Nonlinear Functional Analysis and Applications
Vol. 28, No. 3 (2023), pp. 685-702
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.03.07
http://nfaa.kyungnam.ac.kr/journal-nfaa
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FIXED POINTS OF MULTI-VALUED OSILIKE-BERINDE NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. This paper is concerned with fixed point results of a finite family of multi-valued Osilike-Berinde nonexpansive type mappings in hyperbolic spaces along with some numerical examples. Also strong convergence and $\Delta-$convergence of a sequence generated by Alagoz iteration scheme are investigated.

1. Introduction

Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to K$ is called contraction if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in K$. When $\lambda = 1$, T is called a nonexpansive mapping.

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[0]Received November 10, 2022. Revised May 14, 2023. Accepted May 22, 2023.
[0]2020 Mathematics Subject Classification: 47H09, 47H10.
[0]Keywords: Hyperbolic space, fixed point, Osilike-Berinde nonexpansive mappings, Fejer monotone.
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One of the most important and fruitful result in a metric space was given by Banach [4] called "Banach Contraction Principle". This principle was generalized and its several variants were studied by mathematicians over different spaces.

Note that a subset K of a metric space (X, d) is called proximal if there exists an element $y \in K$ such that

$$d(x, y) = d(x, K) = \inf_{z \in K} d(x, z)$$

for all $x \in X$. Let $CB(K)$ and $P(K)$ be the collection of all nonempty closed bounded subsets and the collection of all nonempty proximal bounded closed subsets of K, respectively.

The concept of Hausdorff metric to approximate fixed points of multi-valued nonexpansive mappings was introduced by Markin [25]. The Hausdorff distance on $CB(K)$ is denoted by $H(.,.)$ and is defined by

$$H(A, B) = \max\{\sup d(x, B), \sup d(A, y)\},$$

where $A, B \in CB(K)$, $d(A, x) = \inf_{a \in A} d(a, x)$.

The concept of contraction mapping was generalized by many researcher. This class of mappings is known as almost contraction mappings. Ali et al. [3] introduced concept of weak contraction. A mapping $T : K \to K$ is called weak contraction if there exists $\zeta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \zeta d(x, y) + Ld(x, Tx) \text{ for all } x, y \in K.$$

Osilike [30] introduced almost contraction in the following manner: there exists $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) + Ld(x, Tx) \text{ for all } x, y \in K.$$

This concept was further extended to multi-valued mappings by Berinde [7] in the following manner: a multi-valued mapping $T : K \to CB(K)$ is called weak contraction if there exists $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$H(Tx, Ty) \leq \lambda d(x, y) + Ld(x, Tx)$$

for all $x, y \in K$. When $\lambda = 1$, T is called Osilike-Berinde nonexpansive mapping. Hence, we observed that a nonexpansive mapping implies Osilike-Berinde nonexpansive mapping. Various generalizations of nonexpansive mappings are given by many authors (refer [1, 13, 19, 27, 29, 32]).

Example 1.1. Let X be a uniformly convex Banach space, whose norm $||.||$ is induced by metric d such that $d(x, y) = ||x-y||$ for all $x, y \in X$. Let $K = [0, 2]$

be a nonempty subset of X and define a mapping $T : K \to P(K)$ by

$$Tx = \begin{cases} [0, \frac{x}{2}],\ 0 \le x \le \frac{1}{2}, \\ \{0\},\ \frac{1}{2} < x \le 1. \end{cases}$$

First, we will show that T is a multi-valued nonexpansive mapping. Consider the following cases:

Case I: when $0 \le x, y \le \frac{1}{2}$. Then $d(x, y) = ||x - y||$ and

$$\begin{aligned} H(Tx, Ty) &= H\left([0, \frac{x}{2}], [0, \frac{y}{2}]\right) \\ &= \frac{1}{2}||x - y|| \\ &\le d(x, y). \end{aligned}$$

Case II: when $\frac{1}{2} < x, y \le 1$. Then $H(Tx, Ty) = 0$. Clearly $H(Tx, Ty) \le d(x, y)$.

Case III: when $0 \le x \le \frac{1}{2}$, $\frac{1}{2} < y \le 1$. Then

$$\begin{aligned} H(Tx, Ty) &= H\left([0, \frac{x}{2}], \{0\}\right) \\ &= ||\frac{x}{2}|| \\ &\le d(x, y). \end{aligned}$$

Case IV: when $0 \le y \le \frac{1}{2}$, $\frac{1}{2} < x \le 1$. By similar procedure as in Case III, we have $H(Tx, Ty) \le d(x, y)$.

