Nonlinear Functional Analysis and Applications
Vol. 28, No. 3 (2023), pp. 623-646
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.03.03
http://nfaa.kyungnam.ac.kr/journal-nfaa

APPROXIMATING COMMON FIXED POINT OF THREE MULTIVALUED MAPPINGS SATISFYING CONDITION (E) IN HYPERBOLIC SPACES

Austine Efut Ofem[1], Godwin Chidi Ugwunnadi[2], Ojen Kumar Narain[3] and Jong Kyu Kim[4]

[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
e-mail: ofemaustine@gmail.com; 222128007@stu.ukzn.ac.za

[2]Department of Mathematics, University of Eswatini, Private Bag 4, Kwaluseni, Eswatini; Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa
e-mail: ugwunnadi4u@yahoo.com; gcugwunnadi@uniswa.sz

[3]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
e-mail: naraino@ukzn.ac.za

[4]Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam, 5176, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this article, we introduce the hyperbolic space version of a faster iterative algorithm. The proposed iterative algorithm is used to approximate the common fixed point of three multi-valued almost contraction mappings and three multi-valued mappings satisfying condition (E) in hyperbolic spaces. The concepts weak w^2-stability involving three multi-valued almost contraction mappings are considered. Several strong and $\triangle$-convergence theorems of the suggested algorithm are proved in hyperbolic spaces. We provide an example to compare the performance of the proposed method with some well-known methods in the literature.

[0]Received July 19, 2022. Revised April 1, 2023. Accepted April 12, 2023.
[0]2020 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Weak w^2-stability, multivalued almost contraction mappings, multivalued mappings satisfying condition (E), strong and $\triangle$-convergence.
[0]Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

1. Introduction

The role play by the ambient spaces in fixed point theory are very paramount. Many problems in different discipline of science are naturally nonlinear. Thus, restructuring linear version of a given problem into its equivalent nonlinear version is very important. Moreover, the study of diverse problems in spaces without linear structure are significant in applied and pure sciences. Many attempts have been made to introduce a convex-like structure on a metric space. One of the spaces that posses this structure is the Hyperbolic space.

In this article, we intent to carry out our studies in the setting of hyperbolic space studied by Kohlenbach [20]. This notion of hyperbolic space is more restrictive than the notion of hyperbolic space considered in [13] and more general than the notion of hyperbolic space studied in [32]. Banach and CAT(0) spaces are well known to be special cases of hyperbolic spaces (see [2], [18]). Also, the class of hyperbolic spaces properly contains Hilbert ball endowed with hyperbolic metric [14], Hadamard manifolds, $\mathbb{R}$-trees and Cartesian product of Hilbert spaces.

Definition 1.1. A hyperbolic space $(\mathcal{Q}, d, \mathcal{K})$ in the sense of Kohlenbach [20] is a metric space $(\mathcal{Q}, d)$ together with a convexity mapping $\mathcal{K} : \mathcal{Q}^2 \times [0,1] \to \mathcal{Q}$ satisfying

(C_1) $d(\eta, \mathcal{K}(m, w, \xi)) \leq \xi d(\eta, m) + (1-\xi) d(\eta, w)$;
(C_2) $d(\mathcal{K}(m, w, \xi), \mathcal{K}(m, w, \upsilon)) \leq |\xi - \upsilon| d(m, w)$;
(C_3) $\mathcal{K}(m, w, \xi) = \mathcal{K}(w, m, (1-\xi))$;
(C_4) $d(\mathcal{K}(m, u, \xi), \mathcal{K}(w, v, \xi)) \leq (1-\xi) d(m, w) + \xi d(u, v)$,

for all $m, w, u, v \in \mathcal{Q}$ and $\xi, \upsilon \in [0,1]$. A nonempty subset $\mathcal{J}$ of a hyperbolic space $\mathcal{Q}$ is termed convex, if $\mathcal{K}(m, w, \xi) \in \mathcal{J}$ for all $m, w \in \mathcal{J}$ and $\xi \in [0,1]$.

Suppose $m, w \in \mathcal{Q}$ and $\xi \in [0,1]$, the notation $(1-\xi)m \oplus \xi w$ is used for $\mathcal{K}(m, w, \xi)$. The following also holds for the more general setting of convex metric space [15]: for any $m, w \in \mathcal{Q}$ and $\xi \in [0,1]$, $d(m, (1-\xi)m \oplus \xi w) = \xi d(m, w)$ and $d(w, (1-\xi)m \oplus \xi w) = (1-\xi) d(m, w)$. Consequently, $1m \oplus 0w = m$, $0m \oplus 1w = w$ and $(1-\xi)m \oplus \xi m = \xi m \oplus (1-\xi)m = m$.

The notion of multivalued contraction mappings and nonexpansive mappings using Hausdorff metric was initiated by Nadler [24] and Markin [23]. The theory of multivalued mappings has several applications in convex optimization, game theory, control theory, economics and differential equations.

Let $\mathcal{Q}$ be a metric space and $\mathcal{J}$ a nonempty subset of $\mathcal{Q}$. The subset $\mathcal{J}$ is called proximal if for all $m \in \mathcal{Q}$, there exists a member w in $\mathcal{J}$ such that

$$d(m, w) = \operatorname{dist}(m, \mathcal{J}) = \inf\{d(m, s) : s \in \mathcal{J}\}.$$

Let $\mathcal{P}(\mathcal{J})$ denote the collection of all nonempty proximal bounded and closed subsets of $\mathcal{J}$, and $\mathcal{BC}(\mathcal{J})$ the collection of all nonempty closed bounded subsets. The Hausdorff distance on $\mathcal{BC}(\mathcal{J})$ is defined by

$$\mathscr{H}(\mathscr{W},\mathscr{V}) = \max\left\{\sup_{m\in\mathscr{W}} d(m,\mathscr{V}), \sup_{w\in\mathscr{V}} d(w,\mathscr{W})\right\},\ \forall \mathscr{W},\mathscr{V} \in \mathcal{BC}(\mathcal{J}).$$

A point $m \in \mathcal{J}$ is called a fixed point of the multivalued mapping $\mathcal{T} : \mathcal{J} \to 2^{\mathcal{J}}$ if $m \in \mathcal{T}m$. Let $\mathcal{F}(\mathcal{T})$ denote the set of all fixed points of $\mathcal{T}$. A multivalued mapping $\mathcal{T} : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ is called nonexpansive if $\mathscr{H}(\mathcal{T}m, \mathcal{T}w) \leq \rho(m, w)$, for all $m, w \in \mathcal{J}$ and it is called quasi-nonexpansive if $\mathcal{F}(\mathcal{T}) \neq \emptyset$ such that $\mathscr{H}(\mathcal{T}m, \mathcal{T}m^\star) \leq \rho(m, m^\star)$, for all $m \in \mathcal{J}$ and $m^\star \in \mathcal{F}(\mathcal{T}) \neq \emptyset$. In 2007, the notion of single-valued almost contraction mapping of Berinde [5] was extended to multivalued almost contractions by Berinde and Berinde [6] as follows:

Definition 1.2. A multivalued mapping $\mathcal{T} : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ is said to be an almost contraction if there exist $\varrho \in [0, 1)$ and $L \geq 0$ such that the following inequality holds:

$$\mathscr{H}(\mathcal{T}m, \mathcal{T}w) \leq \varrho d(m, w) + L\text{dist}(m, \mathcal{T}m),\ \forall m, w \in \mathcal{J}. \tag{1.1}$$

In 2008, Suzuki [34] introduced a generalized class of nonexpansive mappings which is also known as condition (C) and further showed that the class of mapping satisfying condition (C) is more general than the class of nonexpansive mappings [37]. In 2011, Eslami and Abkar [11] defined the multivalued version of condition (C) as follows:

Definition 1.3. A multivalued mapping $\mathcal{T} : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ is said to satisfy condition (C) if the following inequalities hold:

$$\frac{1}{2}\text{dist}(m, \mathcal{T}m) \leq d(m, w) \ \Rightarrow\ \mathscr{H}(\mathcal{T}m, \mathcal{T}w) \leq d(m, w),\ \forall m, w \in \mathcal{J}. \tag{1.2}$$

Very recently, García-Falset et al. [12] defined a new single-valued mapping called condition (E). This class of mappings are weaker than the class of nonexpansive mappings and stronger than the class of quasi-nonexpansive mappings.

Recently, Kim et al. [19] defined the multivalued and hyperbolic space version of the class of mappings satisfying condition (E). The authors also established some existence and convergence results for such mappings.

Definition 1.4. A multivalued mapping $\mathcal{T} : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ is said to satisfy condition (E_μ) if the following inequality holds:

$$\text{dist}(m, \mathcal{T}w) \leq \mu\text{dist}(m, \mathcal{T}m) + d(m, w),\ \forall m, w \in \mathcal{J}. \tag{1.3}$$

The mapping $\mathcal{T}$ is said to satisfy condition (E) whenever $\mathcal{T}$ satisfies condition (E_μ) for some $\mu \geq 1$.

