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OPTIMAL CONTROL PROBLEM FOR HOST-PATHOGEN MODEL

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Abstract. In this paper, we study the distributed optimal control problem of a coupled system of the host-pathogen model. The system consists of the density of the susceptible host, the density of the infected host, and the density of pathogen particles. Our main goal is to minimize the infected density and also to decrease the cost of the drugs administered. First, we prove the existence and uniqueness of solutions for the proposed problem. Then, the existence of the optimal control is established and necessary optimality conditions are also derived.

1. Introduction

For better understanding the mechanism of the spread of infectious disease as well as treatment and prevention, mathematical models play an important role. Bio-mathematicians and mathematical biologists have shown that these diseases can affect the dynamics of communities. The host population is divided into susceptible and infected classes, with one differential equation that represent each class in classical epidemiology. Host-pathogen interactions are the interactions between a host (plants or humans) and pathogen (bacteria or virus).

In 1981, Anderson and May proposed the host-pathogen model and studied about the population of infection without the intra-species competition and also its spatial effects in [2]. Most of the interaction models are formulated

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in relation to time evolution by ordinary differential equations. This present the limitation to not incorporate any spatial position and its possibility that infection can spread over a spatial region. Therefore, many researchers have recognized that spatial structure is also an important factor in the infectious spread. In [8], Dwyer extended the original host-pathogen model with the density-dependent host population dynamics and also incorporated the spatial behavior of the populations. In [23], Wang et al. studied the general system by replacing the constant parameters with spatial dependent parameters and considering the solution dynamics on a general bounded domain with zero-flux boundary conditions. The reaction-diffusion host-pathogen model is analyzed in [21], in which the authors explored how diffusion rates and spatial heterogeneity affect the dynamics of the system by incorporating the frequency-dependent interaction.

Recently, in modern population dynamics, the use of reaction-diffusion (typical spatially extended model) is the simplest mechanism used to model a variety of physical and biological phenomena [3, 9, 11, 12, 17, 22] and the references therein. In this work, we considered the nonlinear coupled system of host-pathogen model with spatial diffusion and also incorporated two control variables as follows:

$$\begin{cases} \partial_t u_1 = D_1 \Delta u_1 + r(1-u_1-u_2)u_1 - \mu u_1 - \beta u_1 u_3 + c_1 u_1, \text{ in } Q_T, \\ \partial_t u_2 = D_2 \Delta u_2 + \beta u_1 u_3 - \nu u_2 - r(u_1+u_2)u_2, \text{ in } Q_T, \\ \partial_t u_3 = \alpha c_2 u_2 - \delta u_3, \text{ in } Q_T, \end{cases} \tag{1.1}$$

with initial and boundary conditions

$$u_i(x,0) = u_{i,0}(x) > 0, \quad i = 1,2,3 \text{ in } \Omega,$$
$$\frac{\partial u_i}{\partial \eta} = 0, \ i = 1,2, \text{ in } \Sigma_T,$$

where $Q_T = \Omega \times (0,T)$, $\Sigma_T = \partial\Omega \times (0,T)$, Ω is an open bounded domain in $\mathbb{R}^d, (d \leq 3)$ with boundary $\partial\Omega$ and η is the unit normal vector on $\partial\Omega$. The mathematical model consists of three physical variables describing the density of susceptible hosts $u_1(x,t)$, the density of infected hosts $u_2(x,t)$ and the density of pathogen particles $u_3(x,t)$ at time t and spatial position x. Here, $D_i > 0, \ i = 1,2$ denotes the constant diffusion coefficients of the corresponding population. The parameters $r, \mu, \beta, \nu, \alpha$ and δ are positive constants, which are shown in Table 1. In (1.1), $u_{i,0}(x), i = 1,2,3$ represent the initial conditions of unknown variables $u_i, i = 1,2,3$ respectively. Further, we have assumed the Neumann boundary conditions on the boundary Σ_T.