Now choose $x = \frac{1}{2}$, $y = 1$. Then

$$\begin{aligned} H(Tx, Ty) &= H\left([0, \frac{x}{2}], \{0\}\right) \\ &= \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} d(x, y) + Ld(x, Tx) &= \frac{1}{2} + Ld\left(x, \frac{x}{2}\right) \\ &= \frac{1}{2} + L\frac{1}{4}. \end{aligned}$$

Clearly $H(Tx, Ty) \le d(x, y) + Ld(x, Tx)$ for $L \ge 0$. It conclude that T is multi-valued Osilike-Berinde nonexpansive mapping.

A mapping T is called multi-valued quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$H(Tx, Ty) \le d(x, y)$$

for all $y \in F(T)$. A point $x \in K$ is called fixed point of multi-valued mapping T if $x \in Tx$. Here we denote the set of fixed point of T by $F(T)$.

Fixed point theory has a vital role in the field of analysis due to its applications in various fields (refer [3, 22, 31, 33, 37]). Several mathematicians studied fixed point results over different spaces. Once the existence result of a fixed point for a mapping is established, an algorithm to find the value of the fixed point is desirable. Banach contraction principle uses Picard iteration to approximate fixed point. In this direction some well-known iterations are Mann [24], Ishikawa [10], Noor [28], Thakur [36], and so on.

In 2020, Sen et al. [13] introduced a new class of nonexpansive mappings, namely generalized $(\alpha, \beta)-$nonexpansive mapping and proved the existence and convergence results for this class of mappings in the framework of uniformly convex Banach spaces by using K iteration scheme given by Hussain et al. [9] as follows:

$$\begin{cases} x_1 \in K, \\ z_k = (1-\beta_k)x_k + \beta_k T x_x, \\ y_k = T((1-\alpha_k)Tx_k + \alpha_k T z_k), \\ x_{k+1} = Ty_k. \end{cases}$$

In 2021, Arshad et al. [12] proved weak and strong convergence of a mapping endowed with property (CSC) in uniformly convex Banach spaces by using K^* iterative scheme given by Ullah and Arshad [38] as follows:

$$\begin{cases} x_1 \in K, \\ z_k = (1-\beta_k)x_k + \beta_k T x_x, \\ y_k = T((1-\alpha_k)z_k + \alpha_k T z_k), \\ x_{k+1} = Ty_k. \end{cases}$$

In 2022, Khan et al. [15] proved some fixed point convergence results for generalized α-nonexpansive mappings in the framework of uniformly convex Banach spaces through KF iteration scheme defined by

$$\begin{cases} x_1 \in K, \\ z_k = T((1-\beta_k)x_k + \beta_k T x_x), \\ y_k = T z_k, \\ x_{k+1} = T((1-\alpha_k)Tx_k + \alpha_k T y_k). \end{cases}$$

In the same year, Junaid et al. [11] proved strong convergence and $\Delta-$convergence of F iteration scheme in hyperbolic space for general class of contractive-like operators. In 2022, Austine et al. [3] proved fixed point results and w^2 stability of Riech-Suzuki type nonexpansive mappings in the framework of Banach space by using AH iterative scheme as follows:

$$\begin{cases} x_1 \in K, \\ z_k = (1-\beta_k)x_k + \beta_k T x_x, \\ y_k = T(Tz_k), \\ w_k = T(Ty_k), \\ x_{k+1} = (1-\alpha_k)w_k + \alpha_k T w_k. \end{cases}$$

Some classical fixed point theorems for single-valued nonexpansive mappings have been extended to multi-valued mappings. The multi-valued version of Banach contraction principle [4] was given by Nadler [26] in 1969. Sastry and Babu [34] introduced multi-valued version of Mann [24] and Ishikawa [10] iteration and proved convergence theorems for nonexpansive mappings in a Hilbert space. In 2016, Kim et al. [21] introduced multi-valued version of Thakur iteration [36] and proved convergence results in a uniformly convex Banach space.

Various iterative schemes are introduced by many authors for single-valued as well as multi-valued mappings. We focus on the iteration scheme given by Alagoz et al. [2] in 2016 for multi-valued mappings. Alagoz et al. [2] studied the convergence of the following iteration scheme: Let K be a nonempty convex subset of a hyperbolic space X. Let $\{T_i : i = 1, 2, ..., k\}$ be a family of multi-valued mappings such that $T_i : K \to P(K)$ and $P_{T_i}(x) = \{y \in T_i x : d(x, y) = d(x, Tx)\}$ is a nonexpansive mapping. Suppose that $\{\alpha_{nk}\}$ is a sequence in $[0, 1]$ for all $n = 1, 2, ...$ and $k = 1, 2, ..., j$. Let for $i = 1, 2, ..., k$, $x_0 \in K$, $\{x_k\}$ be a sequence generated by the following:

$$\begin{cases} x_{k+1} = W(u_{(n-1)k}, y_{(n-1)k}, \alpha_{nk}), \\ y_{(n-1)k} = W(u_{(n-2)k}, y_{(n-2)k}, \alpha_{(n-1)k}), \\ \quad \vdots \\ y_{2k} = W(u_{1k}, y_{1k}, \alpha_{2k}), \\ y_{1k} = W(u_{0k}, y_{0k}, \alpha_{1k}), \end{cases} \tag{1.1}$$

where $u_{ik} \in P_{T_{i+1}}(y_{ik})$, $i = 0, 1, 2, ..., k-1$ and $y_{0k} = x_k$.

