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A FIXED POINT APPROACH TO THE STABILITY OF A GENERAL QUINTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of a general quintic functional equation, $\sum_{k=0}^{6}(-1)^k {}_6\mathrm{C}_k f(x+(3-k)y)=0$, by using the fixed point method.

1. Introduction

In 1940, Ulam [11] asked the following question about the stability of group homomorphisms:

What are the conditions under which an exact solution exists near each approximate solution of a given functional equation?

In 1941, Hyers [6] gave an affirmative answer to this question for additive mappings between Banach spaces. Thereafter, a number of mathematicians came to deal with this question (see [5, 8, 9, 10]).

The following functional equation:

$$\sum_{k=0}^{6}(-1)^k {}_6\mathrm{C}_k f\big(x+(3-k)y\big)=0 \tag{1.1}$$

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is a kind of the general quintic functional equation, where ${}_n\mathrm{C}_k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Every solution of equation (1.1) is called the general quintic mapping. For example, the mapping $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where a_5, a_4, a_3, a_2, a_1, and a_0 are real constants, is a general quintic mapping as we can easily check that it is a solution of the functional equation (1.1).

To the best of our knowledge, Cădariu and Radu were the first to prove the Hyers-Ulam stability of functional equations using the fixed point method (see [1, 2, 3]).

In this paper, we apply the fixed point method to prove the generalized Hyers-Ulam stability of a general quintic functional equation (1.1). More precisely, starting from an arbitrary approximate solution f of a general quintic functional equation (1.1), we will explicitly construct an exact solution F of this equation using the formula:

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_n\mathrm{C}_i \Bigg(\frac{(-1)^{n-i}20^i}{64^n} f_e\big(2^{2n-i}x\big) + \sum_{j=0}^{i} \frac{{}_i\mathrm{C}_j(-42)^{i-j}336^j}{512^n} f_o\big(2^{3n-i-j}x\big) \Bigg)$$

or

$$F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_n\mathrm{C}_i \Bigg(\sum_{j=0}^{i} {}_i\mathrm{C}_j 42^j(-336)^{i-j}512^{n-i} f_o\left(\frac{x}{2^{3n-i-j}}\right) + 20^i(-64)^{n-i} f_e\left(\frac{x}{2^{2n-i}}\right) \Bigg),$$

depending on the given conditions, where f_e is the even part and f_o is the odd part of the function f. Furthermore, we will prove that the constructed function F is quite close to the function f with respect to the sup-norm.

Throughout the paper, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{N}_0$ the set of all nonnegative integers.

2. Preliminaries

Let S be a set. A function $d : S \times S \to [0, \infty]$ is called a generalized metric on S, if d satisfies the following conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in S$;
(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in S$.

In the case, (S, d) is called a generalized metric space. The only real difference between generalized and traditional metrics is that infinity belongs to the range of generalized metrics.

We shall now introduce a theorem of Diaz and Margolis [4]. It has been proved by a number of examples that this theorem is very useful to prove the stability of various functional equations.

Theorem 2.1. ([4]) *Assume that (S, d) is a complete generalized metric space and $J : S \to S$ is a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. Then, for each element $x \in S$, either*

$$d\big(J^n x, J^{n+1} x\big) = \infty \ (\textit{for all } n \in \mathbb{N}_0)$$

or there exists a $k \in \mathbb{N}_0$ that satisfies the following four conditions:

(i) $d\big(J^n x, J^{n+1} x\big) < \infty$ *for all integers $n \geq k$;*
(ii) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(iii) *y^* is the unique fixed point of J in $T = \big\{y \in S : d\big(J^k x, y\big) < \infty\big\}$;*
(iv) $d\big(y, y^*\big) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in T$.*

From now on, we set $c_{21} = 4\sqrt{21}\cos\theta - 14$. The following two lemmas are essential to proving the main theorems of our paper. To prove these lemmas, we only need very basic knowledge. But arriving at these lemmas is not easy.