Our aim is to minimize the infected hosts and to maximize the density of susceptible hosts. To achieve this, we formulate an optimal control problem with two control interventions c_1 and c_2. In this article, we focused on a partial

TABLE 1. Symbols and description of parameters

Symbol	Description
u_1	Density of susceptible hosts
u_2	Density of infected hosts
u_3	Density of pathogen particles
D_1	Constant diffusion coefficient of susceptible hosts
D_2	Constant diffusion coefficient of infected hosts
r	Reproductive rate of the host
t	Time
μ	Natural death rate of susceptible hosts
β	Transmission rate
c_1	Efficaciousness of drug therapy in blocking off the infection of new cells
c_2	Efficacy of drug therapy in decreasing the production of virus
ν	Death rate of infected hosts
α	Production rate of pathogen particles from the infected hosts
δ	Decay rate of pathogen particles

differential equations (PDE) constrained optimal control problem. The PDE incorporates the dynamics of the population and its control strategies. In the literature, many researchers have developed optimal control problems related to epidemic models without spatial distribution [1, 5, 7, 10, 15, 16, 19], the study of optimal control problems constrained by PDE models with spatial behavior of population are much fewer, among which are Zine et al. [24], Liu et al. [13], Sowndarrajan et al. [18] and the references therein [6].

Our work in this paper is different from others so far reported in the literature. We prove the existence and uniqueness of the model with spatial movement and control variables for a model (1.1). It is to note that apart from the existence of optimal control, we also verified the necessary first-order condition satisfied by the optimal control. Other than the literature mentioned above, for optimal control problems constrained by system of PDEs (1.1) with diffusion operators concerned, it should be emphasized that, to the best of authors knowledge, there is no paper available in the literature. Therefore, in this work, we have made an attempt to study an optimal control problem with PDEs (1.1).

The paper is organized as follows. In Section 2, we prove the existence of a global strong solution for the direct system (1.1). Section 3 is devoted to optimal control, in which we study the existence of optimal solution for our proposed optimal control problem. Then we derive the adjoint problem and the first order optimality conditions using the Lagrangian framework. Finally, we obtain the existence of solution to the adjoint problem.

2. Existence of global solutions for direct problem

In this section, we study the existence of global solutions for direct problem (1.1). We consider the Hilbert space $H = L^2(\Omega)$ and the initial condition $u_0 = (u_{1,0}, u_{2,0}, u_{3,0})$. By $A : D(A) \subset H \to H$ the linear operator

$$Au = (D_1 \Delta u_1, D_2 \Delta u_2, 0), \text{ for all } u = (u_1, u_2, u_3) \in D(A), \quad (2.1)$$

$$D(A) = \left\{ u = (u_1, u_2, u_3) \in (H^2(\Omega))^2 \times L^\infty(\Omega), \ \frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = 0, \ x \in \partial\Omega \right\}$$

and by $f(u(t), t) = (f_1(u, t), f_2(u, t), f_3(u, t))$ the nonlinear term in (1.1) as

$$\begin{cases} f_1(u, t) = r(1 - u_1 - u_2)u_1 - \mu u_1 - \beta u_1 u_3 + c_1 u_1, \\ f_2(u, t) = \beta u_1 u_3 - \nu u_2 - r(u_1 + u_2)u_2, \\ f_3(u, t) = \alpha c_2 u_2 - \delta u_3, \end{cases} \quad (2.2)$$

for all $u \in D(f), t \in [0, T]$, where $D(f) = \{u \in L^2(\Omega), f(u, t) \in L^2(\Omega), \text{ for all } t \in [0, T]\}$. Then the problem (1.1) can be rewritten as

$$\begin{cases} u'(t) = Au(t) + f(u(t), t), \ t \in [0, T], \\ u(0) = u_0. \end{cases} \quad (2.3)$$

Theorem 2.1. ([4, 14, 20]) *Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0−semigroup of contractions on a Banach space X, $f : X \times [0, T] \to X$ be a function, measurable in t and Lipschitz continuous in $x \in X$, uniformly with respect to $t \in [0, T]$.*