After that Bello et al. [6] studied some fixed point results and established demiclosedness principle for mean nonexpansive mappings by using iteration scheme (1.1) in hyperbolic spaces. Inspired by work of Bello et al. [6], our aim in this paper is to establish strong convergence and $\Delta-$convergence of the sequence $\{x_k\}$ defined by (1.1) for Osilike-Berinde type nonexpansive mappings in complete hyperbolic spaces.

2. Preliminaries

This section starts with some basic concepts and also contains some useful results which are required to get main results.

Definition 2.1. ([11]) A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X \times X \times [0,1] \to X$ such that for all $x, y, z \in X$ and $\alpha, \beta \in [0,1]$, we have

(i) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, y) + \alpha d(u, x)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, 1-\alpha)$,
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(z, w) + \alpha d(x, y)$.

Remark 2.2. Banach spaces and $CAT(0)$ spaces are examples of hyperbolic spaces.

Example 2.3. Let $X = \mathbb{R}$ be a Banach space. Let $d : X \times X \to [0, \infty)$ be a mapping defined by

$$d(x, y) = ||x - y||.$$

It is clear that d is metric on X. Let $K = [0,1]$ be a subset of X. Further we define a mapping $W : X \times X \times [0,1]$ by

$$W(x, y, \alpha) = \alpha x + (1-\alpha)y$$

for all $x, y \in X$ and $\alpha \in [0,1]$. Then (X, d, W) is hyperbolic space, in fact,

(i)

$$\begin{aligned} d(u, W(x, y, \alpha)) &= ||u - W(x, y, \alpha)|| \\ &= ||u - \alpha x - (1-\alpha)y|| \\ &= ||(1-\alpha)(u-y) + \alpha(u-x)|| \\ &\leq (1-\alpha)d(u, y) + \alpha d(u, x). \end{aligned}$$

(ii)

$$\begin{aligned} d(W(x, y, \alpha), W(x, y, \beta)) &= ||W(x, y, \alpha) - W(x, y, \beta)|| \\ &= ||\alpha x - \alpha y - \beta x + \beta y|| \\ &= ||(\alpha - \beta)x - (\alpha - \beta)y|| \\ &= |\alpha - \beta| d(x, y). \end{aligned}$$

(iii)

$$\begin{aligned} W(y, x, 1-\alpha)) &= (1-\alpha)y + (1-(1-\alpha))x \\ &= W(x, y, \alpha). \end{aligned}$$

(iv)

$$\begin{aligned} d(W(x,z,\alpha),W(y,s,\alpha)) &= ||(W(x,z,\alpha)-W(y,s,\alpha)|| \\ &= ||(1-\alpha)(z-s)+\alpha(x-y)|| \\ &= (1-\alpha)d(z,s)+\alpha d(x,y). \end{aligned}$$

Definition 2.4. ([8]) A nonempty subset K of a hyperbolic space X is said to be convex if $W(x,y,\alpha)\in K$ for all $x,y\in K$ and $\alpha\in[0,1]$.

Definition 2.5. ([35]) A hyperbolic space X is said to be uniformly convex if for any $r>0$ and $\varepsilon\in(0,2]$, there exists a $\delta\in(0,1]$ such that for all $x,y,z\in X$,

$$d(W(x,y,\frac{1}{2}),z)\leq(1-\delta)r,$$

provided $d(x,z)\leq r$, $d(y,z)\leq r$ and $d(x,y)\geq\varepsilon r$.

Definition 2.6. ([16, 17, 18]) Let K be a nonempty closed subset of a $CAT(0)$ space X and $\{x_k\}$ be any bounded sequence in K. For $x\in X$ there is a continuous functional $r(.,\{x_k\}):X\to[0,\infty)$ defined by

$$r(x,\{x_k\})=\limsup_{k\to\infty}d(x_k,x).$$

The asymptotic radius $r(K,\{x_k\})$ of $\{x_k\}$ with respect to K is given by

$$r(K,\{x_k\})=\inf\{r(x,\{x_k\}):x\in K\}.$$

A point $x\in K$ is said to be an asymptotic center of the sequence $\{x_k\}$ with respect to K, if

$$r(x,\{x_k\})=\inf\{r(y,\{x_k\}):y\in K\}.$$

The set of all asymptotic centres of $\{x_k\}$ with respect to K is denoted by $A(K,\{x_k\})$.