Lemma 2.2. *Let θ be a real number with $0 < \theta < \frac{\pi}{4}$ and $\cos 3\theta = -\frac{17}{21\sqrt{21}}$. Then $1 < c_{21} < 2$ and $c_{21}^3 + 42c_{21}^2 + 336c_{21} = 512$.*

Proof. When $0 < \theta < \frac{\pi}{4}$ and $\cos 3\theta = -\frac{17}{21\sqrt{21}}$, it is easy to confirm that $1.7483 < 3\theta < 1.7484$, $0.5827 < \theta < 0.5828$, and hence $0.8349 < \cos\theta < 0.8350$. Thus, $1.3039 < c_{21} < 1.3059$. (WolframAlpha was used to calculate the above numerical results.)

We can also check that the following equality is true:

$$\begin{aligned}
&c_{21}^3 + 42c_{21}^2 + 336c_{21} \\
&\quad = 1344\sqrt{21}\cos^3\theta - 14112\cos^2\theta + 2352\sqrt{21}\cos\theta - 2744 \\
&\qquad + 14112\cos^2\theta - 4704\sqrt{21}\cos\theta + 8232 \\
&\qquad + 1344\sqrt{21}\cos\theta - 4704 \\
&\quad = 336\sqrt{21}\big(4\cos^3\theta - 3\cos\theta\big) + 784 \\
&\quad = 336\sqrt{21}\cos 3\theta + 784 \\
&\quad = 512,
\end{aligned}$$

which establishes the lemma. □

Now we set $c_{77} = 4\sqrt{77}\cos\theta + 14$.

Lemma 2.3. *Let θ be a real number such that $0 < \theta < \frac{\pi}{4}$ and $\cos 3\theta = \frac{669}{77\sqrt{77}}$. Then $49 < c_{77} < 50$ and $\frac{42}{c_{77}} + \frac{336}{c_{77}^2} + \frac{512}{c_{77}^3} = 1$.*

Proof. When $0 < \theta < \frac{\pi}{4}$ and $\cos 3\theta = \frac{669}{77\sqrt{77}}$, it is easy to show that $0.1406 < 3\theta < 0.1407$, $0.0468 < \theta < 0.0469$, and hence $0.998900 < \cos\theta < 0.998906$. Thus, $49.0612 < c_{77} < 49.0615$. (WolframAlpha was used to calculate the above numerical results.)

We also obtain the following equality

$$\frac{42}{c_{77}} + \frac{336}{c_{77}^2} + \frac{512}{c_{77}^3} = 1$$

by using the equality

$$\begin{aligned}
512 + 336c_{77} + 42c_{77}^2 - c_{77}^3 &= 512 + 1344\sqrt{77}\cos\theta + 4704 + 51744\cos^2\theta \\
&\quad + 4704\sqrt{77}\cos\theta + 8232 - 4928\sqrt{77}\cos^3\theta \\
&\quad - 51744\cos^2\theta - 2352\sqrt{77}\cos\theta - 2744 \\
&= -1232\sqrt{77}\big(4\cos^3\theta - 3\cos\theta\big) + 10704 \\
&= -1232\sqrt{77}\cos 3\theta + 10704 \\
&= -1232\sqrt{77}\frac{669}{77\sqrt{77}} + 10704 \\
&= 0.
\end{aligned}$$

This completes the proof of this lemma. □

3. Main results

Throughout this section, let X and Y be a real vector space and a real Banach space, respectively. For any given mappings $f : X \to Y$ and $\varphi : X \times X \to [0,\infty)$, the following abbreviations will be used:

$$\begin{aligned}
f_e(x) &= \frac{1}{2}\big(f(x) + f(-x)\big), \\
f_o(x) &= \frac{1}{2}\big(f(x) - f(-x)\big), \qquad (3.1) \\
\varphi_e(x,y) &= \frac{1}{2}\big(\varphi(x,y) + \varphi(-x,-y)\big),
\end{aligned}$$

$$\begin{aligned} Df(x,y) &= \sum_{k=0}^{6}(-1)^k {}_6\mathrm{C}_k f\big(x+(3-k)y\big) \\ &= f(x+3y)-6f(x+2y)+15f(x+y)-20f(x) \\ &\quad + 15f(x-y)-6f(x-2y)+f(x-3y). \end{aligned}$$

The following theorem proves the generalized Hyers-Ulam stability of the general quintic functional equation (1.1) using the fixed point method. We note that according to Lemma 2.2, c_{21} is a constant whose value lies between 1 and 2.