(1) *If $u_0 \in X$, then problem (2.3) has a unique mild solution $u \in C([0, T]; X)$ which verifies the equality*

$$u(t) = S(t)u_0 + \int_0^t S(t - s) f(u(s), s) ds, \text{ for all } t \in [0, T].$$

(2) *If in addition X is a Hilbert space, A is self-adjoint and dissipative on X and $u_0 \in D(A)$, then the mild solution is in fact a strong solution and $u \in W^{1,2}(0, T; X) \cap L^2(0, T; D(A))$.*

Let U_{ad} be the set of admissible control functions:

$$U_{ad} = \{(c_1, c_2) \in (L^2(Q_T))^2, 0 \le c_1(x, t) \le 1, 0 \le c_2(x, t) \le 1 \ \text{ a.e on } \ Q_T\}.$$

Theorem 2.2. *Let Ω be a bounded domain from $\mathbb{R}^d$, $d \le 3$, with the boundary of class $C^{2+\alpha}$, $\alpha > 0$. If the given parameters are positive, $c_1, c_2 \in U_{ad}$ and $u_0 = (u_{1,0}, u_{2,0}, u_{3,0}) \in D(A)$, $u_{i,0} > 0$ on Ω, $i = 1, 2, 3$, then the problem (1.1) has a unique (global) strong solution $u = (u_1, u_2, u_3) \in W^{1,2}(0, T; L^2(\Omega))$ such* *that* $u_i \in L^2(0,T;H^2(\Omega)\cap L^\infty(0,T;H^1(\Omega)))$, $i=1,2$, *and* $u_i \in L^\infty(Q_T)$, $u_i > 0$ *on* Q_T *for* $i=1,2,3$. *Moreover, there exists* $C>0$ *independent of the control terms* c_1, c_2 *and of the corresponding state solution* u *such that*

$$\begin{cases} \|\partial_t u_i\|_{L^2(Q_T)} + \|u_i\|_{L^2(0,T;H^2(\Omega))} + \|u_i\|_{H^1(\Omega)} + \|u_i\|_{L^\infty(Q_T)} \le C, \ i=1,2, \\ \|\partial_t u_3\|_{L^2(Q_T)} + \|u_3\|_{L^\infty(Q_T)} + \|u_3(t)\|_{L^2(\Omega)} \le C. \end{cases} \tag{2.4}$$

Proof. One associates to problem (2.3) the so-called truncated problem:

$$u_N'(t) = Au_N(t) + f^N(u_N(t),t), \ t\in[0,T], \ u_N(0) = u_0, \tag{2.5}$$

where $N>0$ is large enough and $f^N(u,t) = (f_1^N(u,t), f_2^N(u,t), f_3^N(u,t))$ is obtained from $f(u,t) = (f_1(u,t), f_2(u,t), f_3(u,t))$ from (2.2) in the following way. If $|u_1| \le N$, then u_1 in $f_1(u_1,u_2,u_3,t)$ remains unchanged. If $u_1 > N$, u_1 from (2.2) is replaced by N. If $u_1 < -N$, then u_1 is replaced by $-N$. Similarly one proceeds for u_2 and u_3. Thus function $f^N = (f_1^N, f_2^N, f_3^N)$ becomes Lipschitz continuous in $u = (u_1,u_2,u_3)$ uniformly with respect to $t\in[0,T]$. According to Theorem 2.1, problem (2.5) admits a unique strong solution $u_N = (u_1^N, u_2^N, u_3^N) \in W^{1,2}(0,T;L^2(\Omega))$ with $u_i^N \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$.