Remark 2.7. ([14]) Every bounded sequence in uniformly convex Banach spaces and $CAT(0)$ spaces has a unique asymptotic center with respect to closed convex subset.

Definition 2.8. ([14]) A sequence $\{x_k\}$ in X is said to be $\Delta-$convergent to $x\in X$ if x is the unique asymptotic center of $\{x_{k_n}\}$ of $\{x_k\}$. In this case $\Delta-\lim_{k\to\infty}x_k=x$.

Definition 2.9. ([35]) Let X be a hyperbolic space. A map $\eta:(0,\infty)\times(0,2]\to(0,1]$ which provides a $\delta=\eta(r,\varepsilon)$ for a given $r>0$ and $\varepsilon\in(0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone if it decreases with r.

Lemma 2.10. ([23]) *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_k\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Lemma 2.11. ([8]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_k\}$ be a bounded sequence in X with $A(\{x_k\})(:= A(X, \{x_k\})) = \{x\}$. Suppose that $\{x_{k_n}\}$ is any subsequence of $\{x_k\}$ with $A(\{x_{k_n}\}) = \{x_1\}$ and $\{d(x_k, x_1)\}$ converges. Then $x = x_1$.*

Lemma 2.12. ([20]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_k\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_k\}$ and $\{y_k\}$ are sequences in X such that $\limsup_{k\to\infty} d(x_k, x^*) \leq c$, $\limsup_{k\to\infty} d(y_k, x^*) \leq c$ and $\lim_{k\to\infty} d(W(x_k, y_k, t_k), x^*) \leq c$ for some $c > 0$. Then $\lim_{k\to\infty} d(x_k, y_k) = 0$.*

Lemma 2.13. ([8]) *Let (X, d, W) be a complete uniformly convex hyperbolic space, K be a nonempty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$. Let $P_T : K \to 2^K$ be a multi-valued mapping defined by*

$$P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}, \quad x \in K.$$

Then the following conclusions hold:

(a) *P_T is a multi-valued mapping from K to $P(K)$.*
(b) *$F(T) = F(P_T)$.*
(c) *$P_T(p) = \{p\}$, for each $p \in F(T)$.*
(d) *For each $x \in K$, $P_T(x)$ is a closed subset of Tx and so it is compact.*
(e) *$d(x, Tx) = d(x, P_T(x))$ for each $x \in K$.*

Definition 2.14. ([6]) Let K be a non-empty closed subset of a complete metric space X and $\{x_k\}$ be a sequence in K. Then $\{x_k\}$ is called a Fejer monotone sequence with respect to K if for all $x \in K$ and $k \in \mathbb{N}$,

$$d(x_{k+1}, x) \leq d(x_k, x).$$

Proposition 2.15. ([6]) *Let K be a nonempty closed subset of a complete metric space X and $\{x_k\}$ be a sequence in K. Suppose $T : K \to K$ is any nonlinear mapping and the sequence $\{x_k\}$ is Fejer monotone with respect of K. Then we have the following:*

(i) *$\{x_k\}$ is bounded.*

(ii) *The sequence* $\{d(x_k, x^*)\}$ *is decreasing and converges for all* $x^* \in F(T)$.
(iii) $\lim_{k\to\infty} d(x_k, F(T))$ *exists.*

Lemma 2.16. ([5]) *Let* K *be a nonempty closed subset of a complete metric space* X *and* $\{x_k\}$ *be Fejer monotone with respect to* K*. Then* $\{x_k\}$ *is convergent to some* $x^* \in K$ *if and only if* $\lim_{k\to\infty} d(x_k, K) = 0$.

Vetro [39] established some results related to Hausdorff distance. These results are as follows:

Lemma 2.17. ([39]) *Let* (X, d) *be a metric space. Then for any* $A, B, C \in CB(X)$ *and any* $x, y \in X$*, we have:*
(i) $d(x, B) \leq d(x, b)$ *for* $b \in B$;
(ii) $\delta(A, B) \leq H(A, B)$;
(iii) $d(x, B) \leq H(A, B)$ *for any* $x \in A$;
(iv) $H(A, A) = 0$;
(v) $H(A, B) = H(B, A)$;
(vi) $H(A, C) \leq H(A, B) + H(B, C)$;
(vii) $d(x, A) \leq d(x, y) + d(y, A)$.