The studies involving multivalued nonexpansive mappings are known to be more difficult than the concepts involving single-valued nonexpansive mappings. For the approximation of fixed points of various mappings, iterative methods are well known to be essential.

In recent years, several authors have introduced and studied different iterative algorithms for approximating fixed points of multivalued nonexpansive mappings as well as multivalued mappings satisfying condition (E) (see [8, 19] and the references in them).

In 2007, Agarwal et al. [1] introduced the S-iterative algorithm for single-valued contraction mappings. In 2014, Chang et al. [8] considered the mixed-type S-iterative algorithm in hyperbolic spaces for multivalued nonexpansive mappings as follows:

$$\begin{cases} m_1 \in \mathcal{J}, \\ w_k = \mathcal{K}(m_k, u_k, \eta_k), \\ m_{k+1} = \mathcal{K}(u_k, v_k, \xi_k), \ k \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $v_k \in \mathcal{T}_1 w_k$, $u_k \in \mathcal{T}_2 m$, $\{\xi_k\}$ and $\{\eta_k\}$ are real sequences in (0,1).

Also, Kim et al. [19] considered the multivalued and hyperbolic space version of S-iterative algorithm for fixed points multivalued mappings satisfying condition (E) as follows:

$$\begin{cases} m_1 \in \mathcal{J}, \\ w_k = \mathcal{K}(m_k, u_k, \eta_k), \\ m_{k+1} = \mathcal{K}(u_k, v_k, \xi_k), \ k \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $v_k \in \mathcal{T} w_k$, $u_k \in \mathcal{T} m_k$, $\{\xi_k\}$ and $\{\eta_k\}$ are real sequences in (0,1).

It is worthy to know that the iterative algorithm (1.4) involves two multivalued mappings while the iterative algorithm (1.5) involves one multivalued mapping and the class of mappings considered by Kim et al. [19] is more general than the class of mappings considered by Chang et al. [8].

In 2019, Chuadchawnay et al. [9] studied the iterative algorithm (1.4) for common fixed points of two multivalued mappings satisfying condition (E) in hyperbolic spaces. In 2022, Ahmad et al. [3] constructed the hyperbolic space version of F iterative algorithm [4]. The authors obtained some fixed points convergence results for single-valued mappings satisfying condition (E) and single valued almost contraction mappings. Furthermore, they obtained data dependence and weak w^2-stability results for single-valued almost contraction mappings.

Very recently, Ofem et al. [26], introduced a novel mixed-type iterative algorithm in hyperbolic spaces as follows:

$$\begin{cases} m_1 \in \mathcal{J}, \\ s_k = \mathcal{K}(m_k, u_k, \eta_k), \\ w_k = \mathcal{K}(u_k, t_k, \xi_k), \\ p_k = h_k, \\ m_{k+1} = \ell_k, \ k \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{\xi_k\}$, $\{\eta_k\}$ are real sequences in (0,1) and $\ell_k \in \mathcal{G}_1 p_k$, $h_k, \in \mathcal{T}_2 w_k$, $t_k \in \mathcal{T}_1 s_k$, $u_k \in \mathcal{G}_2 m_k$.

Motivated by the above results, in the study, we introduce an iterative scheme with three mappings satisfying the condition (E):

$$\begin{cases} m_1 \in \mathcal{J}, \\ s_k = \mathcal{K}(m_k, u_k, \eta_k), \\ w_k = t_k, \\ p_k = h_k, \\ m_{k+1} = \ell_k, \ k \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\{\eta_k\}$ is a real sequence in (0,1) and $\ell_k \in \mathcal{T}_3 p_k$, $h_k, \in \mathcal{T}_2 w_k$, $t_k \in \mathcal{T}_1 s_k$, $u_k \in \mathcal{T}_3 m_k$. We provide the strong convergence analysis of the iterative method (1.7) for common fixed points of three multivalued almost-contraction mappings. We show that the new method is weak w^2-stable with respect to three multivalued almost-contraction mappings. We prove strong and $\triangle$-convergence results of (1.7) for common fixed point of three multivalued mappings satisfying the condition (E). Finally, we present a numerical experiment to compare the efficiency of our iterative method (1.7) over some well-known existing methods. Our results provides affirmative answers to some of the open questions raised in [3].

2. Preliminaries

A hyperbolic space $(\mathcal{Q}, d, \mathcal{K})$ is termed uniformly convex [15], if given $s > 0$ and $\varepsilon \in (0, 2]$, there exists $\sigma \in (0, 1]$ such that for any $m, w, p \in \mathcal{Q}$,

$$d(\frac{1}{2}m \oplus \frac{1}{2}w, p) \leq (1 - \sigma)s,$$

provided $d(m, p) \leq s, d(m, p) \leq s$ and $d(m, w) \geq \varepsilon s$. A mapping $\Theta : (0, \infty) \times (0, 2] \to (0, 1]$ which ensures that $\sigma = \Theta(s, \varepsilon)$ for any $s > 0$ and $\varepsilon \in (0, 2]$, is said to be modulus of uniform convexity. The mapping Θ is termed monotone if for fixed ε, it decreases with s, that is, $\Theta(s_2, \varepsilon) \leq \Theta(s_1, \varepsilon)$, for all $s_2 \geq s_1 > 0$.

In 2007, with modulus of uniform convexity $\sigma(s, \varepsilon) = \frac{\varepsilon^2}{8}$ quadratic in ε, Leustean [21] showed that CAT(0) space are uniformly convex hyperbolic spaces. This implies that the class of uniformly convex hyperbolic spaces are natural generalization of both CAT(0) space and uniformly convex Banach spaces [15].

Next, we give the definition of $\triangle$-convergence. In view of this, we consider the following concept which will be useful in the definition. Let $\mathcal{J}$ denote a nonempty subset of the metric space $(\mathcal{Q}, d)$ and $\{m_k\}$ be any bounded sequence in $\mathcal{Q}$. For all $m \in \mathcal{Q}$, we define:

- asymptotic radius of $\{m_k\}$ at m as
$$r_a(\{m_k\}, m) = \limsup_{k\to\infty} d(m_k, m);$$
- asymptotic radius of $\{m_k\}$ relative to $\mathcal{J}$ as
$$r_a(\{m_k\}, \mathcal{J}) = \inf\{r_a(\{m_k\}, m); m \in \mathcal{J}\};$$
- asymptotic center of $\{m_k\}$ relative to $\mathcal{J}$ as
$$AC(\{m_k\}, \mathcal{J}) = \{m \in \mathcal{J}; r_a(\{m_k\}, m) = r_a(\{m_k\}, \mathcal{J})\}. \tag{2.1}$$

It is known that every sequence that is bounded has a unique asymptotic center with respect to each closed convex subset in Banach spaces and $CAT(0)$ spaces. If the asymptotic center is taken with rest to $\mathcal{Q}$, then we simplify denote it by $AC(\{m_k\})$ [33].

The following lemma by Leustean [21] shows that the above property holds in a complete uniformly convex hyperbolic space.

Lemma 2.1. ([21]) *Let $(\mathcal{Q}, d, \mathcal{K})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity Θ. Then for any sequence $\{m_k\}$ that is bounded in $\mathcal{Q}$, has a unique asymptotic center with respect to any nonempty closed convex subset $\mathcal{J}$ of $\mathcal{Q}$.*

Now, we further consider some definitions and lemmas that will be useful in proving our main results as follows:

Definition 2.2. A sequence $\{m_k\}$ in $\mathcal{Q}$ is said to be $\triangle$-convergent to an element m in $\mathcal{Q}$, if m is the unique asymptotic center of every subsequence $\{m_{k_l}\}$ of $\{m_k\}$. For this, we write $\triangle - \lim_{k\to\infty} m_k = m$ and say m is the $\triangle$-limit of $\{m_k\}$.

Lemma 2.3. ([17]) *Assume that $\mathcal{Q}$ is a uniformly convex hyperbolic space with the monotone modulus of uniform convexity Θ. Let $m \in \mathcal{Q}$ and $\{\vartheta_k\}$ be a sequence in $[d, e]$ for some $d, e \in (0, 1)$. Suppose $\{m_k\}$ and $\{w_k\}$ are sequences in $\mathcal{Q}$ such that $\limsup_{k\to\infty} d(m_k, m) \leq c$, $\limsup_{k\to\infty} d(w_k, m) \leq c$ and $\lim_{k\to\infty} d(\mathcal{K}(m_k, w_k, \vartheta_k), m) = c$ for some $c \geq 0$. Then we get $\lim_{k\to\infty} d(m_k, w_k) = 0$.*

Definition 2.4. ([7]) Two sequences $\{m_k\}$ and $\{w_k\}$ are said to be equivalent if

$$d(m_k, w_k) \to 0 \text{ as } k \to \infty.$$

Definition 2.5. ([36]) Let $(\mathcal{Q}, d)$ be a metric space, $\mathcal{T} : \mathcal{Q} \to \mathcal{Q}$ be a self-map and for arbitrary $m_1 \in \mathcal{Q}$, $\{m_k\}$ be the iterative algorithm defined by

$$m_{k+1} = f(\mathcal{T}, m_k), \ k \geq 0. \tag{2.2}$$

Assume that $m_k \to m^\star$ as $k \to \infty$, for all $m^\star \in \mathcal{F}(\mathcal{T})$ and for any sequence $\{y_k\} \subset \mathcal{Q}$ which is equivalent to $\{m_k\}$, we have

$$\lim_{k\to\infty} d(y_{k+1}, f(\mathcal{T}, y_k)) = 0 \implies \lim_{k\to\infty} y_k = m^\star.$$

Then we say that the iterative algorithm (2.2) is weak w^2-stable with respect to $\mathcal{T}$.