To prove the boundedness of u_N, we take

$$M = \max\{\|f_i^N\|_{L^\infty(Q_T)}, \|u_{i,0}\|_{L^\infty(\Omega)}, \ i=1,2,3\}$$

and therefore function $v_1^N(x,t) = u_1^N(x,t) - Mt - \|u_{1,0}\|_{L^\infty(\Omega)}$ satisfies

$$\begin{cases} \frac{d}{dt} v_1^N(t) = D_1 \Delta v_1^N + f_1^N(u^N,t) - M, \ t\in[0,T], \\ v_1^N(0) = u_{1,0} - \|u_{1,0}\|_{L^\infty(\Omega)}. \end{cases}$$

Then the strong solution of the problem can be written as

$$v_1^N(t) = S(t)(u_{1,0} - \|u_{1,0}\|_{L^\infty}) + \int_0^t S(t-s)(f_1^N(u^N,s) - M)ds,$$

where $\{S(t), \ t\ge 0\}$ is the C_0-semigroup generated by the operator $B : D(B) \subset H \to H$,

$$Bu_1 = D_1 \Delta u_1, \ D(B) = \left\{ u_1 \in H^2(\Omega), \ \frac{\partial u_1}{\partial \eta} = 0 \ a.e. \ on \ \partial\Omega \right\}.$$

Since $u_{1,0} - \|u_{1,0}\|_{L^\infty(\Omega)} \le 0$ and $f_1^N(u^N,s) - M \le 0$, it follows that

$$v_1^N(x,t) \le 0 \ \text{ for all } (x,t) \in Q_T.$$

Similarly, we prove that $w_1^N(x,t) - u_1^N(x,t) + Mt + \|u_{1,0}\|_{L^\infty(\Omega)}$ is nonnegative. Then

$$|u_i^N(x,t)| \le Mt + \|u_{i,0}\|_{L^\infty(\Omega)} \ \text{ for all } (x,t)\in Q_T, \ i=1,2,3. \tag{2.6}$$

Thus we have proved that $u_i^N \in L^\infty(Q_T),\ i=1,2,3$, the upper bound being dependent only on N. To show the positiveness of u_1^N, we write the problem as

$$\begin{cases} \frac{\partial u_1^N}{\partial t} v_1^N(t) = D_1 \Delta u_1^N + f_1^N(u^N,t),\ (x,t)\in Q_T, \\ \frac{\partial u_1^N}{\partial \eta} = 0,\ (x,t)\in\Sigma, \\ u_1^N(x,0) = u_{1,0},\ x\in\Omega. \end{cases} \tag{2.7}$$

We set $u_1^N = (u_1^N)^+ - (u_1^N)^-$ with $(u_1^N)^+(x,t) = \sup\{u_1^N(x,t),0\},\ (u_1^N)^-(x,t) = -\inf\{u_1^N(x,t),0\}$. One multiplies (2.7) by $(u_1^N)^-$, integrating over $\Omega\times[0,t]$ and using Green's formula, we obtain

$$\int_\Omega |(u_1^N)^-|^2 dx \le c\int_0^t\int_\Omega |(u_1^N)^-|^2 dxds,$$

for some constant $c>0$. Gronwall's inequality leads to $u_1^N(x,t)\ge 0$ on Q_T. Since $u_{1,0}>0$ on Ω, one deduces that $u_1^N(x,t)>0,$ for all $(x,t)\in Q_T$. Similarly, we get $u_2^N(x,t)>0$ and $u_3^N(x,t)>0,$ for all $(x,t)\in Q_T$.

If we choose $N > 2\max\{\|u_{i,0}\|_{L^\infty(\Omega)},\ i=1,2,3\}$, there exists $\theta\in(0,T)$ such that $M\theta + \|u_{i,0}\|_{L^\infty(\Omega)} \le \frac{N}{2},\ i=1,2,3$. From (2.6) we derive that $|u_i^N(x,t)|\le N,$ for all $t\in(0,\theta),\ x\in\Omega,\ i=1,2,3$. Thus $f^N=f$ for $t\in(0,\theta)$, so $u^N=(u_1^N,u_2^N,u_3^N)$ is a solution of problem (1.1) defined on $\Omega\times(0,\theta)$. Now, we prove that this local solution is in fact a global solution to our problem. To this end, it sufficient to show that $u_i^N,\ i=1,2,3$ are bounded on $\Omega\times(0,\theta)$. By the third equation from (1.1), together with $u_1,u_3>0$ on Q_T and $u_{3,0}>0$ on Ω, we get $0<u_3(x,t)\le u_{3,0}(x)e^{-\delta t},\ (x,t)\in Q_T$. Hence $u_3\in L^\infty(\Omega\times(0,\theta))$.