3. Main results

3.1. **Structure of fixed point set of multi-valued Osilike-Berinde nonexpansive mapping.**

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a complete hyperbolic space* X*. Let* $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ *be a finite family of multi-valued quasi-nonexpansive mappings such that* $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ *and* $P_{T_i} : K \to 2^K$ *are multi-valued Osilike-Berinde nonexpansive mappings. Then* $F(T)$ *is closed and convex.*

Proof. To show that $F(T)$ is closed, let $\{x_k\}$ be a sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in K$ and $p \in F(T)$. Then from Lemma 2.13, we have $p \in F(P_T)$ and $P_T(p) = \{p\}$. By using quasi-nonexpansiveness of T and Lemma 2.17, we have

$$\begin{aligned} d(x_k, Ty) &\leq d(x_k, p) + d(p, Ty) \\ &\leq H(P_T(x_k), P_T(p)) + d(p, Ty) \\ &\leq d(x_k, p) + Ld(p, Tp) + d(p, Ty) \\ &\leq d(x_k, p) + d(p, y) \\ &\leq d(x_k, y). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Ty) = 0.$$

By uniqueness of limit, we have $y \in Ty$. Hence $F(T)$ is closed.

Next we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0,1]$. By using Lemma 2.17, we have

$$\begin{aligned} d(x, T(W(x,y,\alpha))) &\le H(P_T(x), P_T(W(x,y,\alpha)) \\ &\le d(x, W(x,y,\alpha)) + Ld(x, Tx) \\ &\le d(x, W(x,y,\alpha)). \end{aligned}$$

Hence

$$d(x, T(W(x,y,\alpha))) \le d(x, W(x,y,\alpha)). \tag{3.1}$$

Using similar argument, we have

$$d(y, T(W(x,y,\alpha))) \le d(y, W(x,y,\alpha)). \tag{3.2}$$

By using Lemma 2.17, (3.1) and (3.2), we have

$$\begin{aligned} d(x,y) &\le d(x, T(W(x,y,\alpha))) + d(T(W(x,y,\alpha)), y) \\ &\le H(P_T(x), P_T(W(x,y,\alpha))) + H(P_T(W(x,y,\alpha)), P_T(y)) \\ &\le (d(x, W(x,y,\alpha)) + d(y, W(x,y,\alpha))) + L(d(x,Tx) + d(y,Ty)) \\ &\le d(x, W(x,y,\alpha)) + d(y, W(x,y,\alpha)) \\ &= d(x,y). \end{aligned}$$

Therefore,

$$d(x,y) \le d(x,y). \tag{3.3}$$

Hence, we conclude that (3.1) and (3.2) are

$$d(x, T(W(x,y,\alpha))) = d(x, W(x,y,\alpha))$$

and $d(y, T(W(x,y,\alpha))) = d(y, W(x,y,\alpha))$, respectively, because if we take strictly less than sign $<$, then from (3.3) we get the contradiction that $d(x,y) < d(x,y)$. Therefore,

$$T(W(x,y,\alpha)) = W(x,y,\alpha)$$

for all $x, y \in F(T)$ and $\alpha \in [0,1]$. Thus $W(x,y,\alpha) \in F(T)$ which implies that $F(T)$ is convex. $\square$

Corollary 3.2. *Let K be a nonempty closed convex subset of a complete hyperbolic space X. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued quasi-nonexpansive mappings such that $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $P_{T_i} : K \to 2^K$ are multi-valued Osilike-Berinde nonexpansive mappings.*

Let $\{x_k\}$ *be a bounded sequence in* K *such that* $\lim_{k\to\infty} d(x_k, Tx_k) = 0$. *Then* $F(T)$ *is closed and convex.*

Proof. Let $\{x_k\}$ be a bounded sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in K$ and $p \in F(T)$. Using quasi-nonexpansiveness of T, we have

$$\begin{aligned} d(x_k, Ty) &\le d(x_k, Tx_k) + d(Tx_k, p) + d(p, Ty) \\ &\le d(x_k, Tx_k) + d(x_k, p) + d(p, y) \\ &\le d(x_k, Tx_k) + d(x_k, y). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Ty) = 0.$$

Hence $F(T)$ is closed. The rest is the same as of the proof in Lemma 3.1. □

Theorem 3.3. *Let* K *be a nonempty closed convex subset of a complete uniformly convex hyperbolic space* X *with monotone modulus of convexity* η. *Let* $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ *be a finite family of multi-valued quasi-nonexpansive mappings such that* $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ *and* $P_{T_i} : K \to 2^K$ *are multi-valued Osilike-Berinde nonexpansive mappings. Let* $\{x_k\}$ *be a bounded sequence in* K *such that* $\lim_{k\to\infty} d(x_k, Tx_k) = 0$ *and* $\Delta - \lim_{k\to\infty} x_k = x^*$. *Then* $x^* \in F(T)$.