Proposition 2.6. ([19]) *Suppose $\mathcal{T} : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ is a multivalued mapping satisfying condition (E) such that $\mathcal{F}(\mathcal{T}) \neq \emptyset$. Then $\mathcal{T}$ is multivalued quasi-nonexpansive mapping.*

Lemma 2.7. ([19]) *Let $(\mathcal{Q}, d, \mathcal{K})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity Θ, $\mathcal{J}$ be a nonempty closed convex subset of $\mathcal{Q}$. Let $\mathcal{T} : \mathcal{J} \to \mathcal{P}(\mathcal{J})$ be a multivalued mapping which satisfies condition (E) with convex values. Suppose $\{m_k\}$ is a sequence in $\mathcal{J}$ with $\triangle - \lim_{k\to\infty} m_k = m$ and $\lim_{k\to\infty} dist(m_k, \mathcal{T}m_k) = 0$. Then $m \in \mathcal{F}(\mathcal{T})$.*

Lemma 2.8. ([8]) *Let $(\mathcal{Q}, d, \mathcal{K})$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity Θ and $\{m_k\}$ a sequence which is bounded in $\mathcal{Q}$ such that $AC(\{m_k\}) = \{m\}$. Suppose that $\{u_k\}$ is a subsequence of $\{m_k\}$ such that $AC(\{u_k\}) = \{u\}$, and the sequence $\{d(m_k, u)\}$ is convergent. Then we have $m = u$.*

3. Convergence results for two multivalued almost contraction mappings

Theorem 3.1. *Let $\mathcal{J}$ be a nonempty closed convex subset of a hyperbolic space $\mathcal{Q}$ and $\mathcal{T}_i : \mathcal{J} \to \mathcal{P}(\mathcal{J})$ $(i = 1, 2, 3)$ be three multivalued almost contraction mappings. Let $\mathscr{F} = \bigcap_{i=1}^{3} \mathcal{F}(\mathcal{L}_i) \neq \emptyset$ and $\mathcal{T}_i m^\star = \{m^\star\}$ for each $m^\star \in \mathscr{F}$ $(i = 1, 2, 3)$. Let $\{m_k\}$ be the sequence defined by (1.7). Then, $\{m_k\}$ converges to a point in $\mathscr{F}$.*

Proof. Let $m^\star \in \mathscr{F}$. From (1.1) and (1.7), we have

$$\begin{aligned} d(s_k, m^\star) &= d(\mathcal{K}(m_k, u_k, \eta_k), m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k d(u_k, m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k \text{dist}(u_k, \mathcal{T}_3 q^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k \mathscr{H}(\mathcal{T}_3 m_k, \mathcal{T}_3 m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k \varrho d(m_k, m^\star) \\ &= (1-(1-\varrho)\eta_k)d(m_k, m^\star). \end{aligned} \tag{3.1}$$

Since $0 \leq \varrho < 1$ and $0 < \eta_k < 1$, it follows that $(1-(1-\varrho)\eta_k) < 1$. Thus, (3.1) becomes

$$d(s_k, m^\star) \leq d(m_k, m^\star). \tag{3.2}$$

Using (1.7) and (3.2), we have

$$\begin{aligned} d(w_k, m^\star) &= d(t_k, m^\star) \\ &\leq \text{dist}(t_k, \mathcal{T}_1 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_1 s_k, \mathcal{T}_1 m^\star) \\ &\leq \varrho d(s_k, m^\star) \\ &\leq \varrho d(m_k, m^\star). \end{aligned} \tag{3.3}$$

Also, from (1.7) and (3.3), we have

$$\begin{aligned} d(p_k, m^\star) &= d(h_k, m^\star) \\ &\leq \text{dist}(h_k, \mathcal{T}_2 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_2 w_k, \mathcal{T}_2 m^\star) \\ &\leq \varrho d(w_k, m^\star) \\ &\leq \varrho^2 d(m_k, m^\star). \end{aligned} \tag{3.4}$$

Finally, by (1.7) and (3.4), we have

$$\begin{aligned} d(m_{k+1}, m^\star) &= d(\ell_k, m^\star) \\ &\leq \text{dist}(\ell_k, \mathcal{T}_1 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) \\ &\leq \varrho d(p_k, m^\star) \\ &\leq \varrho^3 d(m_k, m^\star). \end{aligned} \tag{3.5}$$

Inductively, we obtain

$$d(m_{k+1}, m^\star) \leq \varrho^{3(k+1)} d(m_0, m^\star).$$

Since $0 \leq \varrho < 1$, it follows that $\lim_{k\to\infty} m_k = m^\star$. □

4. Weak w^2–stability results

In this section, firstly, we give the definition of w^2-stability involving two mappings in hyperbolic space. After this, we prove that our new iterative algorithm (1.7) is weak w^2-stable with respect to two multivalued almost contraction mappings.

Definition 4.1. Let $(\mathcal{Q}, d, \mathcal{K})$ be a hyperbolic space, $\mathcal{T}_i : \mathcal{Q} \to \mathcal{Q}$ $(i = 1, 2, 3)$ be three self-maps and for arbitrary $m_1 \in \mathcal{Q}$, $\{m_k\}$ be the iterative algorithm defined by

$$m_{k+1} = f(\mathcal{T}_i, m_k)\ (i = 1, 2, 3),\ k \geq 0. \tag{4.1}$$

Assume that $m_k \to m^\star$ as $k \to \infty$, for all $m^\star \in \mathscr{F} = \bigcap_{i=1}^{3} \mathcal{F}(\mathcal{T}_i)$ and for any sequence $\{x_k\} \subset \mathcal{Q}$ which is equivalent to $\{m_k\}$, we have

$$\lim_{k\to\infty} \epsilon_k = \lim_{k\to\infty} d(x_{k+1}, f(\mathcal{T}_i, x_k)) = 0 \implies \lim_{k\to\infty} x_k = m^\star.$$

Then we say that the iterative algorithm (4.1) is weak w^2-stable with respect to $\mathcal{T}_i$ $(i = 1, 2, 3)$.

Theorem 4.2. *Suppose that all the assumptions in Theorem* 3.1 *are satisfied. Then, the sequence* $\{m_k\}$ *defined by* (1.7) *is weak* w^2*-stable with respect to* $\mathcal{T}_1$*,* $\mathcal{T}_2$ *and* $\mathcal{T}_3$*.*

Proof. Suppose $\{m_k\}$ is the sequence defined by (1.7) and $\{x_k\} \subset \mathcal{J}$ is an equivalent sequence of $\{m_k\}$. We define $\{\epsilon_k\} \in \mathbb{R}^+$ by

$$\begin{cases} x_1 \in \mathcal{W}, \\ c_k = \mathcal{K}(x_k, g_k, \eta_k), \\ b_k = i_k, \\ a_k = f_k, \\ \epsilon_k = d(x_{k+1}, e_k),\ k \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\eta_k\}$ is a real sequence in (0,1) and $a_k \in \mathcal{T}_1 a_k,\ f_k, \in \mathcal{G}_2 b_k,\ i_k \in \mathcal{T}_1 c_k$, $g_k \in \mathcal{T}_3 x_k$. Suppose $\lim_{k\to\infty} \epsilon_k = 0$ and $m^\star \in \mathscr{F}$. Then, from (1.7) and (4.2), we have

$$\begin{aligned} d(s_k, c_k) &= d(\mathcal{K}(m_k, u_k, \eta_k), \mathcal{K}(x_k, g_k, \eta_k)) \\ &\leq (1 - \eta_k) d(m_k, x_k) + \eta_k \mathscr{H}(\mathcal{T}_3 m_k, \mathcal{T}_3 x_k) \\ &\leq (1 - \eta_k) d(m_k, x_k) + \eta_k \varrho d(m_k, x_k) + \eta_k L \mathrm{dist}(m_k, \mathcal{G}_3 m_k) \\ &\leq (1 - (1 - \varrho)\eta_k) d(m_k, x_k) + \eta_k L d(m_k, m^\star) + \eta_k L \mathrm{dist}(\mathcal{T}_3 m_k, m^\star) \\ &\leq (1 - (1 - \varrho)\eta_k) d(m_k, x_k) + \eta_k L d(m_k, m^\star) + \eta_k L \mathscr{H}(\mathcal{T}_2 m_k, \mathcal{T}_3 m^\star) \\ &\leq (1 - (1 - \varrho)\eta_k) d(m_k, x_k) + \eta_k L d(m_k, m^\star) + \eta_k L \varrho d(m_k, m^\star) \\ &\leq (1 - (1 - \varrho)\eta_k) d(m_k, x_k) + \eta_k L (1 + \varrho) d(m_k, m^\star). \end{aligned} \tag{4.3}$$