By the first equation of (1.1), it follows that $0<u_1(x,t)\le \tilde u_1(x,t),\ (x,t)\in \Omega\times(0,\theta)$, where $\tilde{u_1}$ is the solution of the boundary value problem

$$\begin{cases} \frac{\partial \tilde u_1}{\partial t} = D_1\Delta\tilde u_1 + r\tilde u_1,\ (x,t)\in\Omega\times(0,\theta), \\ \frac{\partial \tilde u_1}{\partial \eta} = 0,\ (x,t)\in\partial\Omega\times(0,\theta), \\ \tilde u_1(x,0) = u_{1,0},\ x\in\Omega. \end{cases}$$

Since $\tilde u_1\in L^\infty(\Omega\times(0,\theta))$, we get $u_1\in L^\infty(\Omega\times(0,\theta))$. Similarly, we can also find that $u_2\in L^\infty(\Omega\times(0,\theta))$. Therefore the solution $u=(u_1,u_2,u_3)$ is defined on the whole set $Q_T,\ u_i\in L^\infty(Q_T),\ u_i\in W^{1,2}(0,T;L^2(\Omega),\ u_i>0)$ on $Q_T,\ i=1,2,3$ and $u_i\in L^2(0,T;H^2(\Omega)),\ i=1,2$.

By the first equation of (1.1) one obtains via Green's formula:

$$\int_0^t\int_\Omega |\partial_t u_1|^2 dsdx + D_1^2\int_0^t\int_\Omega |\Delta u_1|^2 dsdx + 2D_1\int_\Omega |\nabla u_1|^2 dx$$
$$-2D_1\int_\Omega |\nabla u_{1,0}|^2 dx = \int_0^t\int_\Omega u_1^2(r(1-u_1-u_2)-\mu-\beta u_3+c_1)^2 dsdx.$$

Since $u_{1,0} \in H^2(\Omega)$ and $\|u_i\|_{L^\infty(Q_T)},\ i = 1, 2, 3$ are bounded independently of the control terms $c_i,\ i = 1, 2$, we yield that $u_1 \in L^\infty(0, T; H^1(\Omega))$ and the first inequality in (2.4) holds for the case $i = 1$. Case $i = 2$ can be treated similarly as previous. Multiplying the third equation of (1.1) by u_3 and integrating over $\Omega \times [0, t]$. This completes the proof. □

3. Optimal control problem

In this section, we study the existence of optimal control, the derivation of the adjoint problem and also the optimality conditions. Further, the existence of solution for the adjoint problem is also proved. First, we prove the existence of solution for the following optimal control problem:

$$\hat{J}(u_2, c_1, c_2) = \frac{\alpha_1}{2} \int_{Q_T} |u_2 - u_{2Q}|^2 dxdt + \frac{1}{2} \int_{Q_T} (Ac_1^2 + Bc_2^2) dxdt, \tag{3.1}$$

where (c_1, c_2) belongs to the admissible set U_{ad}. Here, J is the cost functional and u_2 is the state variable and c_1 and c_2 are the control variables, while u_{2Q} is the corresponding desired rates belong to $L^2(Q_T)$. Moreover, α_1 and A, B are the positive parameters used to change the relative importance of the terms that appear in the definition of the functional. The goal is to minimize the functional (3.1) subject to state equations with respect to input rate.

3.1. **Existence of control.**

Theorem 3.1. *If the parameters $r, \mu, \beta, \nu, \alpha$ and δ are positive and $u_0 \in D(A)$, $u_{i,0} > 0$, $i = 1, 2, 3$ on Ω, then the optimal control problem* (3.1) *subject to* (1.1) *admits an optimal solution* $(u_1^*, u_2^*, u_3^*, c_1^*, c_2^*)$.

