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CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH SYMMETRIC q-DERIVATIVE OPERATOR

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Abstract. The aim of this paper is to study certain subclass $\widetilde{\mathcal{S}}_{\Sigma}^{q}(\lambda, \alpha, t, s, p, b)$ of analytic and bi-univalent functions which are defined by using symmetric q-derivative operator. We estimate the second and third coefficients of the Taylor-Maclaurin series expansions belonging to the subclass and upper bounds for Feketo-Szegö inequality. Furthermore, some relevant connections of certain special cases of the main results with those in several earlier works are also pointed out.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in the unit disk $\mathbb{U}$.

For analytic functions f and g with $f(0) = g(0)$, f is said to be subordinate to g if there exists an analytic function ω on $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. The subordination will be denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad \text{in } \mathbb{U}.$$

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Note that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ when g is univalent in $\mathbb{U}$.

The well-known Koebe one-quarter theorem [10] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Hence every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z(z \in U)$ and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f); r_0(f) \geq 1/4),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-uninalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of the class Σ, see [24].

In 1967, Lewin [17] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Netanyahu [21] showed that $\max_{f\in\Sigma} |a_2| = 4/3$ and Suffridge [26] has given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. A brief summery of functions in the family Σ can be found in the study of Srivastava *et al.* [24], which is a basic research on the bi-univalent function family Σ (also, see the references cited therein). In a number of sequels to [24], bounds for the first two coefficients $|a_2|$ and $|a_3|$ of different subclasses of bi-univalent functions were given, for example, see [1, 2, 11, 12, 18, 23, 25]. But the coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \cdots\})$ is still an open problem. In recent years, Srivastava *et al.*'s pioneering research on the subject [24] has successfully revitalized the study of bi-univalent functions to have produced numerous bi-univalent function papers. There are also several papers dealing with bi-univalent functions defined by subordination, for example, see [3, 4, 7].

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. The Chebyshev polynomials of the first and second kinds are well known (see [9, 19, 20, 22]). Recently, Kizilateş *et al.* [16] defined (p, b)-Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

For any integer $n \geq 2$ and $0 < b < p \leq 1$, the (p, b)-Chebyshev polynomials of the second kind is defined by the following recurrence relation:

$$U_n(t, s, p, b) = (p^n + b^n)tU_{n-1}(t, s, p, b) + (pb)^{n-1}sU_{n-2}(t, s, p, b) \qquad (1.3)$$

with the initial values $U_0(t,s,p,b) = 1$, $U_1(t,s,p,b) = (p+b)t$ and s is a real variable. Also, it follows readily from (1.3) that

$$U_2(t,s,p,b) = t^2(p+b)(p^2+b^2) + pbs, \tag{1.4}$$
$$U_3(t,s,p,b) = t^3(p+b)(p^2+b^2)(p^3+b^3) + pbst(p^3+b^3) + (pb)^2st(p+b), \cdots .$$

By assuming various values of t, s, p and b, we get some interesting polynomials as follows (see [8, 16]):

(i) When $t = t/2$, $s = s$, $p = p$ and $b = q$, the (p,b)-Chebyshev polynomials of the second kind becomes (p,q)-Fibonacci polynomials.

(ii) When $t = t$, $s = -1$, $p = 1$ and $b = q$, the (p,b)-Chebyshev polynomials of the second kind becomes q-Chebyshev polynomials of the second kind.

(iii) When $t = t$, $s = 1$, $p = 1$ and $b = 1$, the (p,b)-Chebyshev polynomials of the second kind becomes Pell polynomials.

(iv) When $t = 1/2$, $s = 2y$, $p = 1$ and $b = 1$, the (p,b)-Chebyshev polynomials of the second kind becomes Jacobi polynomials.

The generating function of the (p,b)-Chebyshev polynomials of the second kind is as follows:

$$\begin{aligned} G_{p,b}(z) &= \frac{1}{1 - tpz\eta_p - tbz\eta_b - spbz^2\eta_{p,b}} \\ &= \sum_{n=0}^{\infty} U_n(t,s,p,b)z^n \qquad (z \in \mathbb{U}), \end{aligned} \tag{1.5}$$

where the Fibonacci operator η_b was introduced by Mason and Handscomb [20], by $\eta_b f(z) = f(bz)$. Similarly, we define another operator $\eta_{p,b}f(z) = f(pbz)$.