Since $0 \leq \varrho < 1$ and $0 < \eta_k < 1$, it follows that $(1-(1-\varrho)\eta_k) < 1$. Thus, (4.3) becomes

$$d(s_k, c_k) \leq d(m_k, x_k) + \eta_k L(1+\varrho)d(m_k, m^\star). \tag{4.4}$$

By (1.7), (4.2) and (4.4), we obtain

$$\begin{aligned} d(w_k, b_k) &= d(t_k, i_k) \\ &= \mathscr{H}(\mathcal{T}_1 s_k, \mathcal{T}_1 c_k) \\ &\leq \varrho d(s_k, c_k) + L\text{dist}(s_k, \mathcal{T}_1 s_k) \\ &\leq \varrho d(s_k, c_k) + Ld(s_k, m^\star) + L\text{dist}(\mathcal{T}_1 s_k, w^\star) \\ &\leq \varrho d(s_k, c_k) + Ld(s_k, m^\star) + L\mathscr{H}(\mathcal{T}_1 s_k, \mathcal{T}_1 m^\star) \\ &\leq \varrho d(s_k, c_k) + Ld(s_k, m^\star) + L\varrho d(s_k, m^\star) \\ &\leq \varrho d(s_k, c_k) + L(1+\varrho)d(s_k, m^\star) \\ &\leq \varrho[d(m_k, x_k) + \eta_k L(1+\varrho)d(m_k, m^\star)] \\ &\quad + L(1+\varrho)d(s_k, m^\star). \end{aligned} \tag{4.5}$$

By (1.7), (4.2) and (4.5), we obtain

$$\begin{aligned} d(p_k, a_k) &= d(h_k, f_k) \\ &= \mathscr{H}(\mathcal{T}_2 w_k, \mathcal{T}_2 b_k) \\ &\leq \varrho d(w_k, b_k) + L\text{dist}(w_k, \mathcal{T}_2 w_k) \\ &\leq \varrho d(w_k, b_k) + Ld(w_k, m^\star) + L\text{dist}(\mathcal{T}_2 w_k, m^\star) \\ &\leq \varrho d(w_k, b_k) + Ld(w_k, m^\star) + L\mathscr{H}(\mathcal{T}_2 w_k, \mathcal{T}_2 m^\star) \\ &\leq \varrho d(w_k, b_k) + Ld(w_k, m^\star) + L\varrho d(w_k, m^\star) \\ &\leq \varrho d(w_k, b_k) + L(1+\varrho)d(w_k, m^\star) \\ &\leq \varrho^2[d(m_k, x_k) + \eta_k L(1+\varrho)d(m_k, m^\star)] \\ &\quad + \varrho L(1+\varrho)d(s_k, m^\star) + L(1+\varrho)d(w_k, m^\star). \end{aligned} \tag{4.6}$$

Using (1.7), (4.2) and (4.6), we obtain

$$\begin{aligned} d(x_{k+1}, m^\star) &\leq d(x_{k+1}, m_{k+1}) + d(m_{k+1}, m^\star) \\ &\leq d(x_{k+1}, e_k) + d(e_k, m_{k+1}) + d(m_{k+1}, m^\star) \\ &\leq \epsilon_k + d(e_k, \ell_k) + d(m_{k+1}, m^\star) \\ &\leq \epsilon_k + \mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 a_k) + d(m_{k+1}, m^\star) \\ &\leq \epsilon_k + \varrho d(p_k, a_k) + L\text{dist}(p_k, \mathcal{T}_1 p_k) + d(m_{k+1}, m^\star) \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon_k + \varrho d(p_k, a_k) + Ld(p_k, m^\star) \\
&\quad + L\mathrm{dist}(\mathcal{T}_1 p_k, m^\star) + d(m_{k+1}, m^\star) \\
&\leq \epsilon_k + \varrho d(p_k, a_k) + Ld(p_k, m^\star) \\
&\quad + L\mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) + d(m_{k+1}, m^\star) \\
&\leq \epsilon_k + \varrho^3[d(m_k, x_k) + \eta_k L(1+\varrho)d(m_k, m^\star)] \\
&\quad + \varrho^2 L(1+\varrho)d(s_k, m^\star) + \varrho L(1+\varrho)d(w_k, m^\star) \\
&\quad + Ld(p_k, m^\star) + L\mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) + d(m_{k+1}, m^\star). \qquad (4.7)
\end{aligned}$$

By Theorem 3.1, $\lim_{k\to\infty} d(m_k, m^\star) = 0$. Consequently, we have

$$\lim_{m\to\infty} d(m_{k+1}, m^\star) = 0.$$

Also, by the equivalence of $\{m_k\}$ and $\{x_k\}$, we have $\lim_{m\to\infty} d(m_k, x_k) = 0$. Thus, taking the limit of both sides of (4.7), we have

$$\lim_{k\to\infty} d(x_k, q^\star) = 0.$$

Hence, our new iterative sequence (1.7) is weak w^2-stable with respect to $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. This completes the proof. □

5. △-convergence and strong convergence results for three multivalued mappings.

In this section, we state and prove the △-convergence and strong convergence theorems of the proposed iterative algorithm (1.7) for common fixed points of three multivalued mappings satisfying condition (E). Throughout the remaining part of this article, let $(\mathcal{Q}, d, \mathcal{K})$ denote a complete uniformly convex hyperbolic space with a monotone modulus of convexity Θ and $\mathcal{J}$ be a nonempty closed convex subset of $\mathcal{Q}$.

Theorem 5.1. *Let $\mathcal{J}$ be a nonempty closed convex subset of $\mathcal{Q}$ and $\mathcal{T}_i : \mathcal{J} \to \mathcal{P}(\mathcal{J})$ $(i = 1, 2, 3)$ be three multivalued mappings satisfying condition (E) with convex values. Assume $\mathscr{F} = \bigcap_{i=1}^3 \mathcal{F}(\mathcal{T}_i) \neq \emptyset$, $\lim_{k\to\infty} d(u_k, t_k) = 0$, $\lim_{k\to\infty} dist(u_k, \mathcal{T}_2 w_k) = 0$ and $\mathcal{T}_i m^\star = \{m^\star\}$ for each $m^\star \in \mathscr{F}$ $(i = 1, 2, 3)$. If $\{m_k\}$ is the sequence defined by (1.7), then, $\{m_k\}$ △-converges to a common fixed point of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof. We will divide the proof into the following three steps:

Step 1: First, we show that $\lim_{k\to\infty} d(m_k, m^\star)$ exists for each $m^\star \in \mathscr{F}$. By

Proposition 2.6, we know that T_i $(i = 1, 2, 3)$ are multivalued quasi-nonexpansive mappings. Therefore, for all $m^\star \in \mathscr{F}$ and by (1.7), we obtain

$$\begin{aligned} d(s_k, m^\star) &= d(\mathcal{K}(m_k, u_k, \eta_k), m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k d(u_k, m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k \text{dist}(u_k, \mathcal{T}_3 m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k \mathscr{H}(\mathcal{T}_3 m_k, \mathcal{T}_3 m^\star) \\ &\leq (1-\eta_k)d(m_k, m^\star) + \eta_k d(m_k, m^\star) \\ &= d(m_k, m^\star). \end{aligned} \tag{5.1}$$

Again, from (1.7) and (5.1), we have

$$\begin{aligned} d(w_k, m^\star) &= d(h_k, q^\star) \\ &\leq \text{dist}(t_k, \mathcal{G}_3 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_3 s_k, \mathcal{T}_3 m^\star) \\ &\leq d(s_k, m^\star) \\ &\leq d(m_k, m^\star). \end{aligned} \tag{5.2}$$

From (1.7) and (5.2), we have

$$\begin{aligned} d(p_k, q^\star) &= d(h_k, m^\star) \\ &\leq \text{dist}(h_k, \mathcal{T}_2 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_2 w_k, \mathcal{T}_3 m^\star) \\ &\leq d(w_k, m^\star) \\ &\leq d(m_k, m^\star). \end{aligned} \tag{5.3}$$

Finally, by (1.7) and (5.3), we have

$$\begin{aligned} d(m_{k+1}, m^\star) &= d(\ell_k, m^\star) \\ &\leq \text{dist}(\ell_k, \mathcal{T}_1 m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) \\ &\leq d(p_k, m^\star) \\ &\leq d(m_k, m^\star). \end{aligned} \tag{5.4}$$

This implies that the sequence $\{d(m_k, m^\star)\}$ is non-increasing and bounded below. Thus, $\lim_{m\to\infty} d(m_k, m^\star)$ exists for each $m^\star \in \mathscr{F}$.