Proof. We denote $\inf J(u, c_1, c_2) = m$, where m is finite, $c_1, c_2 \in U_{ad}$ and u is the solution of (1.1). Therefore, there exists a sequence (u^n, c_1^n, c_2^n) with $c_i^n \in U_{ad},\ i = 1, 2,\ u^n \in W^{1,2}(0, T; L^2(\Omega))$ such that

$$\begin{cases} \partial_t u_1^n = D_1 \Delta u_1^n + r(1 - u_1^n - u_2^n) u_1^n - \mu u_1^n - \beta u_1^n u_3^n + c_1^n u_1^n & \text{in } Q_T, \\ \partial_t u_2^n = D_2 \Delta u_2^n + \beta u_1^n u_3^n - \nu u_2^n - r(u_1^n + u_2^n) u_2^n & \text{in } Q_T, \\ \partial_t u_3^n = \alpha c_2^n u_2^n - \delta u_3^n & \text{in } Q_T, \end{cases} \tag{3.2}$$

with initial and boundary conditions

$$u_i^n(x, 0) = u_{i,0}^n(x), \quad i = 1, 2, 3 \text{ in } \Omega, \text{ and } \frac{\partial u_i^n}{\partial \eta} = 0,\ i = 1, 2 \text{ in } \Sigma_T. \tag{3.3}$$

Therefore, there exists a sequence such that $J(u^n, c_1^n, c_2^n) \to m$ as $m \in [0, +\infty)$, and $m \le J(u^n, c_1^n, c_2^n) \le m + \frac{1}{n}$, for all $n \ge 1$. By Theorem 2.2, there exists a constants $C > 0$ independent of n such that

$$\begin{cases} \|\partial_t u_i^n\|_{L^2(Q_T)} + \|u_i^n\|_{L^2(0,T;H^2(\Omega))} + \|u_i^n(t)\|_{H^1(\Omega)} \leq C, \ i = 1,2, \\ \|u_3^n(t)\|_{L^2(\Omega)} \leq C, \end{cases} \tag{3.4}$$

for all $n \geq 1, \ t \in [0,T]$. The sequence $\{u_i^n\}$ is bounded in $C([0,T;L^2(\Omega))$ and $\{\partial_t u_i^n\}, \ i = 1,2$ is bounded in $L^2(Q_T)$. The third equation from (3.2) gives

$$\int_\Omega (u_3^n)^2(x,t)dx = \int_\Omega (u_{3,0})^2 dx + 2\int_0^t\int_\Omega u_3^n(\alpha c_2^n u_2^n - \delta u_3^n)dxdt \ \text{ for all } t \in [0,T].$$

This implies that for all $t, s \in [0,T]$,

$$\left|\int_\Omega (u_3^n)^2(x,t)dx - \int_\Omega (u_3^n)^2(x,s)dx\right| \leq k|t-s|,$$

so by the Ascoli-Arzela Theorem we derive the existence of a function u_3^* such that $u_3^n \to u_3^*$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$, at least on a subsequence denoted again u_3^n. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we infer that $u_i^n(t)$ is compact in $L^2(\Omega)$, for any $t \in [0,T]$ and for $i = 1,2$. The Ascoli-Arzela Theorem implies that $\{u_i^n\}$ is compact in $C([0,T];L^2(\Omega)), \ i = 1,2$. Hence, selecting further sequences, if necessary, we have $u_i^n \to u_i^*$ in $L^2(\Omega), \ i = 1,2$ uniformly with respect to $t \in [0,T]$. The boundedness of $\{\Delta u_i^n\}$ in $L^2(Q_T)$ implies its weak convergence, namely $\Delta u_i^n \rightharpoonup \Delta u_i^*$ in $L^2(Q_T), \ i = 1,2$. Estimates (3.4) lead to

