp(x),$$

$\forall u \in K_\eta,\ y \in K_\eta,\ f \in W^{-1}(R^\infty)$.

By using the augmented Lagrangian multipliers method, we find a discrete solution of (3.6) as follows: The Lagrange functional is defined as follows:

$$\begin{aligned}\hat{L}(u,\mu) =& \frac{1}{2}(A(u),u) + \int\limits_{R^\infty} |D_k u + u|\, dp(x) - \int\limits_{R^\infty} f \cdot u dp(x) \\ &+ \int\limits_{R^\infty} \mu(|D_k u + u| - 1)\, dp(x).\end{aligned}$$

For $\varepsilon \geq 0$, an augmented Lagrangian L_r is defined by

$$L_r(u,\mu) = \hat{L}(u,\mu) + \frac{\varepsilon}{2} \int\limits_{R^\infty} \mu(|D_k u + u - 1|^2)\, dp(x).$$

For variational inequalities systems, augmented Lagrangian multipliers methods have been introduced by Glowinski and Marrocco (see [11]). Theorem 2.1 on p. 168 in ([10]) shows the existence of a solution of this optimization problem ([1]). So, we can write the following linear problem:

$$\begin{aligned}&((1+\mu_h)Ay_h, A(v - y_h)) + \int\limits_{R^\infty} |D_k v + v|\, dp(x) - \int\limits_{R^\infty} |D_k y_h + y_h| \\ &\int\limits_{R^\infty} f \cdot (u - y_h) dp(x) = (\mu_h, A(v - y_h)),\end{aligned}$$

which is obtained by the variational calculus. □

References

[1] M.A. Akinlar, *Application of a finite element method for variational inequalities*, J. Ineq. Appl., **44** (2013).

[2] V. Barbu, *Optimal Control of Variational Inequalities*, Research, Notes in Mathematics 100, Boston-London-Melbourne: Pitman, 1984.

[3] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Mathematics in Science and Engineering; Boston-San Diego-New York, **189**, 1993.

[4] Ju.M. Berezanskii, *Expansion in Eigenfunctions of Self-Adjoint Operators "Naukova Dumka" Kiev*, 1965. English Trans., Math. Monographs, Amer. Math. Soc., **17**, 1968.

[5] S.A. El-Zahaby, *Necessary Conditions for Control Problems Governed by Elliptic Variational Inequalities With an Infinite Number of Variables*, J. Advan. Mod. Anal., **44** (1994), 47–55.

[6] S.A. El-Zahaby and Gh.H. Mostafa, *Boundary Control Problem with Nonlinear State Equations with an Infinite Number of Variables*, AMSE Press, France, 2007.

[7] S.A. El-Zahaby and Gh.H. Mostafa, *Necessary Conditions for Distributed Control Problems Governed by Variational Inequalities With an Infinite Number of Variables*, The Mediterrenean J. Measurment and Control, **1**(2005), 191–197.

[8] I.M. Gali and H.A. El-Saify, *Optimal Control of System Governed by Self-Adjoint Elliptic Operator With an Infinite Number of Variables*, Proc. of the Second International Conference of Functional Differential Systems and Related Topics, II, Warsaw, Poland, (1981), 126–133.

[9] I.M. Gali and H.A. El-Saify, *Distributed Control of a System Governed by Dirichlet and Neumann Problems for a Self-Adjoint Elliptic Operator With an Infinite Number of Variables*, J. Optim. Theory Appl., **39**(2) (1983), 293–298.

[10] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems.* Springer, Berlin, 2008.

[11] R. Glowinski and A. Marrocco, *Numerical solution of two-dimensional magneto-static problems by augmented Lagrangian methods*, Comput. Methods Appl. Mech. Eng., **12** (1977), 33–46.

[12] J.L. Lions and E. Magnes, *Non-homogeneous Boundary Value Problems and Applications*, **I, II**, Springer-Verlag, New York, 1972.

[13] A.H. Siddiqi, *Functional Analysis and Applications*, Springer Nature Singapore Pte Ltd, 2018.