Step 2: Next, we show that

$$\lim_{k\to\infty} \text{dist}(m_k, \mathcal{G}_i m_k) = 0 \quad \text{for all } i = 1, 2, 3. \tag{5.5}$$

From **Step 1**, it is established that for all $m^\star \in \mathscr{F}$, $\lim_{k\to\infty} d(m_k, m^\star)$ exists. Let

$$\lim_{k\to\infty} d(m_k, m^\star) = \gamma \geq 0. \tag{5.6}$$

If $\gamma = 0$, then we get

$$\begin{aligned} \mathrm{dist}(m_k, \mathcal{T}_i m_k) &\leq d(m_k, m^\star) + \mathrm{dist}(\mathcal{T}_i m_k, m^\star) \\ &\leq d(m_k, m^\star) + \mathscr{H}(\mathcal{T}_i m_k, \mathcal{T}_i m^\star) \\ &\leq d(m_k, m^\star) + d(m_k, m^\star) \\ &= 2d(m_k, m^\star) \to 0 \ \text{ as } k \to \infty. \end{aligned}$$

Hence, $\lim_{k\to\infty} \mathrm{dist}(m_k, \mathcal{T}_i m_k) = 0$ for all $i = 1, 2, 3$. If $\gamma > 0$, now from (5.1), (5.2), (5.3) and (5.4), we have

$$\limsup_{k\to\infty} d(s_k, m^\star) \leq \gamma, \tag{5.7}$$

$$\limsup_{k\to\infty} d(w_k, m^\star) \leq \gamma, \tag{5.8}$$

$$\limsup_{k\to\infty} d(p_k, m^\star) \leq \gamma \tag{5.9}$$

and

$$\limsup_{k\to\infty} d(\ell_m, m^\star) \leq \gamma. \tag{5.10}$$

Consequently, we obtain the following inequalities

$$\begin{aligned} \limsup_{k\to\infty} d(u_k, m^\star) &\leq \limsup_{k\to\infty} \mathscr{H}(\mathcal{T}_3 m_k, \mathcal{T}_3 m^\star) \\ &\leq \limsup_{k\to\infty} d(m_k, m^\star) = \gamma, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \limsup_{k\to\infty} d(t_k, m^\star) &\leq \limsup_{k\to\infty} \mathscr{H}(\mathcal{T}_1 s_k, \mathcal{T}_1 m^\star) \\ &\leq \limsup_{k\to\infty} d(s_k, m^\star) \leq \gamma \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \limsup_{k\to\infty} d(\ell_k, m^\star) &\leq \limsup_{k\to\infty} \mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) \\ &\leq \limsup_{k\to\infty} d(p_k, m^\star) \leq \gamma. \end{aligned} \tag{5.13}$$

Now, from (1.7) we get

$$\begin{aligned} d(m_{k+1}, m^\star) &= d(\ell_k, m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_1 p_k, \mathcal{T}_1 m^\star) \\ &\leq d(p_k, m^\star), \end{aligned}$$

this yields

$$\gamma \leq \liminf_{k\to\infty} d(p_k, m^\star). \tag{5.14}$$

By (5.9) and (5.14), we have

$$\lim_{k\to\infty} d(p_k, m^\star) = \gamma. \tag{5.15}$$

Now, using (1.7), we obtain

$$\begin{aligned} d(p_k, m^\star) &= d(h_k, m^\star) \\ &\leq \mathscr{H}(\mathcal{T}_2 w_k, \mathcal{T}_2 m^\star) \\ &\leq d(w_k, m^\star), \end{aligned} \tag{5.16}$$

which yields

$$\gamma \leq \liminf_{k\to\infty} d(w_k, m^\star). \tag{5.17}$$

From (5.8) and (5.17), we have

$$\lim_{k\to\infty} d(w_k, m^\star) = \gamma. \tag{5.18}$$

By (1.7) and our hypothesis, we have

$$d(w_k, m^\star) \leq d(u_k, m^\star) + d(t_k, u_k),$$

which gives

$$\gamma \leq \liminf_{k\to\infty} d(u_k, m^\star). \tag{5.19}$$

Using (5.11) and (5.19), we have

$$\lim_{k\to\infty} d(u_k, m^\star) = \gamma. \tag{5.20}$$

Also,

$$\begin{aligned} d(u_k, m^\star) &\leq d(u_k, t_k) + d(t_k, m^\star) \\ &\leq d(u_k, t_k) + \mathscr{H}(\mathcal{T}_1 s_k, \mathcal{T}_1 m^\star) \\ &\leq d(u_k, t_k) + d(s_k, m^\star), \end{aligned}$$

implies that

$$\gamma \leq \liminf_{k\to\infty} d(s_k, m^\star). \tag{5.21}$$

From (5.7) and (5.21), we obtain

$$\lim_{k\to\infty} d(s_k, m^\star) = \gamma. \tag{5.22}$$

Finally, by (1.7), we obtain

$$\lim_{k\to\infty} d(s_k, m^\star) = \lim_{k\to\infty} d(\mathcal{K}(m_k, u_k, \eta_k), m^\star) = \gamma. \tag{5.23}$$

Now, due to (5.6), (5.11), (5.23) and Lemma 2.3, we have

$$\lim_{k\to\infty} d(m_k, u_k) = 0. \tag{5.24}$$

Since $\mathrm{dist}(m_k, \mathcal{T}_3 m_k) \le d(m_k, u_k)$, we get

$$\lim_{k\to\infty} \mathrm{dist}(m_k, \mathcal{T}_3 m_k) = 0. \tag{5.25}$$

On the other hand, by (1.7) and (5.24), we have

$$d(s_k, m_k) = d(\mathcal{K}(m_k, u_k, \eta_k), m_k) \le \eta_k d(m_k, u_k) \tag{5.26}$$

and

$$\begin{aligned} \mathrm{dist}(s_k, \mathcal{T}_1 s_k) &\le d(s_k, t_k) \\ &= d(\mathcal{K}(m_k, u_k, \eta_k), t_k) \\ &\le (1-\eta_k) d(m_k, t_k) + \eta_k d(u_k, t_k) \\ &\le (1-\eta_k)[d(m_k, u_k) + d(u_k, t_k)] + \eta_k d(u_k, t_k). \end{aligned} \tag{5.27}$$

Now, using our hypothesis and (5.24), we have

$$\lim_{k\to\infty} \mathrm{dist}(s_k, \mathcal{T}_1 s_k) = 0. \tag{5.28}$$

Since $\mathcal{T}_1$ satisfies condition (E), we obtain

$$\begin{aligned} \mathrm{dist}(m_k, \mathcal{T}_1 m_k) &\le d(m_k, s_k) + \mathrm{dist}(s_k, \mathcal{T}_1 m_k) \\ &\le d(m_k, s_k) + \mu \mathrm{dist}(s_k, \mathcal{T}_1 s_k) + d(s_k, m_k) \\ &\le 2d(u_k, w_k) + \mu \mathrm{dist}(s_k, \mathcal{T}_1 s_k). \end{aligned}$$

By (5.24), (5.26) and (5.28), we have

$$\lim_{k\to\infty} \mathrm{dist}(m_k, \mathcal{T}_1 m_k) = 0. \tag{5.29}$$

Finally, from our hypothesis, we have

$$\mathrm{dist}(m_k, \mathcal{T}_2 m_k) \le d(m_k, u_k) + \mathrm{dist}(u_k, \mathcal{T}_2 m_k) \to 0 \text{ as } k \to \infty.$$

Hence, $\lim_{k\to\infty} \mathrm{dist}(m_k, \mathcal{T}_i m_k) = 0$, $i = 1, 2, 3$.

Step 3: Finally, we show that the sequence $\{m_k\}$ is $\triangle$-convergent to a point in $\mathscr{F}$. In view of this, it suffices to show that

$$\mathcal{K}_{\triangle}(\{m_k\}) = \bigcup_{\{u_k\}\subset\{m_k\}} \subset \mathscr{F} \tag{5.30}$$

and $\mathcal{K}_{\triangle}(\{m_k\})$ has only one point. Set $u \in \mathcal{K}_{\triangle}(\{m_k\})$. Then a subsequence $\{u_k\}$ of $\{m_k\}$ exists such that $AC(\{u_k\}) = \{u\}$. From Lemma 2.1, a subsequence $\{v_k\}$ of $\{u_k\}$ exists such that $\triangle - \lim_{k\to\infty} v_k = v \in \mathcal{J}$. Since $\lim_{k\to\infty} \mathrm{dist}(v_k, \mathcal{T}_i v_k) = 0$ $(i = 1, 2, 3)$, by Lemma 2.7, we know that $v \in \mathscr{F}$. By the convergence of $\{d(u_k, v)\}$, then from Lemma 2.8, we obtain $u = v$. This implies that $\mathcal{K}_{\triangle}(\{m_k\}) \subset \mathscr{F}$.