Theorem 3.1. *Let θ be given as in Lemma 2.2 and let $\varphi : X \times X \to [0,\infty)$ be a mapping for which there exists a constant $0 < L < 1$ such that*

$$\varphi(2x,2y) \le c_{21} L\varphi(x,y) \tag{3.2}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.3}$$

for all $x, y \in X$, then there exists a unique solution $F : X \to Y$ to the general quintic functional equation (1.1) that satisfies $F(0) = 0$ and the inequality

$$\|f(x)-F(x)\| \le \frac{1}{1-L}\Phi(x) \tag{3.4}$$

for all $x \in X$, where $\Phi(x) = \frac{1}{512}\big(\varphi_e(2x,2x) + 6\varphi_e(3x,x) + 36\varphi_e(2x,x) + 78\varphi_e(x,x)+24\varphi_e(0,x)\big)$ (see (3.1) for φ_e). In particular, the F can be expressed as

$$\begin{aligned} F(x) = \lim_{n\to\infty}\sum_{i=0}^{n} {}_n\mathrm{C}_i \Bigg(&\frac{(-1)^{n-i}20^i}{64^n} f_e\big(2^{2n-i}x\big) \\ &+ \sum_{j=0}^{i} \frac{{}_i\mathrm{C}_j(-42)^{i-j}336^j}{512^n} f_o\big(2^{3n-i-j}x\big)\Bigg) \end{aligned} \tag{3.5}$$

for all $x \in X$.

Proof. Let S be the set of all functions $g : X \to Y$ with $g(0) = 0$. Then we can define a generalized metric on S by

$$d(g,h) = \inf\big\{K \ge 0 : \|g(x)-h(x)\| \le K\Phi(x) \ \text{ for all } x \in X\big\}.$$

It is not difficult to verify that (S,d) is a complete generalized metric space (see the proof of [7, Theorem 3.1]).

Now we consider the mapping $J : S \to S$ defined by

$$
\begin{aligned}
Jg(x) = {} & \frac{496}{1024}g(2x) - \frac{176}{1024}g(-2x) - \frac{50}{1024}g(4x) + \frac{34}{1024}g(-4x) \\
& + \frac{1}{1024}g(8x) - \frac{1}{1024}g(-8x)
\end{aligned}
$$

for all $x \in X$. We can apply the mathematical induction to prove the following equality

$$
J^n g(x) = \sum_{i=0}^{n} {}_n\mathrm{C}_i \left(\frac{(-1)^{n-i}20^i}{64^n} g_e\big(2^{2n-i}x\big) + \sum_{j=0}^{i} \frac{{}_i\mathrm{C}_j(-42)^{i-j}336^j}{512^n} g_o\big(2^{3n-i-j}x\big) \right)
$$

for all $n \in \mathbb{N}_0$ and $x \in X$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of generalized metric d, we have