A q-analog of the class of starlike functions was first introduced in [13] by means of the q-difference operator $\mathcal{D}_q f(z)$ acting on functions $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by (see [14, 15])

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0), \tag{1.6}$$

where, $\mathcal{D}_q f(0) = f'(0)$ and $\mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z))$. From (1.6), we have

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. If $q \to 1^-$, $[n]_q \to n$. For a function $h(z) = z^n$, we observe that

$$\mathcal{D}_q(h(z) = \mathcal{D}_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

$$\lim_{q \to 1^{-1}} \left(\mathcal{D}_q(h(z))\right) = \lim_{q \to 1^{-1}} \left([n]_q z^{n-1}\right) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative.

With the aid of above definition, Brahim and Sidomou [5] introduced the symmetric q-derivative $\widetilde{\mathcal{D}}_q f$ which is defined as follows:

$$\widetilde{\mathcal{D}}_q f(z) = \begin{cases} \dfrac{f(qz) - f(q^{-1}z)}{(q-q^{-1})z} & (z \neq 0), \\ f'(0) & (z = 0). \end{cases} \tag{1.7}$$

From (1.7), we deduce that $\widetilde{\mathcal{D}}_q z^n = \widetilde{[n]}_q z^{n-1}$ and a power series of $\widetilde{\mathcal{D}}_q f$ is

$$\widetilde{\mathcal{D}}_q f(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1},$$

when f has the form (1.1) and the symbol $\widetilde{[n]}_q$ denotes the number $\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. It is easy to check that the following properties hold:

$$\begin{aligned} \widetilde{\mathcal{D}}_q(f(z) + g(z)) &= \widetilde{\mathcal{D}}_q f(z) + \widetilde{\mathcal{D}}_q g(z), \\ \widetilde{\mathcal{D}}_q(f(z)g(z)) &= g(q^{-1}z)\widetilde{\mathcal{D}}_q f(z) + f(qz)\widetilde{\mathcal{D}}_q g(z) \\ &= g(qz)\widetilde{\mathcal{D}}_q f(z) + f(q^{-1}z)\widetilde{\mathcal{D}}_q g(z). \end{aligned}$$

We note that

$$\widetilde{\mathcal{D}}_q f(z) = \mathcal{D}_{q^2} f(q^{-1}z).$$

Indeed, it follows From (1.2) and (1.7) that

$$\begin{aligned} \widetilde{\mathcal{D}}_q\ g(w) &= \frac{g(qw) - g(q^{-1}w)}{(q-q^{-1})w} \\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q (2a_2^2 - a_3) w^2 - \widetilde{[4]}_q (5a_2^3 - 5a_2 a_3 + a_4) w^3 + \cdots . \end{aligned}$$

Definition 1.1. Let $0 \leq \lambda \leq 1$, $0 < b < p \leq 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $\frac{1}{2} < t < 1$. A function $f \in \Sigma$ is said to be in the subclass $\widetilde{\mathcal{S}}_{\Sigma}^{q}(\lambda, \alpha, t, s, p, b)$, if the following conditions are satisfied:

$$1+\frac{1}{\alpha}\left(\frac{z\widetilde{\mathcal{D}}_q f(z)}{(1-\lambda)z+\lambda f(z)}-1\right)\prec G_{p,b}(z)\quad(z\in\mathbb{U})$$

and

$$1+\frac{1}{\alpha}\left(\frac{w\widetilde{\mathcal{D}}_q g(w)}{(1-\lambda)w+\lambda g(w)}-1\right)\prec G_{p,b}(w)\quad(w\in\mathbb{U}),$$

where $g=f^{-1}$ and $G_{p,b}$ is given by (1.5).

The object of the present paper is to study the Chebyshev polynomial expansions to provide estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclass $\widetilde{\mathcal{S}}^q_\Sigma(\lambda,\alpha,t,s,p,b)$. Also, we investigate the Feketo-Szegö inequalities for the class $\widetilde{\mathcal{S}}^q_\Sigma(\lambda,\alpha,t,s,p,b)$.

2. Coefficient estimates for the subclass $\widetilde{\mathcal{S}}^q_\Sigma(\lambda,\alpha,t,s,p,b)$

In order to establish our results, we shall need the following lemma.