Now, we show that the set $\mathcal{K}_{\triangle}(\{m_k\})$ contain exactly one element. For this, let $\{u_k\}$ be a subsequence of $\{m_k\}$ with $AC(\{u_k\}) = \{u\}$ and $AC(\{m_k\}) = \{m\}$. Already, we have that $u = v$ and $v \in \mathscr{F}$. Conclusively, by the convergence of $\{d(m_k, m^\star)\}$, then by Lemma 2.8, we obtain $m = v \in \mathscr{F}$. It follows that $\mathcal{K}_{\triangle}(\{m_k\}) = \{m\}$. This completes the proof. $\square$

Next, we establish some strong convergence theorems.

Theorem 5.2. *Let $\mathcal{J}$ be a nonempty closed compact subset of $\mathcal{Q}$ and $\mathcal{T}_i : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ $(i = 1, 2, 3)$ be three multivalued mappings satisfying condition (E) with convex values. Let $\mathscr{F} = \bigcap_{i=1}^{3} \mathcal{F}(\mathcal{T}_i) \neq \emptyset$ and $\mathcal{T}_i m^\star = \{m^\star\}$ for each $m^\star \in \mathscr{F}$ $(i = 1, 2, 3)$. Let $\{m_k\}$ be the sequence defined by* (1.7)*. Then, $\{m_k\}$ converges strongly to a point in $\mathscr{F}$.*

Proof. For all $m \in \mathcal{J}$ and $i = 1, 2, 3$, we can assume that $\mathcal{T}_i$ is a bounded closed and convex subset of $\mathcal{J}$. By the compactness of $\mathcal{J}$, we know that $\mathcal{T}_i$ is a nonempty compact convex subset and bounded proximal subset in $\mathcal{J}$. It follows that $\mathcal{T}_i : \mathcal{J} \to \mathcal{P}(\mathcal{J})$. Thus, all the assumptions in Theorem 5.1 are performed. Hence, from Theorem 5.1, we have that $\lim_{k\to\infty} (m_k, m^\star)$ exists and $\lim_{k\to\infty} \text{dist} d(m_k, \mathcal{T}_i m_k) = 0$, for each $m^\star \in \mathscr{F}$ and $i = 1, 2, 3$. By the compactness of $\mathcal{J}$, we are sure of the existence of a subsequence $\{m_{k_i}\}$ of $\{m_k\}$ with $\lim_{k\to\infty} m_{k_i} = \chi \in \mathcal{J}$. Using condition (E) for some $\mu \geq 1$ and for each $i = 1, 2, 3$, we have

$$
\begin{aligned}
\text{dist}(\chi, \mathcal{G}_i \chi) &\leq \text{dist}(\chi, m_{k_i}) + \text{dist}(m_{k_i}, \mathcal{G}_i \chi) \\
&\leq \mu \text{dist}(m_{k_i}, \mathcal{G}_i m_{k_i}) + 2d(\chi, m_{k_i}) \\
&\to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

It shows that $\chi \in \mathscr{F}$. By the strong convergence of $\{m_{k_i}\}$ to χ and the existence of $\lim_{k\to\infty} d(m_k, \chi)$ from Theorem 5.1, it implies that the sequence $\{m_k\}$ converges strongly to χ. $\square$

Theorem 5.3. *Let $\mathcal{J}$ be a nonempty closed compact subset of $\mathcal{Q}$ and $\mathcal{T}_i : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ $(i = 1, 2, 3)$ be three multivalued mappings satisfying condition (E) with convex values. Let $\mathscr{F} = \bigcap_{i=1}^{3} \mathcal{F}(\mathcal{T}_i) \neq \emptyset$ and $\mathcal{T}_i m^\star = \{m^\star\}$ for each $m^\star \in \mathscr{F}$ $(i = 1, 2, 3)$. Let $\{m_k\}$ be the sequence defined by* (1.7)*. Then, $\{m_k\}$ converges strongly to a point in $\mathscr{F}$ if and only if $\liminf_{k\to\infty} dist(m_k, \mathscr{F}) = 0$.*

Proof. Suppose that $\liminf_{k\to\infty} \text{dist}(m_k, \mathscr{F}) = 0$. From (5.4), we have $d(m_{k+1}, m^\star) \leq d(m_k, m^\star)$ for all $m^\star \in \mathscr{F}$. It follows that $\text{dist}(m_{k+1}, \mathscr{F}) \leq \text{dist}(m_k, \mathscr{F})$.

Therefore, $\lim_{k\to\infty} \operatorname{dist}(m_{k+1}, \mathscr{F})$ exists and $\lim_{k\to\infty} \operatorname{dist}(m_{k+1}, \mathscr{F}) = 0$. Thus, there exists a subsequence $\{m_{k_r}\}$ of the sequence $\{m_k\}$ such that $d(m_{k_r}, t_r) \le \frac{1}{2^r}$ for all $r \ge 1$, where $\{t_r\}$ is a sequence in $\mathscr{F}$. In view of (5.4), we obtain

$$d(m_{k_{r+1}}, t_r) \le d(m_{k_r}, t_r) \le \frac{1}{2^r}. \tag{5.31}$$

Using (5.31) and the concept of triangle inequality, then we get

$$\begin{aligned} d(t_{r+1}, t_r) &\le d(t_{r+1}, w_{k_{r+1}}) + d(w_{k_{r+1}}, t_r) \\ &\le \frac{1}{2^{r+1}} + \frac{1}{2^r} < \frac{1}{2^{r-1}}. \end{aligned}$$

It follows clearly that $\{t_r\}$ is a Cauchy sequence in $\mathcal{J}$ and moreover, it is convergent to some $p \in \mathcal{J}$. Since for all $i = 1, 2, 3$,

$$\operatorname{dist}(t_r, \mathcal{T}_i p) \le \mathscr{H}(\mathcal{T}_i t_r, \mathcal{T}_i p) \le d(p, t_r)$$

and $t_r \to p$ as $k \to \infty$, it implies that $\operatorname{dist}(p, \mathcal{T}_i p) = 0$ and hence, $p \in \mathscr{F}$ and $\{m_{k_r}\}$ strongly converges to p. Since $\lim_{k\to\infty} d(m_k, p)$ exists, it implies that $\{m_k\}$ converges strongly to p. □

Theorem 5.4. *Let $\mathcal{J}$ be a nonempty closed compact subset of $\mathcal{Q}$ and $\mathcal{G}_i : \mathcal{J} \to \mathcal{BC}(\mathcal{J})$ $(i = 1, 2, 3)$ be three multivalued mappings satisfying condition (E) with convex values. Let $\mathscr{F} = \bigcap_{i=1}^{3} \mathcal{F}(\mathcal{T}_i) \neq \emptyset$ and $\mathcal{T}_i m^\star = \{m^\star\}$ for each $m^\star \in \mathscr{F}$ $(i = 1, 2, 3)$. Let $\{m_k\}$ be the sequence defined by (1.7). Assume that there exists an increasing self-function f defined on $[0, \infty)$ such that $f(0) = 0$ with $f(l) > 0$ for all $l > 0$ and $i = 1, 2, 3$, we have*

$$dist(m_k, \mathcal{T}_i m_k) \ge f(dist(m_k, \mathscr{F})).$$

Then, the sequence $\{m_k\}$ converges strongly to a point in $\mathscr{F}$.

Proof. It is established in Theorem 5.1 that $\operatorname{dist}(m_k, \mathcal{T}_i m_k) = 0$. Hence one can assume that

$$\lim_{k\to\infty} f(\operatorname{dist}(m_k, \mathscr{F})) \le \lim_{k\to\infty} \operatorname{dist}(m_k, \mathcal{T}_i m_k) = 0.$$

Thus, it implies that $\lim_{k\to\infty} f(\operatorname{dist}(m_k, \mathscr{F})) = 0$. Since f is an increasing self-function defined on $[0, \infty)$ with $f(0) = 0$, we know that $\lim_{k\to\infty} \operatorname{dist}(m_k, \mathscr{F}) = 0$. The conclusion of the proof follows from Theorem 5.3. □

6. Numerical example

In this section, we presents examples of mappings which satisfy condition (E). We carry out numerical experiment to compare the efficiency and applicability of the new method (1.7) with some existing iterative methods.