Proof. Since $\{x_k\}$ is a bounded sequence in K, from Lemma 2.10, $\{x_k\}$ has a unique asymptotic center in K. Since $\Delta - \lim_{k\to\infty} x_k = x^*$, we have $A(K, \{x_k\}) = \{x^*\}$. Hence for $p \in F(T)$, we have

$$\begin{aligned} d(x_k, Tx^*) &\le d(x_k, Tx_k) + d(Tx_k, Tx^*) \\ &\le d(x_k, Tx_k) + d(Tx_k, p) + d(p, Tx^*) \\ &\le d(x_k, Tx_k) + d(x_k, p) + d(p, x^*). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Tx^*) \le \lim_{k\to\infty} d(x_k, x^*).$$

Since

$$\begin{aligned} r(Tx^*, \{x_k\}) &= \limsup_{k\to\infty} d(x_k, Tx^*) \\ &\le \limsup_{k\to\infty} d(x_k, x^*) \\ &= r(x^*, \{x_k\}). \end{aligned}$$

By uniqueness of asymptotic center of $\{x_k\}$, we have $Tx^* = x^*$. Hence $x^* \in F(T)$. □

3.2. **Strong convergence and $\Delta-$convergence of a sequence in hyperbolic spaces.**

Lemma 3.4. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued quasi-nonexpansive mappings such that $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $P_{T_i} : K \to 2^K$ are multi-valued Osilike-Berinde nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by* (1.1) *and let $y_{0k} = x_k$. Then*

(i) $d(y_{ik}, p) \leq d(x_k, p)$ *for* $i = 1, 2, ..., k-1$,
(ii) $\lim_{k\to\infty} d(x_k, p)$ *exists for all* $p \in F(T)$,
(iii) $\lim_{k\to\infty} d(x_k, F(T))$ *exists.*

Proof. (i) We proceed by induction on i.

$$\begin{aligned} d(y_{1k}, p) &= d(W(u_{0k}, y_{0k}, \alpha_{1k}), p) \\ &\leq (1-\alpha_{1k})d(u_{0k}, p) + \alpha_{1k}d(y_{0k}, p) \\ &\leq (1-\alpha_{1k})H(P_{T_1}(y_{0k}), P_{T_1}(p)) + \alpha_{1k}d(y_{0k}, p) \\ &\leq (1-\alpha_{1k})(d(y_{0k}, p) + Ld(p, Tp)) + \alpha_{1k}d(y_{0k}, p) \\ &= d(y_{0k}, p) \\ &= d(x_k, p). \end{aligned}$$

Hence, we have $d(y_{1k}, p) \leq d(x_k, p)$. Assuming that $d(y_{ik}, p) \leq d(x_k, p)$ holds for some $1 \leq i \leq k-2$. Now

$$\begin{aligned} d(y_{(i+1)k}, p) &= d(W(u_{ik}, y_{ik}, \alpha_{(i+1)k}), p) \\ &\leq (1-\alpha_{(i+1)k})d(u_{ik}, p) + \alpha_{(i+1)k}d(y_{ik}, p) \\ &\leq (1-\alpha_{(i+1)k})H(P_{T_{(i+1)}}(y_{ik}), P_{T_{(i+1)}}(p)) + \alpha_{(i+1)k}d(y_{ik}, p) \\ &\leq d(x_k, p). \end{aligned}$$

We now show that $d(y_{ik}, p) \leq d(x_k, p)$ for $i = 1, 2, ..., k-1$.

$$\begin{aligned} d(y_{(k-1)k}, p) &= d(W(u_{(k-2)k}, y_{(k-2)k}, \alpha_{(k-1)k}), p) \\ &\leq (1-\alpha_{(k-1)k})d(u_{(k-2)k}, p) + \alpha_{(k-1)k}d(y_{(k-2)k}, p) \\ &\leq (1-\alpha_{(k-1)k})H(P_{T_{(k-1)}}(y_{(k-2)k}), P_{T_{(k-1)k}}(p)) \\ &\quad + \alpha_{(k-1)k}d(y_{(k-2)k}, p) \\ &\leq (1-\alpha_{(k-1)k})(d(y_{(k-2)k}, p) + Ld(p, Tp)) + \alpha_{(k-1)k}d(y_{(k-2)k}, p) \\ &\leq d(x_k, p). \end{aligned}$$

Thus by induction, $d(y_{ik}, p) \leq d(x_k, p)$ for $i = 1, 2, ..., k-1$.

(ii)

$$\begin{aligned} d(x_{k+1},p) &= d(W(u_{(n-1)k},y_{(n-1)k},\alpha_{nk}),p) \\ &\le (1-\alpha_{nk})d(u_{(n-1)k},p)+\alpha_{nk}d(y_{(n-1)k},p) \\ &\le (1-\alpha_{nk})H(P_{T_n}(y_{(n-1)k}),P_{T_n}(p))+\alpha_{nk}d(y_{(n-1)k},p) \\ &\le (1-\alpha_{nk})(d(y_{(n-1)k},p)+Ld(p,Tp))+\alpha_{nk}d(y_{(n-1)k},p) \\ &\le d(x_k,p). \end{aligned}$$

This implies that $\{x_k\}$ is Fejer monotone with respect to $F(T)$, so by Proposition 2.15, $\lim_{k\to\infty} d(x_k,p)$ exists.