Lemma 2.1. ([10]) *Let $\mathcal{P}$ be the class of all analytic functions h in $\mathbb{U}$ of the form:*

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n,$$

which satisfy $\operatorname{Re}(h(z))>0$ *for all $z\in\mathbb{U}$. Then if $h\in\mathcal{P}$, then $|c_n|\le 2$ $(n\in\mathbb{N})$.*

We begin by proving the following result.

Theorem 2.2. *Let $0\le\lambda\le1$, $0<b<p\le1$, $\alpha\in\mathbb{C}\setminus\{0\}$ and $\frac{1}{2}<t<1$. If the function $f(z)$ given by* (1.1) *belongs to $\widetilde{\mathcal{S}}^q_\Sigma(\lambda,\alpha,t,s,p,b)$, then*

$$|a_2|\le\frac{|\alpha|(p+b)^{\frac{3}{2}}\ t\sqrt{t}}{\sqrt{|\varphi^\alpha_\lambda(t,s,p,b)|}}\tag{2.1}$$

and

$$|a_3|\le\frac{(p+b)^2|\alpha|^2t^2}{(\widetilde{[2]}_q-\lambda)^2}+\frac{(p+b)|\alpha|t}{(\widetilde{[3]}_q-\lambda)},\tag{2.2}$$

where

$$\begin{aligned}\varphi^\alpha_\lambda(t,s,p,b)=&(p+b)t^2[\alpha(p+b)(\lambda^2-(1+\widetilde{[2]}_q)\lambda+\widetilde{[3]}_q)-(\widetilde{[2]}_q-\lambda)^2(p^2+b^2)]\\&+(\widetilde{[2]}_q-\lambda)^2[(p+b)t-pbs].\end{aligned}\tag{2.3}$$

Proof. Let $f \in \widetilde{\mathcal{S}}^q_\Sigma(\lambda,\alpha,t,s,p,b)$ and g be the analytic function of f^{-1} to $\mathbb{U}$. Then there exist two functions ϕ and ψ, analytic in $\mathbb{U}$ with $\phi(0)=\psi(0)=0$, $|\phi(z)|<1$ and $|\psi(w)|<1$ $(z,w\in\mathbb{U})$ such that

$$1+\frac{1}{\alpha}\left(\frac{z\widetilde{\mathcal{D}}_q f(z)}{(1-\lambda)z+\lambda f(z)}-1\right)=G_{p,b}(\phi(z)) \quad (z\in\mathbb{U}) \tag{2.4}$$

and

$$1+\frac{1}{\alpha}\left(\frac{w\widetilde{\mathcal{D}}_q g(w)}{(1-\lambda)w+\lambda g(w)}-1\right)=G_{p,b}(\psi(w)) \quad (w\in\mathbb{U}). \tag{2.5}$$

Next, we define the function $p,q\in\mathcal{P}$ by

$$p(z)=\frac{1+\phi(z)}{1-\phi(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$q(w)=\frac{1+\psi(w)}{1-\psi(w)}=1+q_1w+q_2w^2+\cdots$$

or equivalently,

$$\phi(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\cdots \tag{2.6}$$

and

$$\psi(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{1}{2}q_1^2\right)w^2+\cdots. \tag{2.7}$$

Using (2.6) and (2.7) together with (1.5), it follows that

$$\begin{aligned}G_{p,b}(\phi(z)) =&1+\frac{U_1(t,s,p,b)}{2}p_1z\\ &+\left(\frac{U_1(t,s,p,b)}{2}(p_2-\frac{1}{2}p_1^2)+\frac{U_2(t,s,p,b)}{4}p_1^2\right)z^2+\cdots\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}G_{p,b}(\psi(w)) =&1+\frac{U_1(t,s,p,b)}{2}q_1w\\ &+\left(\frac{U_1(t,s,p,b)}{2}(q_2-\frac{1}{2}q_1^2)+\frac{U_2(t,s,p,b)}{4}q_1^2\right)w^2+\cdots.\end{aligned} \tag{2.9}$$