Example 6.1. Let $\mathcal{Q} = \mathbb{R}$ with the distance metric $d(m,w) = |m-w|$ and $\mathcal{J} = [0,\infty)$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{J} \to \mathcal{P}(\mathcal{J})$ be defined by

$$\mathcal{T}_1 m = \begin{cases} [0, \frac{4m}{5}], & \text{if } m \in [\frac{1}{5}, \infty), \\ \\ \{0\}, & \text{if } \ m \in [0, \frac{1}{5}), \end{cases}$$

$$\mathcal{T}_2 m = \begin{cases} [0, \frac{m}{3}], & \text{if } m \in (3, \infty], \\ \\ \{0\}, & \text{if } \ m \in [0, 3], \end{cases}$$

and

$$\mathcal{T}_3 m = \begin{cases} [0, \frac{m}{2}], & \text{if } m \in (2, \infty], \\ \\ \{0\}, & \text{if } \ m \in [0, 2], \end{cases}$$

for all $m \in \mathcal{J}$. Then, clearly,

$$\mathscr{F} = \mathcal{F}(\mathcal{T}_1) \cap \mathcal{F}(\mathcal{T}_2) \cap \mathcal{F}(\mathcal{T}_3) = \{0\}.$$

Since $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are not continuous at $\frac{1}{5}$, 2 and 3, respectively, so $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are not nonexpansive mappings. Next, we show that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are multivalued mappings satisfying condition (E). Firstly, we consider $\mathcal{T}_1$ and the following possible cases:

Case 1: If $m, w \in [\frac{1}{5}, \infty)$, then

$$\text{dist}(m, \mathcal{T}_1 m) = \text{dist}\left(m, \left[0, \frac{4m}{5}\right]\right) = \left|m - \frac{4m}{5}\right| = \left|\frac{m}{5}\right|.$$

Therefore,

$$\begin{aligned} \text{dist}(m, \mathcal{T}_1 w) &= \text{dist}\left(m, \left[0, \frac{4w}{5}\right]\right) \\ &= \left|m - \frac{4w}{5}\right| \\ &= \left|m - \frac{4m}{5} + \frac{4m}{5} - \frac{4w}{5}\right| \\ &\leq \left|m - \frac{4m}{5}\right| + \left|\frac{4m}{5} - \frac{4w}{5}\right| \\ &\leq 5\left|\frac{m}{5}\right| + \frac{4}{5}|m-w| \\ &\leq 5\left|\frac{m}{5}\right| + |m-w| \\ &= 5\text{dist}(m, \mathcal{T}_1 m) + d(m,w). \end{aligned}$$

Case 2: If $m, w \in [0, \frac{1}{5})$, then

$$\mathrm{dist}(m, \mathcal{T}_1 m) = \mathrm{dist}\,(m, \{0\}) = |m - 0| = |m|\,.$$

Therefore,

$$\begin{aligned}\mathrm{dist}(m, \mathcal{T}_1 w) &= \mathrm{dist}\,(m, \{0\}) \\ &= |m| \\ &\leq 5|m| + |m - w| \\ &= 5\mathrm{dist}(m, \mathcal{T}_1 m) + d(m, w).\end{aligned}$$

Case 3: If $m \in [\frac{1}{5}, \infty)$ and $w \in [0, \frac{1}{5})$, then

$$\mathrm{dist}(m, \mathcal{T}_1 m) = \mathrm{dist}\left(m, \left[0, \frac{4m}{5}\right]\right) = \left|m - \frac{4m}{5}\right| = \left|\frac{m}{5}\right|.$$

Therefore,

$$\begin{aligned}\mathrm{dist}(m, \mathcal{T}_1 w) &= \mathrm{dist}\,(m, \{0\}) \\ &= |m| \\ &= 5\left|\frac{m}{5}\right| \\ &\leq 5\left|\frac{m}{5}\right| + |m - w| \\ &= 5\mathrm{dist}(m, \mathcal{T}_1 m) + d(m, w).\end{aligned}$$

Case 4: If $m \in [0, \frac{1}{5})$ and $w \in [\frac{1}{5}, \infty)$, then

$$\mathrm{dist}(m, \mathcal{T}_1 m) = \mathrm{dist}\,(m, \{0\}) = |m - 0| = |m|\,.$$

Therefore,

$$\begin{aligned}\mathrm{dist}(m, \mathcal{T}_1 w) &= \mathrm{dist}\left(m, \left[0, \frac{4w}{5}\right]\right) \\ &= \left|m - \frac{4w}{5}\right| \\ &= \left|m - \frac{4m}{5} + \frac{4m}{5} - \frac{4w}{5}\right| \\ &\leq \left|m - \frac{4m}{5}\right| + \left|\frac{4m}{5} - \frac{4w}{5}\right| \\ &= \left|\frac{m}{5}\right| + \frac{4}{5}\,|m - w| \\ &\leq |m| + |m - w| \\ &\leq 5\,|m| + |m - w| \\ &= 5\mathrm{dist}(m, \mathcal{T}_1 m) + d(m, w).\end{aligned}$$

For all $m, w \in \mathcal{J}$, we have shown that $\mathcal{T}_1$ satisfies (1.1) for some $\mu = 5$. Hence, $\mathcal{T}_1$ is a multivalued mapping satisfying condition (E).

Following the same approach above, we can show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are multivalued mappings satisfying condition (E) for some $\mu = 3$ and $\mu = 2$, respectively.

Now, for control parameters $\xi_k = \eta_k = \zeta_k = \frac{2}{5}$, for all $k \in \mathbb{N}$ and starting point $m_1 = 7$, then by using MATLAB R2015a, we obtain the following Tables 1–2 and Figures 1–2.

TABLE 1. Convergence behavior of various iterative algorithms.

m_k	Mann	Ishikawa	S	Picard-Mann	F	New
m_1	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
m_2	6.37500000	6.14062500	5.51562500	5.28125000	2.84570313	0.99609375
m_3	4.82812500	4.42895508	3.47192383	1.15332031	0.58132401	0.19844055
m_4	4.34960938	3.83960342	2.73807144	1.41311646	0.25150437	0.03953308
m_5	2.93090820	2.35154659	1.22208148	0.92735767	0.09284048	0.00787573
m_6	2.56454468	1.94737452	0.85927604	0.60857847	0.03427119	0.00156899
m_7	2.24397659	1.61266952	0.60417847	0.39937962	0.01265089	0.00031257
m_8	1.96347952	1.33549195	0.42481298	0.26209288	0.00466996	0.00006227
m_9	1.71804458	1.10595427	0.29869663	0.17199845	0.00172387	0.00001241
m_{10}	1.50328901	0.91586838	0.21002107	0.11287398	0.00063635	0.00000247
m_{11}	1.31537788	0.75845350	0.14767106	0.07407355	0.00023490	0.00000049
m_{12}	1.15095565	0.62809431	0.10383122	0.04861077	0.00008671	0.00000010
m_{13}	1.00708619	0.52014060	0.07300632	0.03190082	0.00003201	0.00000002
m_{14}	0.88120042	0.43074143	0.05133257	0.02093491	0.00001182	0.00000000

TABLE 2. Convergence behavior of various iterative algorithms.

m_k	Noor	CR	Thakur	Picard-S	M	New
m_1	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000	7.00000000
m_2	6.05273438	3.11914063	3.63671875	2.81640625	3.46093750	0.99609375
m_3	4.28493118	2.89815140	2.39045715	2.65986633	2.21124268	0.19844055
m_4	2.66259070	0.38066182	0.73324889	0.23971707	0.59615850	0.03953308
m_5	2.15815458	0.16133519	0.38667422	0.08708472	0.29342176	0.00787573
m_6	1.74928545	0.06837839	0.20391023	0.03163624	0.14441852	0.00156899
m_7	1.41787785	0.02898068	0.10753079	0.01149285	0.07108099	0.00031257
m_8	1.14925646	0.01228283	0.05670569	0.00417514	0.03498518	0.00006227
m_9	0.93152623	0.00520581	0.02990339	0.00151675	0.01721927	0.00001241
m_{10}	0.75504568	0.00220637	0.01576937	0.00055101	0.00847511	0.00000247
m_{11}	0.61199991	0.00093512	0.00831588	0.00020017	0.00417134	0.00000049
m_{12}	0.49605462	0.00039633	0.00438533	0.00007272	0.00205308	0.00000010
m_{13}	0.40207552	0.00016798	0.00231257	0.00002642	0.00101050	0.00000002
m_{14}	0.32590106	0.00007119	0.00121952	0.00000960	0.00049736	0.00000000

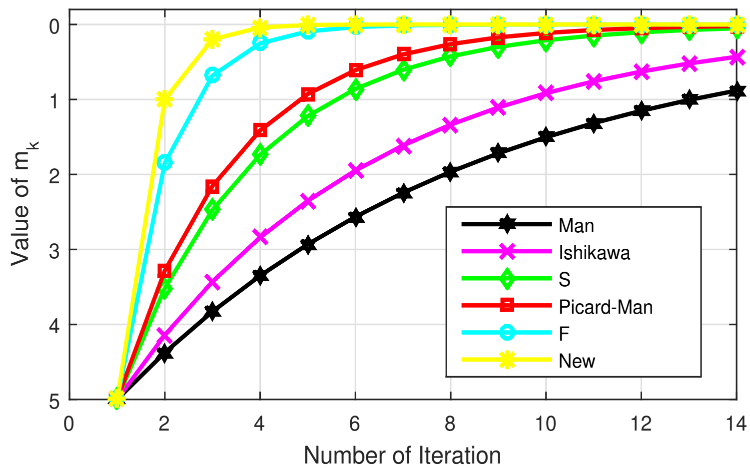

FIGURE 1. Graph corresponding to Table 1.