$$
\begin{aligned}
\|Jg(x) - Jh(x)\| &\le \frac{496}{1024}\|g(2x) - h(-2x)\| + \frac{176}{1024}\|g(-2x) - h(-2x)\| \\
&\quad + \frac{50}{1024}\|g(4x) - h(4x)\| + \frac{34}{1024}\|g(-4x) - h(-4x)\| \\
&\quad + \frac{1}{1024}\|g(8x) - h(8x)\| + \frac{1}{1024}\|g(-8x) - h(-8x)\| \\
&\le K\left(\frac{336}{512}\Phi(2x) + \frac{42}{512}\Phi(4x) + \frac{1}{512}\Phi(8x)\right) \\
&\le K\left(\frac{336c_{21}L}{512} + \frac{42c_{21}^2L^2}{512} + \frac{c_{21}^3L^3}{512}\right)\Phi(x) \\
&\le K\left(\frac{336c_{21}}{512} + \frac{42c_{21}^2}{512} + \frac{c_{21}^3}{512}\right)L\Phi(x) \\
&\le LK\Phi(x)
\end{aligned}
$$

for all $x \in X$, which implies that

$$
d(Jg, Jh) \le Ld(g, h)
$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0 < L < 1$.

Moreover, it follows from (3.3) that

$$\begin{aligned}
&\|f(x) - Jf(x)\| \\
&\quad \le \frac{1}{64}\|Df_e(x,x) + 3Df_e(0,x)\| \\
&\qquad + \frac{1}{512}\|Df_o(2x,2x) + 6Df_o(3x,x) + 36Df_o(2x,x) + 70Df_o(x,x)\| \\
&\quad \le \frac{1}{512}\big(\varphi_e(2x,2x) + 6\varphi_e(3x,x) + 36\varphi_e(2x,x) + 78\varphi_e(x,x) + 24\varphi_e(0,x)\big) \\
&\quad \le \Phi(x)
\end{aligned}$$

for all $x \in X$, which implies that $d(f, Jf) \le 1$ by the definition of d. Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of J in the set $T = \{g \in S : d(f,g) < \infty\}$, which is represented by (3.5) for all $x \in X$. We note that

$$d(f,F) \le \frac{1}{1-L} d(f,Jf) \le \frac{1}{1-L},$$

which implies (3.4).

Using the conditions (3.2) and (3.3), we have the following inequalities

$$\begin{aligned}
&\lim_{n\to\infty}\left\|\sum_{i=0}^{n} {}_n\mathrm{C}_i \frac{(-1)^{n-i}20^i}{64^n} Df_e\big(2^{2n-i}x, 2^{2n-i}y\big)\right\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{64^n}\sum_{i=0}^{n} {}_n\mathrm{C}_i 20^i \varphi_e\big(2^{2n-i}x, 2^{2n-i}y\big) \\
&\quad \le \lim_{n\to\infty} \frac{1}{64^n}\sum_{i=0}^{n} {}_n\mathrm{C}_i 20^i L^{n-i} c_{21}^{n-i} \varphi_e(2^n x, 2^n y) \\
&\quad \le \lim_{n\to\infty} \frac{1}{64^n}(c_{21}L + 20)^n \varphi_e(2^n x, 2^n y) \\
&\quad \le \lim_{n\to\infty} \frac{1}{64^n}(c_{21}L + 20)^n c_{21}^n L^n \varphi_e(x,y) \\
&\quad \le \lim_{n\to\infty} \frac{1}{64^n}\big(c_{21}^2 + 20c_{21}\big)^n L^n \varphi_e(x,y) \\
&\quad \le \lim_{n\to\infty} \frac{1}{512^n}\big(8c_{21}^2 + 160c_{21}\big)^n L^n \varphi_e(x,y) \\
&\quad \le \lim_{n\to\infty} L^n \left(\frac{c_{21}^3 + 42c_{21}^2 + 336c_{21}}{512}\right)^n \varphi_e(x,y) \\
&\quad = \lim_{n\to\infty} L^n \varphi_e(x,y) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{512^n}\left\|\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j(-42)^{i-j}336^j Df_o\big(2^{3n-i-j}x,2^{3n-i-j}y\big)\right)\right\| \\
&\le \lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j 42^{i-j}336^j\varphi_e\big(2^{3n-i-j}x,2^{3n-i-j}y\big)\right) \\
&\le \lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}{}_i\mathrm{C}_j 42^{i-j}c_{21}^{i-j}336^j\right)\varphi_e\big(2^{3n-2i}x,2^{3n-2i}y\big) \\
&\le \lim_{n\to\infty}\frac{1}{512^n}\sum_{i=0}^{n}{}_n\mathrm{C}_i c_{21}^{2n-2i}(42c_{21}+336)^i\varphi_e(2^n x,2^n y) \\
&\le \lim_{n\to\infty}\frac{1}{512^n}\big(c_{21}^2+42c_{21}+336\big)^n\varphi_e(2^n x,2^n y) \\
&= \lim_{n\to\infty}L^n\left(\frac{c_{21}^3+42c_{21}^2+336c_{21}}{512}\right)^n\varphi_e(x,y) \\
&= \lim_{n\to\infty}L^n\varphi_e(x,y)=0
\end{aligned}$$