(iii) By Proposition 2.15 and Lemma 2.16, $\lim_{k\to\infty} d(x_k,F(T))$ exists. □

Theorem 3.5. *Let K be a nonempty closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i = 1,2,...,k)$ be a finite family of multi-valued quasi-nonexpansive mappings such that $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $P_{T_i} : K \to 2^K$ are multi-valued Osilike-Berinde nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then $\lim_{k\to\infty} d(x_k,T_ix_k) = 0$ for $i = 1,2,...,k$.*

Proof. From Lemma 3.4, we have $\lim_{k\to\infty} d(x_k,p)$ exists for all $p \in F(T)$. So suppose that $\lim_{k\to\infty} d(x_k,p) = c$, where $c \ge 0$. If $c = 0$, then we have results. Let $c > 0$. Since

$$\lim_{k\to\infty} d(x_k,p) = c \quad \Rightarrow \quad \limsup_{k\to\infty} d(x_k,p) \le c.$$

Also from Lemma 3.4,

$$d(y_{ik},p) \le d(x_k,p),$$

we have

$$\limsup_{k\to\infty} d(y_{ik},p) \le c \quad \text{for} \quad i = 1,2,...,k-1. \tag{3.4}$$

Note that for $i = 1,2,...,k$

$$\begin{aligned} d(u_{(i-1)k},p) &\le H(P_{T_i}(y_{(i-1)k}),P_{T_i}(p)) \\ &\le d(y_{(i-1)k},p). \end{aligned}$$

Which implies that

$$\limsup_{k\to\infty} d(u_{(i-1)k},p) \le c. \tag{3.5}$$

Since $\lim_{k\to\infty} d(x_{k+1},p) = c$, we have

$$\lim_{k\to\infty} d(W(u_{(n-1)k},y_{(n-1)k},\alpha_{nk}),p) = c. \tag{3.6}$$

From Lemma 2.12, (3.4), (3.5) and (3.6), we have

$$\lim_{k\to\infty} d(y_{(k-1)k},u_{(k-1)k}) = 0.$$

Note that for $i = 1, 2, ..., k-1$, we have

$$d(x_{k+1}, p) \leq d(y_{ik}, p),$$

therefore

$$c \leq \liminf_{k\to\infty} d(y_{ik}, p).$$

Also

$$d(W(u_{(i-2)k}, y_{(i-2)k}, \alpha_{(i-1)k}), p) = d(y_{(i-1)k}, p),$$

therefore

$$\lim_{k\to\infty} d(W(u_{(i-2)k}, y_{(i-2)k}, \alpha_{(i-1)k}), p) = c.$$

Thus by induction, we have

$$\lim_{k\to\infty} d(y_{(i-1)k}, u_{(i-1)k}) = 0 \quad \text{for} \quad i = 1, 2, ..., k. \tag{3.7}$$

Also we have

$$\begin{aligned} d(y_{ik}, y_{(i-1)k}) &= d(W(u_{(i-1)k}, y_{(i-1)k}, \alpha_{ik}), y_{(i-1)k}) \\ &\leq (1-\alpha_{ik}) d(u_{(i-1)k}, y_{(i-1)k}) + \alpha_{ik} d(y_{(i-1)k}, y_{(i-1)k}), \end{aligned}$$

it implies that $\lim_{k\to\infty} d(y_{ik}, y_{(i-1)k}) = 0$ and

$$\begin{aligned} d(x_k, y_{1k}) &= d(x_k, W((u_{0k}, y_{0k}, \alpha_{1k}))) \\ &\leq (1-\alpha_{1k}) d(x_k, u_{0k}) + \alpha_{1k} d(x_k, y_{0k}) \\ &= (1-\alpha_{1k}) d(x_k, u_{0k}) + \alpha_{1k} d(x_k, x_k), \end{aligned}$$

it implies that $\lim_{k\to\infty} d(x_k, y_{1k}) = 0$.
Since

$$d(x_k, y_{ik}) \leq d(x_k, y_{1k}) + d(y_{1k}, y_{12}) + \cdots + d(y_{(i-1)k}, y_{ik}),$$

we have

$$\lim_{k\to\infty} d(x_k, y_{ik}) = 0 \quad \text{for} \quad i = 1, 2, ..., k-1. \tag{3.8}$$

Now from (3.7) and (3.8), we have

$$\begin{aligned} d(x_k, T_i x_k) &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) + d(u_{(i-1)k}, T_i x_k) \\ &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) + H(P_{T_i} y_{(i-1)k}, P_{T_i} x_k) \\ &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) + d(y_{(i-1)k}, x_k) \\ &\quad + L d(x_k, T_i x_k). \end{aligned}$$