By equating the coefficients from (2.4), (2.5), (2.8) and (2.9), we have

$$\frac{1}{\alpha}(\widetilde{[2]}_q-\lambda)a_2=\frac{U_1(t,s,p,b)}{2}p_1, \tag{2.10}$$

$$\frac{1}{\alpha}\left[(\widetilde{[3]}_q-\lambda)a_3+(\lambda^2-\widetilde{[2]}_q\lambda)a_2^2\right]=\frac{U_1(t,s,p,b)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t,s,p,b)}{4}p_1^2, \tag{2.11}$$

$$-\frac{1}{\alpha}(\widetilde{[2]}_q - \lambda)a_2 = \frac{U_1(t,s,p,b)}{2}q_1, \tag{2.12}$$

and

$$\frac{1}{\alpha}\left[(\lambda^2 - \widetilde{[2]}_q\lambda)a_2^2 + (\widetilde{[3]}_q - \lambda)(2a_2^2 - a_3)\right] = \frac{U_1(t,s,p,b)}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{U_2(t,s,p,b)}{4}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we find that

$$p_1 = -q_1, \tag{2.14}$$

$$\frac{2}{\alpha^2}(\widetilde{[2]}_q - \lambda)^2 a_2^2 = \frac{U_1^2(t,s,p,b)}{4}(p_1^2 + q_1^2). \tag{2.15}$$

If we add (2.11) to (2.13), then

$$\frac{2}{\alpha}\left(\lambda^2 - (\widetilde{[2]}_q + 1)\lambda + \widetilde{[3]}_q\right)a_2^2 = \frac{U_1(t,s,p,b)}{2}(p_2 + q_2) + \frac{U_2(t,s,p,b) - U_1(t,s,p,b)}{4}(p_1^2 + q_1^2). \tag{2.16}$$

By using (2.15) in equality (2.16), we observe that

$$\frac{2}{\alpha}\left(\lambda^2 - (\widetilde{[2]}_q + 1)\lambda + \widetilde{[3]}_q - \frac{(\widetilde{[2]}_q - \lambda)^2(U_2(t,s,p,b) - U_1(t,s,p,b))}{\alpha U_1^2(t,s,p,b)}\right)a_2^2 = \frac{U_1(t,s,p,b)}{2}(p_2 + q_2). \tag{2.17}$$

Then, by applying (1.3), (1.4) and Lemma 2.1 to (2.17), we obtain the inequality (2.1).

Next, if we subtract (2.13) from (2.11), we have

$$\frac{2}{\alpha}(\widetilde{[3]}_q - \lambda)(a_3 - a_2^2) = \frac{U_1(t,s,p,b)}{2}(p_2 - q_2) + \frac{U_2(t,s,p,b) - U_1(t,s,p,b)}{4}(p_1^2 - q_1^2). \tag{2.18}$$

By applying (2.14), (2.15) and (2.18), it is evident that

$$a_3 = \frac{\alpha^2 U_1^2(t,s,p,b)}{8(\widetilde{[2]}_q - \lambda)^2}(p_1^2 + q_1^2) + \frac{\alpha U_1(t,s,p,b)}{4(\widetilde{[3]}_q - \lambda)}(p_2 - q_2). \tag{2.19}$$

Hence, by using (1.3) and Lemma 2.1 to (2.19), we get the inequality (2.2). This completes the proof. □

By putting $p = 1$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let* $0 \leq \lambda \leq 1$, $0 < b < 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\widetilde{\mathcal{S}_{\Sigma}^{q}}(\lambda, \alpha, t, s, 1, b)$, *then*

$$|a_2| \leq \frac{|\alpha|(1+b)^{\frac{3}{2}}\ t\sqrt{t}}{\sqrt{|\zeta_{\lambda}^{\alpha}(t,s,b)|}}$$

and

$$|a_3| \leq \frac{(1+b)^2|\alpha|^2t^2}{(\widetilde{[2]}_q - \lambda)^2} + \frac{(1+b)|\alpha|t}{(\widetilde{[3]}_q - \lambda)},$$

where

$$\begin{aligned}\zeta_{\lambda}^{\alpha}(t,s,b) =&(1+b)t^2[\alpha(1+b)(\lambda^2 - (1+\widetilde{[2]}_q)\lambda + \widetilde{[3]}_q) - (\widetilde{[2]}_q - \lambda)^2(1+b^2)] \\ &+ (\widetilde{[2]}_q - \lambda)^2[(1+b)t - bs]. \end{aligned} \tag{2.20}$$

Remark 2.4. Taking $\lambda = 0$, $\alpha = b = 1$ and $s = -1$ in Corollary 2.3, we obtain a recent result due to Altinkaya and Yalçin [3, Theorem 7].