for all $x,y\in X$, which imply that

$$\begin{aligned}
DF(x,y) &= \lim_{n\to\infty}\left(\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(\sum_{j=0}^{i}\frac{{}_i\mathrm{C}_j(-42)^{i-j}336^j}{512^n}Df_o\big(2^{3n-i-j}x,2^{3n-i-j}y\big)\right.\right. \\
&\qquad\left.\left.+\frac{(-1)^{n-i}20^i}{64^n}Df_e\big(2^{2n-i}x,2^{2n-i}y\big)\right)\right) \\
&= 0
\end{aligned}$$

for all $x,y\in X$.

We note that if F is a solution of the functional equation (1.1) and $F(0)=0$, then F is a fixed point of J due to the equality $F(x)-JF(x)=\frac{1}{64}\big(DF_e(x,x)+3DF_e(0,x)\big)+\frac{1}{512}\big(DF_o(2x,2x)+6DF_o(3x,x)+36DF_o(2x,x)+70DF_o(x,x)\big)$. □

Similar to Theorem 3.1, we apply the fixed point method to prove the generalized Hyers-Ulam stability of the general quintic function equation (1.1). We note that according to Lemma 2.3, c_{77} is a constant whose value lies between 49 and 50.

From now on, for the sake of simplicity, we write $\frac{x}{\alpha}$ instead of $\frac{1}{\alpha}x$ for all vectors x and nonzero real numbers α.

Theorem 3.2. *Let θ be given as in* Lemma 2.3 *and let $\varphi : X \times X \to [0, \infty)$ be a mapping for which there exists a constant $0 < L < 1$ such that*

$$L\varphi(2x, 2y) \geq c_{77}\varphi(x, y) \tag{3.6}$$

for all $x, y \in X$. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and inequality (3.3) *for all $x, y \in X$, then there exists a unique solution $F : X \to Y$ to the general quintic functional equation* (1.1) *that satisfies $F(0) = 0$ and the inequality*

$$\|f(x) - F(x)\| \leq \frac{1}{1 - L}\Psi(x) \tag{3.7}$$

for all $x \in X$, where $\Psi(x) = 2\varphi_e\big(\frac{x}{4}, \frac{x}{4}\big) + 3\varphi_e\big(0, \frac{x}{4}\big) + 64\varphi_e\big(\frac{3x}{8}, \frac{x}{8}\big) + 36\varphi_e\big(\frac{x}{4}, \frac{x}{8}\big) + 70\varphi_e\big(\frac{x}{8}, \frac{x}{8}\big)$ (see (3.1) *for φ_e). In particular, F is expressed as*

$$\begin{aligned} F(x) = \lim_{n\to\infty} \sum_{i=0}^{n} {}_n\mathrm{C}_i \Bigg(\sum_{j=0}^{i} {}_i\mathrm{C}_j 42^j (-336)^{i-j} 512^{n-i} f_o\bigg(\frac{x}{2^{3n-i-j}}\bigg) \\ + 20^i(-64)^{n-i} f_e\bigg(\frac{x}{2^{2n-i}}\bigg)\Bigg) \end{aligned} \tag{3.8}$$

for all $x \in X$.