$$\partial_t u_i^n \rightharpoonup \partial_t u_i^* \text{ in } L^2(Q_T), \ i = 1,2,3,$$

$$u_i^n \rightharpoonup u_i^* \ \text{ weakly star in } \ L^\infty(0,T;H^1(\Omega)), \ i = 1,2,$$

$$u_i^n \rightharpoonup u_i^* \ \text{ in } \ L^2(0,T;H^2(\Omega)), \ i = 1,2.$$

Writing $u_1^n u_2^n - u_1^* u_2^* = (u_1^n - u_1^*)u_2^n + u_1^*(u_2^n - u_2^*)$ and $u_1^n u_3^n - u_1^* u_3^* = (u_1^n - u_1^*)u_3^n + u_1^*(u_3^n - u_3^*)$ and making use of the convergences $u_i^n - u_i^* \to 0, \ i = 1,2,3$ in $L^2(Q_T)$ and of the boundedness of $\{u_2^n\}, \{u_3^n\}$ in $L^\infty(Q_T)$, we get

$$u_1^n u_2^n \to u_1^* u_2^*, \quad u_1^n u_3^n \to u_1^* u_3^* \ \text{ in } L^2(Q_T).$$

We also have $c_1^n \rightharpoonup c_1^*$ and $c_2^n \rightharpoonup c_2^*$ in $L^2(Q_T)$ on a subsequence denoted again $c_i^n, \ i = 1,2$. Since, U_{ad} is a closed and convex set in $L^2(Q_T)$, it is weakly closed, so $(c_1^*, c_2^*) \in U_{ad}$ and as above $c_1^n u_1^n \to c_1^* u_1^*$ and $c_2^n u_2^n \to c_2^* u_2^*$ in $L^2(Q_T)$. Now we pass to the limit in $L^2(Q_T)$ as $n \to \infty$ in (3.2) to deduce that (u^*, c_1^*, c_2^*) is an optimal solution. This completes the proof. □

3.2. **Optimality conditions and dual problem.**

Theorem 3.2. *If* (u_1^*, u_2^*, u_3^*) *is an optimal solution of the direct problem* (1.1) *and* (c_1^*, c_2^*) *is an optimal control pair of* (3.1), *then there exists a solution* (p_1, p_2, p_3) *for the adjoint system subject to boundary and final conditions:*

$$\frac{\partial p_1}{\partial \eta} = \frac{\partial p_2}{\partial \eta} = 0 \text{ on } \Sigma_T \quad \text{and} \quad p_1(x,T) = p_2(x,T) = p_3(x,T) = 0 \text{ in } \Omega.$$

Further, the optimality conditions are given by

$$c_1^* = \max\left\{\min\{-\frac{p_1 u_1}{A}, 1\}, 0\right\} \quad \text{and} \quad c_2^* = \max\left\{\min\{-\frac{\alpha p_3 u_2}{B}, 1\}, 0\right\}.$$

Proof. Defining the Lagrangian function as follows:

$$\begin{aligned} &L\left(u_1, u_2, u_3, p_1, p_2, p_3, c_1, c_2\right) \\ &= \frac{\alpha_1}{2}\int_{Q_T} |u_1 - u_{1Q}|^2 dxdt + \frac{1}{2}\int_{Q_T} (Ac_1^2 + Bc_2^2) dxdt \\ &\quad - \int_{Q_T} p_1[\partial_t u_1 - D_1 \Delta u_1 - r(1 - u_1 - u_2)u_1 + \mu u_1 + \beta u_1 u_3 - c_1 u_1] \\ &\quad - \int_{Q_T} p_2[\partial_t u_2 - D_2 \Delta u_2 - \beta u_1 u_3 + \nu u_2 + r(u_1 + u_2)u_2] \\ &\quad - \int_{Q_T} p_3[\partial_t u_3 - \alpha c_2 u_2 + \delta u_3]. \end{aligned}$$