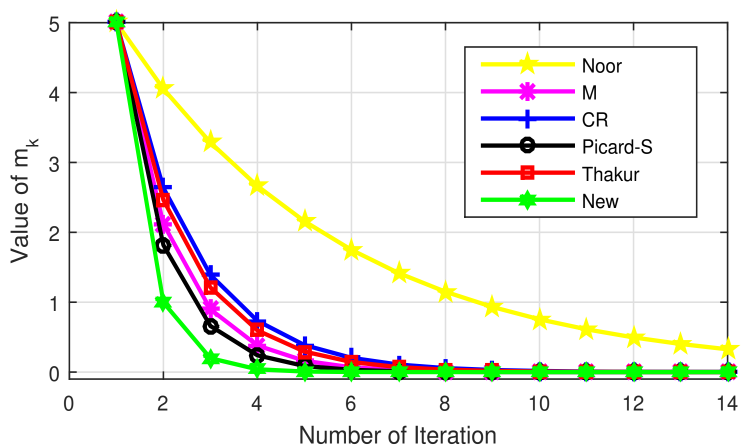

FIGURE 2. Graph corresponding to Table 2.

7. Conclusion

(i) In this work, we have introduced a new iterative algorithm (1.7) in hyperbolic spaces.

(ii) We have proved the strong convergence of the newly defined iterative algorithm (1.7) to the common fixed point of three multivalued almost contraction mappings.

(iii) We studied the concepts of weak w^2-stability results involving three multivalued almost contraction mappings.

(iv) We have proved several strong and $\triangle$-convergence results of (1.7) for common fixed point of multivalued mappings satisfying condition (E).

(v) We presented interesting examples of mappings which satisfy condition (E). We further performed numerical experiment to compare the efficiency and applicability of our iterative method with some leading iterative algorithm.

(vi) The results in this article extend and generalize the results in [27, 28, 29, 30] and several others from the setting of Banach spaces to the setting hyperbolic spaces. Moreover, our results improve and generalize the results in [4, 29] and several others from the setting of single-valued mappings to the setting of multivalued mappings. Also, our improve and extend the results in [4, 29] from the setting of fixed points of single mapping to the setting common fixed points of two mappings.

(vii) Our results give affirmative answers to the two interesting open questions raised by Ahmad et al. [3].

(viii) The main results derived in this article continue to be true in linear and CAT(0) spaces, since the hyperbolic space properly includes these spaces.

Acknowledgments: The fourth author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Nonlinear Convex Anal., **8**(1) (2007), 61–79.

[2] S. Aggarwal, I. Uddin and S. Mujahid, *Convergence theorems for SP-iteration scheme in a ordered hyperbolic metric space* , Nonlinear Funct Anal. Appl., **26**(5) (2021), 961–969.

[3] J. Ahmad, K. Ullah and M. Arshad, *Convergence weak w^2 stability, and data dependence results for the F iterative scheme in hyperbolic spaces*, Nume. Algo., **91** (2022), 1755–1778, https://doi.org/10.1007/s11075-022-01321-y.

[4] J. Ali, M. Jubair and F. Ali, *Stability and convergence of F iterative scheme with an application to the fractional differential equation*, Engineering with Computers, (2020), https://doi.org/10.1007/s00366-020-01172-y.

[5] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration,* Nonlinear Anal. Forum, **9** (2004), 43–53.

[6] M. Berinde and V. Berinde, *On a general class of multivalued weakly Picard mappings,* J. Math. Anal. Appl., **326** (2007), 772–782.

[7] T. Cardinali and P. Rubbioni, *A generalization of the Caristi fixed point theorem in metric spaces*, Fixed Point Theory, **11**(1) (2010), 3–10.

[8] S. Chang, G. Wanga, L. Wanga, Y.K. Tang and Z.L. Mab, *$\triangle$-convergence theorems for multivalued nonexpansive,* Appl. Math. Comput., **249** (2014), 535–540.

[9] P. Chuadchawnay, A. Farajzadehz and A. Kaewcharoeny, *On convergence theorems for two generalized nonexpansive multivalued mappings in hyperbolic spaces*, Thai J. Math., **17**(2) (2019), 445–461.

[10] R. Chugh, V. Kumar and S. Kumar, *Strong convergence of a new three step iterative scheme in Banach spaces*, Amer. J. Comput. Math., **2** (2012), 345–357.

[11] M. Eslamian and A. Abkar, *One-step iterative process for finite family of multivalued mappings,* Math. Comput. Mod., **54**(1-2) (2011), 105–111.

[12] J. García-Falset, E. Llorens-Fuster and T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings.* J. Math. Anal. Appl., **375** (2011), 185–195.

[13] K. Goebel and W.A. Kirk, *Iteration processes for nonexpansive mappings Topological Methods in Nonlinear Functional Analysis.* In: Singh SP, Thomeier S, Watson B (eds.) Contemp. Math. Amer. Math. Soc. AMS, Providence, RI., **21** (1983), 115–123.

[14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings.* Marcel Dekker, New York, 1984.

[15] M. Imdad and S. Dashputre, *Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces,* Math. Sci., **10** (2016), 131–138, DOI 10.1007/s40096-016-0187-8.

[16] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space,* Proc. Amer. Math. Soc., **59**(1) (1976), 65–71.

[17] A.R Khan, H. Fukhar-ud-din and M.A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., **2012**(54) (2012).

[18] J.K. Kim, S. Dashputre, Padmavati and K. Sakure, *Generalized α-nonexpansive mappings in hyperbolic spaces*, Nonlinear Funct Anal. Appl., **27**(3) (2022), 449–469.

[19] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R. Gupta, *Convergence theorems for generalized nonexpansive multivalued mappings in hyperbolic spaces,* SpringerPlus, **5** (2016), 912, DOI 10.1186/s40064-016-2557-y.

[20] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Tran. Amer. Math. Soc., **357**(1) (2005), 89–128.

[21] L. Leuştean, *A quadratic rate of asymptotic regularity for CAT(0) space,* J. Math. Anal. Appl., **325**(1) (2007), 386–399.

[22] W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., **4** (1953), 506–510.

[23] J. Markin, *Continuous dependence of fixed point sets,* Proc. Amer. Math. Soc., **38**(1973), 545–547.

[24] S.B. Nadler, *Multivalued contraction mappings,* Pacific J. Math., **30** (1969), 475–488.

[25] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251**(1) (2000), 217–229.

[26] A.E. Ofem, J.A. Abuchu, R. George, G.C. Ugwunnadi and O.K. Narain, *Some New Results on Convergence, Weak w2-Stability and Data Dependence of Two Multivalued Almost Contractive Mappings in Hyperbolic Spaces*, Mathematics, **10**(20) (2022), 3720.

[27] A.E. Ofem, J.A. Abuchu, G.C. Ugwunnadi, H. Isik and O.K. Narian, *On a four-step iterative algorithm and its application to delay integral equations in hyperbolic spaces,* Rend. Circ. Mat. Palermo, II. Ser., (2023), https://doi.org/10.1007/s12215-023-00908-1.

[28] A.E. Ofem, H. Isik, G.C. Ugwunnadi, R. George and O.K. Narain, *Approximating the solution of a nonlinear delay integral equation by an efficient iterative algorithm in hyperbolic spaces*, AIMS Mathematics, **8**(7) (2023), 14919–14950.

[29] A.E. Ofem, U.E. Udofia and D.I. Igbokwe, *A robust iterative approach for solving nonlinear Volterra Delay integro-differential equations,* Ural Math. J., **7**(2) (2021), 59–85.

[30] G.A. Okeke, A.E. Ofem, T. Abdeljawad, M.A. Alqudah and A. Khan, *A solution of a nonlinear Volterra integral equation with delay via a faster iteration method,* AIMS Mathematics, **8**(1) (2023), 102–124.

[31] W. Phuengrattana and S. Suantai, *On the rate off convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval,* J. Comput. Appl. Math., **235** (2011), 3006–3014.

[32] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces,* Nonlinear Anal., **15** (1990), 537–558.

[33] K. Sokhuma and K. Sokhuma, *Convergence theorems for two nonlinear mappings in $CAT(0)$ spaces*, Nonlinear Funct Anal. Appl., **27**(3) (2021), 499–512.

[34] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340**(2) (2008), 1088–1095.

[35] B.S. Thakurr, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning of fixed points of Suzuki's generalized nonexpansive mappings,* Appl. Math. Comput., **275**(15) (2016), 147–155.

[36] I. Timis, *On the weak stability of Picard iteration for some contractive type mappings,* Annals of the University of Craiova, Math. Comput. Sci. Series, **37**(2) (2010), 106–114.

[37] V. Vairaperumal, *Common fixed point theorems under rational contractions in complex valued extended b-metric spaces*, Nonlinear Funct Anal. Appl., **26**(4) (2021), 685–700.