Proof. Let S be the set of all mappings $g : X \to Y$ with $g(0) = 0$. We define a generalized metric on S by

$$d(g, h) = \inf\big\{K \geq 0 : \|g(x) - h(x)\| \leq K\Psi(x) \ \text{ for all } x \in X\big\}.$$

It is not difficult to verify that (S, d) is a complete generalized metric space (see the proof of [7, Theorem 3.1]).

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) = 31g\bigg(\frac{x}{2}\bigg) - 11g\bigg(\frac{-x}{2}\bigg) - 200g\bigg(\frac{x}{4}\bigg) + 136g\bigg(\frac{-x}{4}\bigg) + 256g\bigg(\frac{x}{8}\bigg) - 256g\bigg(\frac{-x}{8}\bigg)$$

for all $x \in X$. By a similar way to the proof of Theorem 3.1, we can show the truth of the following equality

$$\begin{aligned} J^n g(x) = \sum_{i=0}^{n} {}_n\mathrm{C}_i \Bigg(\sum_{j=0}^{i} {}_i\mathrm{C}_j 42^j (-336)^{i-j} 512^{n-i} g_o\bigg(\frac{x}{2^{3n-i-j}}\bigg) \\ + 20^i(-64)^{n-i} g_e\bigg(\frac{x}{2^{2n-i}}\bigg)\Bigg) \end{aligned}$$

for all $n \in \mathbb{N}_0$ and $x \in X$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of generalized metric d, we have

$$\begin{aligned}
&\|Jg(x) - Jh(x)\| \\
&\leq 31\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + 11\left\|g\left(\frac{-x}{2}\right) - h\left(\frac{-x}{2}\right)\right\| \\
&\quad + 200\left\|g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right)\right\| + 136\left\|g\left(\frac{-x}{4}\right) - h\left(\frac{-x}{4}\right)\right\| \\
&\quad + 256\left\|g\left(\frac{x}{8}\right) - h\left(\frac{x}{8}\right)\right\| + 256\left\|g\left(\frac{-x}{8}\right) - h\left(\frac{-x}{8}\right)\right\| \\
&\leq 42K\Psi\left(\frac{x}{2}\right) + 336K\Psi\left(\frac{x}{4}\right) + 512K\Psi\left(\frac{x}{8}\right) \\
&\leq \frac{42LK\Psi(x)}{c_{77}} + \frac{336L^2K\Psi(x)}{c_{77}^2} + \frac{512L^3K\Psi(x)}{c_{77}^3} \\
&\leq \left(\frac{42}{c_{77}} + \frac{336}{c_{77}^2} + \frac{512}{c_{77}^3}\right)LK\Psi(x) \\
&\leq LK\Psi(x)
\end{aligned}$$

for all $x \in X$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant $0 < L < 1$.

Moreover, by (3.3) we see that

$$\begin{aligned}
&\|f(x) - Jf(x)\| \\
&= \left\|Df_e\left(\frac{x}{4}, \frac{x}{4}\right) + 3Df_e\left(0, \frac{x}{4}\right) + Df_o\left(\frac{x}{4}, \frac{x}{4}\right)\right. \\
&\qquad \left. + 6Df_o\left(\frac{3x}{8}, \frac{x}{8}\right) + 36Df_o\left(\frac{x}{4}, \frac{x}{8}\right) + 70Df_o\left(\frac{x}{8}, \frac{x}{8}\right)\right\| \\
&\leq 2\varphi_e\left(\frac{x}{4}, \frac{x}{4}\right) + 3\varphi_e\left(0, \frac{x}{4}\right) + 6\varphi_e\left(\frac{3x}{8}, \frac{x}{8}\right) + 36\varphi_e\left(\frac{x}{4}, \frac{x}{8}\right) + 70\varphi_e\left(\frac{x}{8}, \frac{x}{8}\right) \\
&= \Psi(x)
\end{aligned}$$

for all $x \in X$. It means that $d(f, Jf) \leq 1 < \infty$ by the definition of d. Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (3.8) for all $x \in X$. We notice that

$$d(f, F) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{1}{1 - L},$$

which implies (3.7).