The first order optimality system is given by the Karush-Kuhn-Tucker(KKT) conditions which results from equating the partial derivatives of the Lagrangian $L\left(u_1, u_2, u_3, p_1, p_2, p_3, c_1, c_2\right)$ with respect to u_1, u_2 and u_3 equal to zero. Now,

$$\begin{cases} \partial_t p_1 = -D_1 \Delta p_1 - r(1 - 2u_1 - u_2)p_1 + \mu p_1 + \beta u_3 p_1 - c_1 p_1 - \beta u_3 p_2 \\ \qquad\quad + r u_2 p_2 - \alpha_1(u_1 - u_{1Q}), \\ \partial_t p_2 = -D_2 \Delta p_2 + \nu p_2 + 2r u_2 p_2 - p_3 \alpha c_2 + p_1 r u_1, \\ \partial_t p_3 = p_3 \delta + p_1 \beta u_1 - p_2 \beta u_1, \end{cases} \tag{3.5}$$

with boundary and final conditions:

$$\frac{\partial p_i}{\partial \eta} = 0, \ i = 1, 2 \text{ on } \Sigma_T, \tag{3.6}$$

$$p_i(T) = 0, \ i = 1, 2, 3 \text{ in } \Omega. \tag{3.7}$$

The system (3.5)-(3.6) is the required adjoint problem for the given optimal control problem (3.1) with system of PDE constraints (1.1). Further, to find the optimality conditions, we calculate the gradient of the $J(c_1, c_2)$:

$$\nabla J(c_1) = \frac{\partial L}{\partial c_1} = \int_{Q_T} (Ac_1 + p_1 u_1) dxdt$$

and

$$\nabla J(c_2) = \frac{\partial L}{\partial c_2} = \int_{Q_T} (Bc_2 + \alpha p_3 u_2) dxdt.$$

Using the property of control space and the compact notation, the optimality condition can be written as

$$c_1^* = \max\left\{\min\{-\frac{p_1 u_1}{A}, 1\}, 0\right\} \quad \text{and} \quad c_2^* = \max\left\{\min\{-\frac{\alpha p_3 u_2}{B}, 1\}, 0\right\}.$$

□

3.3. Existence of the solution of adjoint problem.

Theorem 3.3. *Under the hypotheses of Theorem* 2.2, *if* (u^*, c_1^*, c_2^*) *is an optimal pair, then the adjoint system* (3.5)-(3.6) *admits a unique strong solution* $p = (p_1, p_2, p_3) \in W^{1,2}(0, T; L^2(\Omega))$ *with* $p \in L^\infty(\Omega)$ *and* $p_i \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)),\ i = 1, 2.$

It is easy to establish the existence of a strong solution to the given adjoint system. This can be proved by making the change of variables $s = T - t$ and the change of functions $q_i(x, s) = p_i(x, T - s) = p_i(x, t),\ (x, t) \in Q_T,\ i = 1, 2, 3$ and applying the same method as in the proof of Theorem 2.2.

4. Conclusion

In this paper, we have examined the distributed optimal control problem for the host-pathogen interaction model constrained by a reaction-diffusion system. The mathematical model consists of three coupled equations involving the density of susceptible hosts, the density of infected hosts, and the density of pathogen particles. It describes the importance of the effect of spatial movement and distinct diffusion rates on the dynamics of a diffusive host-pathogen system. We aim to minimize the density of infected cells and also to decrease the cost of the drugs administered. First, we proved the existence of a global strong solution for the direct problem. Further, we proved the existence of optimal control and derived the necessary optimality conditions. Finally, we discussed the existence of a strong solution to the adjoint system. We note that obtaining the first-order necessary condition for an optimal control problem is more important because it is useful for designing feedback controls and also in the development of more efficient and faster numerical simulations of optimal control algorithms. Moreover, it would be interesting to explore the effects of controls using the numerical simulations with additional optimal control measures and perform a sensitivity analysis on the parameters can be our future direction of the work. Based on the ideas in the current paper, we shall investigate this more realistic problem in the near future.

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