Theorem 2.5. *Let* $0 \leq \lambda \leq 1$, $0 < b < p \leq 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$, *and let* $\mu \in \mathbb{R}$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\widetilde{\mathcal{S}_{\Sigma}^{q}}(\lambda, \alpha, t, s, p, b)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(p+b)|\alpha|t}{(\widetilde{[3]}_q - \lambda)}, \ |\mu - 1| \leq \dfrac{|\varphi_{\lambda}^{\alpha}(t,s,p,b)|}{(\widetilde{[3]}_q - \lambda)(p+b)^2|\alpha|t^2}, \\ \\ \dfrac{(p+b)^3|\alpha|^2|1-\mu|t^3}{|\varphi_{\lambda}^{\alpha}(t,s,p,b)|}, \ |\mu - 1| \geq \dfrac{|\varphi_{\lambda}^{\alpha}(t,s,p,b)|}{(\widetilde{[3]}_q - \lambda)(p+b)^2|\alpha|t^2}, \end{cases} \tag{2.21}$$

where $\varphi_{\lambda}^{\alpha}(t,s,p,b)$ *is given by* (2.3).

Proof. From (2.17) and (2.18), it yields

$$\begin{aligned} &a_3 - \mu a_2^2 \\ &= \frac{\alpha\ U_1(t,s,p,b)}{4(\widetilde{[3]}_q - \lambda)}(p_2 - q_2) \\ &\quad + \frac{(1-\mu)\alpha^2 U_1^3(t,s,p,b)(p_2+q_2)}{4[\alpha\ U_1^2(t,s,p,b)(\lambda^2 - (1+\widetilde{[2]}_q)\lambda + \widetilde{[3]}_q) - (\widetilde{[2]}_q - \lambda)^2\ U(t,s,p,b)]} \\ &= \alpha\ U_1(t,s,p,b)\left[\left(h(\mu) + \frac{1}{4(\widetilde{[3]}_q - \lambda)}\right)p_2 + \left(h(\mu) - \frac{1}{4(\widetilde{[3]}_q - \lambda)}\right)q_2\right], \end{aligned}$$

where

$$U(t,s,p,b) = U_2(t,s,p,b) - U_1(t,s,p,b)$$

and

$$h(\mu) = \frac{(1-\mu)\alpha\ U_1^2(t,s,p,b)}{4[\alpha\ U_1^2(t,s,p,b)(\lambda^2 - (1+\widetilde{[2]}_q)\lambda + \widetilde{[3]}_q) - (\widetilde{[2]}_q - \lambda)^2\ U(t,s,p,b)]}.$$

Then, by using Lemma 2.1, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\alpha|\ U_1(t,s,p,b)}{\widetilde{[3]}_q - \lambda}, & 0 \leq |h(\mu)| \leq \dfrac{1}{4(\widetilde{[3]}_q - \lambda)}, \\ \\ 4|\alpha|\ U_1(t,s,p,b)\ |h(\mu)|, & |h(\mu)| \geq \dfrac{1}{4(\widetilde{[3]}_q - \lambda)}. \end{cases} \quad (2.22)$$

So (2.21) can be easily obtained from (1.3) and (2.22). This evidently completes the proof. □

By taking $p = 1$ in Theorem 2.5, we get the following corollary.

Corollary 2.6. *Let $0 \leq \lambda \leq 1$, $0 < b < 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $\frac{1}{2} < t < 1$, and let $\mu \in \mathbb{R}$. If the function $f(z)$ given by (1.1) belongs to $\widetilde{\mathcal{S}}_\Sigma^q(\lambda, \alpha, t, s, 1, b)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(1+b)|\alpha| t}{(\widetilde{[3]}_q - \lambda)}, & |\mu - 1| \leq \dfrac{|\zeta_\lambda^\alpha(t,s,b)|}{(\widetilde{[3]}_q - \lambda)(1+b)^2|\alpha| t^2}, \\ \\ \dfrac{(1+b)^3|\alpha|^2|1-\mu| t^3}{|\zeta_\lambda^\alpha(t,s,b)|}, & |\mu - 1| \geq \dfrac{|\zeta_\lambda^\alpha(t,s,b)|}{(\widetilde{[3]}_q - \lambda)(1+b)^2|\alpha| t^2}, \end{cases}$$

where $\zeta_\lambda^\alpha(t,s,b)$ is given by (2.20).

Remark 2.7. Taking $\lambda = 0$, $\alpha = b = 1$ and $s = -1$ in Corollary 2.6, we get a recent result due to Altinkaya and Yalçin [3, Theorem 9].

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