By the definition of F, together with (3.3) and (3.6), we have the inequalities

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\sum_{i=0}^{n}{}_n\mathrm{C}_i\sum_{j=0}^{i}{}_i\mathrm{C}_j42^j(-336)^{i-j}512^{n-i}Df_o\left(\frac{x}{2^{3n-i-j}},\frac{y}{2^{3n-i-j}}\right)\right\|\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i\sum_{j=0}^{i}{}_i\mathrm{C}_j42^j336^{i-j}512^{n-i}\varphi_e\left(\frac{x}{2^{3n-i-j}},\frac{y}{2^{3n-i-j}}\right)\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i\sum_{j=0}^{i}{}_i\mathrm{C}_j42^j\left(\frac{336L}{c_{77}}\right)^{i-j}512^{n-i}\varphi_e\left(\frac{x}{2^{3n-2i}},\frac{y}{2^{3n-2i}}\right)\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(42+\frac{336L}{c_{77}}\right)^{i}512^{n-i}\varphi_e\left(\frac{x}{2^{3n-2i}},\frac{y}{2^{3n-2i}}\right)\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i\left(42+\frac{336L}{c_{77}}\right)^{i}\left(\frac{512L^2}{c_{77}^2}\right)^{n-i}\varphi_e\left(\frac{x}{2^{n}},\frac{y}{2^{n}}\right)\\
&\leq\lim_{n\to\infty}\left(42+\frac{336L}{c_{77}}+\frac{512L^2}{c_{77}^2}\right)^{n}\left(\frac{L}{c_{77}}\right)^{n}\varphi_e(x,y)\\
&\leq\lim_{n\to\infty}\left(\frac{42}{c_{77}}+\frac{336}{c_{77}^2}+\frac{512}{c_{77}^3}\right)^{n}L^n\varphi_e(x,y)\\
&=\lim_{n\to\infty}L^n\varphi_e(x,y)=0
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{n\to\infty}\left\|\sum_{i=0}^{n}{}_n\mathrm{C}_i20^i(-64)^{n-i}Df_e\left(\frac{x}{2^{2n-i}},\frac{y}{2^{2n-i}}\right)\right\|\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i20^i64^{n-i}\varphi_e\left(\frac{x}{2^{2n-i}},\frac{y}{2^{2n-i}}\right)\\
&\leq\lim_{n\to\infty}\sum_{i=0}^{n}{}_n\mathrm{C}_i20^i\left(\frac{64L}{c_{77}}\right)^{n-i}\varphi_e\left(\frac{x}{2^{n}},\frac{y}{2^{n}}\right)\\
&\leq\lim_{n\to\infty}\left(20+\frac{64L}{c_{77}}\right)^{n}\left(\frac{L}{c_{77}}\right)^{n}\varphi_e(x,y)\\
&\leq\lim_{n\to\infty}\left(\frac{42}{c_{77}}+\frac{336}{c_{77}^2}+\frac{512}{c_{77}^3}\right)^{n}L^n\varphi_e(x,y)\\
&=\lim_{n\to\infty}L^n\varphi_e(x,y)=0
\end{aligned}
$$

for all $x, y \in X$.

Hence, in view of the equality

$$DF(x,y) = \lim_{n\to\infty}\sum_{i=0}^{n} {}_n\mathrm{C}_i \sum_{j=0}^{i} {}_i\mathrm{C}_j 42^j(-336)^{i-j}512^{n-i} f_o\left(\frac{x}{2^{3n-i-j}}, \frac{y}{2^{3n-i-j}}\right) + \lim_{n\to\infty}\sum_{i=0}^{n} {}_n\mathrm{C}_i 20^i(-64)^{n-i} Df_e\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right)$$

for all $x, y \in X$, we may conclude that F is a solution to functional equation (1.1). Notice that if F is a solution to functional equation (1.1), then the equality $F(x) - JF(x) = DF_e\big(\frac{x}{4}, \frac{x}{4}\big) + 3Df_e\big(0, \frac{x}{4}\big) + DF_o\big(\frac{x}{4}, \frac{x}{4}\big) + 6DF_o\big(\frac{3x}{8}, \frac{x}{8}\big) + 36DF_o\big(\frac{x}{4}, \frac{x}{8}\big) + 70DF_o\big(\frac{x}{8}, \frac{x}{8}\big)$ implies that F is a fixed point of J. □

4. Discussion

In this paper, we proved for the first time the generalized Hyers-Ulam stability of a generalized quintic functional equation (1.1) using Diaz and Margolis' theorem (Theorem 2.1), which was first used by Cădariu and Radu to prove the Hyers-Ulam stability of functional equations (see [1, 2, 3]). In this process, the concept of a generalized metric was used.

5. Conclusions

The main results of this paper are summarized as follows: Let θ be a constant that satisfies $\cos 3\theta = -\frac{17}{21\sqrt{21}}$ and $0 < \theta < \frac{\pi}{4}$. We set $c_{21} = 4\sqrt{21}\cos\theta - 14$. Then, $1 < c_{21} < 2$. Let $\varphi : X \times X \to [0, \infty)$ be a mapping for which there exists a constant $0 < L < 1$ such that

$$\varphi(2x, 2y) \le c_{21} L \varphi(x, y)$$

for all $x, y \in X$, and let us define

$$\Phi(x) = \frac{1}{512}\big(\varphi_e(2x, 2x) + 6\varphi_e(3x, x) + 36\varphi_e(2x, x) + 78\varphi_e(x, x) + 24\varphi_e(0, x)\big)$$

for all $x \in X$.

We have proved in Theorem 3.1 the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$. More precisely, Theorem 3.1 states that if a mapping $f : X \to Y$ satisfies $f(0) = 0$ and the inequality

$$\|Df(x, y)\| \le \varphi(x, y)$$

for all $x, y \in X$, then there exists a unique solution $F : X \to Y$ of $DF(x, y) = 0$ that satisfies $F(0) = 0$ and the inequality

$$\|f(x) - F(x)\| \le \frac{1}{1 - L}\Phi(x)$$

for all $x \in X$.

On the other hand, let θ be a constant that satisfies $\cos 3\theta = \frac{669}{77\sqrt{77}}$ and $0 < \theta < \frac{\pi}{4}$. We set $c_{77} = 4\sqrt{77}\cos\theta + 14$. Then, we have $49 < c_{77} < 50$. Assume that $\varphi : X \times X \to [0, \infty)$ is a mapping for which there exists a constant $0 < L < 1$ such that

$$L\varphi(2x, 2y) \geq c_{77}\varphi(x, y)$$

for all $x, y \in X$.

Furthermore, we define

$$\Psi(x) = 2\varphi_e\left(\frac{x}{4}, \frac{x}{4}\right) + 3\varphi_e\left(0, \frac{x}{4}\right) + 64\varphi_e\left(\frac{3x}{8}, \frac{x}{8}\right) + 36\varphi_e\left(\frac{x}{4}, \frac{x}{8}\right) + 70\varphi_e\left(\frac{x}{8}, \frac{x}{8}\right)$$

for any $x \in X$.

We have proved in Theorem 3.2 the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$. More precisely, Theorem 3.2 states that if a mapping $f : X \to Y$ satisfies $f(0) = 0$ and inequality (3.3) for all $x, y \in X$, then there exists a unique solution $F : X \to Y$ of $DF(x, y) = 0$ that satisfies $F(0) = 0$ and the inequality

$$\|f(x) - F(x)\| \leq \frac{1}{1 - L}\Psi(x)$$

for all $x \in X$.

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