Hence we have $\lim_{k\to\infty} d(x_k, T_i x_k) = 0$. □

Theorem 3.6. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i - 1, 2, ..., k)$ be a finite family of multi-valued quasi-nonexpansive mappings such that $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $P_{T_i} : K \to 2^K$ are multi-valued Osilike-Berinde nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.1). Then $\{x_k\}$ converges strongly to $p \in F(T)$ if and only if $\lim_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x_k, F(T) = \inf\{d(x_k, p) : p \in F(T)\}$.*

Proof. If $\{x_k\}$ converges strongly to $p \in F(T)$, then $\lim_{k\to\infty} d(x_k, p) = 0$. Since $0 \le d(x_k, F(T)) = \inf\{d(x_k, p) : p \in F(T)\}$, we have

$$\lim_{k\to\infty} d(x_k, F(T)) = 0.$$

Conversely, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. From Lemma 3.4, we have

$$d(x_{k+1}, p) \le d(x_k, p),$$

which implies that

$$d(x_{k+1}, F(T)) \le d(x_k, F(T)).$$

This implies that $\lim_{k\to\infty} d(x_k, F(T))$ exists. Therefore, by our assumption $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Next we will show that $\{x_k\}$ is a Cauchy sequence in K. For $k > n$,

$$\begin{aligned} d(x_k, x_n) &\le d(x_k, p) + d(p, x_n) \\ &\le 2d(x_k, p). \end{aligned}$$

Taking inf on right hand side, we have

$$d(x_k, x_n) \le 2d(x_k, F(T)).$$

Hence, we have $d(x_k, x_n) \to 0$ as $k, n \to \infty$. Hence $\{x_k\}$ is a Cauchy sequence in K, therefore it converges to some $q \in K$. Next we show that $q \in F(T_1)$. Since $d(x_k, F(T_1)) = \inf_{y\in F(T_1)} d(x_k, y)$, so for each $\varepsilon > 0$, there exists $p_k \in F(T_1)$ such that

$$d(x_k, p_k) < d(x_k, F(T_1)) + \frac{\varepsilon}{2}.$$

Since $d(p_k, q) \le d(x_k, p_k) + d(x_k, q)$, $\lim_{k\to\infty} d(p_k, q) \le \frac{\varepsilon}{2}$. Hence, we obtain that

$$\begin{aligned} d(T_1q, q) &\le d(T_1q, p_k) + d(p_k, q) \\ &\le H(P_{T_1}p_k, P_{T_1}q) + d(p_k, q) \\ &\le d(p_k, q) + Ld(p, T_1p) + d(p_k, q) \\ &\le 2d(p_k, q), \end{aligned}$$

which implies that $d(T_1q, q) \le \varepsilon$. Hence $d(T_1q, q) = 0$. Similarly, $d(T_iq, q) = 0$ for $i = 1, 2, ..., k$. Since $F(T)$ is closed, we have $q \in F(T)$. □

Theorem 3.7. *Let K be a nonempty closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued quasi-nonexpansive mappings such that $F(T) = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$ and $P_{T_i} : K \to 2^K$ are multi-valued Osilike-Berinde nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by* (1.1)*. Then $\{x_k\}$ is Δ-convergent to a common fixed point $p \in F(T)$.*

Proof. Let $p \in F(T)$. Then $p \in F(T_i)$, for $i = 1, 2, ..., k$. Also the sequence $\{x_k\}$ has unique asymptotic center, so suppose that $A(K, \{x_k\}) = \{x\}$. From Lemma 3.4, sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} d(x_k, p)$ exists, so we can find a subsequence $\{w_k\}$ of the sequence $\{x_k\}$ such that $A(K, \{w_k\}) = \{x^*\}$ for some $x^* \in K$. From the Theorem 3.5, $\lim_{k\to\infty} d(w_k, T_i w_k) = 0, i = 1, 2, ..., k$. We claim that x^* is a fixed point of T_1. For this, let $\{v_k\}$ be an another sequence in $T_1 x^*$. Then

$$\begin{aligned}
r(v_k, \{w_k\}) &= \limsup_{k\to\infty} d(v_k, w_k) \\
&\leq \limsup_{k\to\infty} (d(v_k, T_1 w_k) + d(T_1 w_k, w_k)) \\
&\leq \limsup_{k\to\infty} (H(P_{T_1} x^*, P_{T_1} w_k) + d(T_1 w_k, w_k)) \\
&\leq \limsup_{k\to\infty} ((x^*, w_k) + Ld(x^*, T_1 x^*)) + d(T_1 w_k, w_k)) \\
&\leq \limsup_{k\to\infty} d(x^*, w_k) \\
&= r(x^*, \{w_k\}).
\end{aligned}$$

Hence we have $|r(v_k, \{w_k\}) - r(x^*, \{w_k\})| \to 0$ as $k \to \infty$. From Lemma 2.11, we have $\lim_{k\to\infty} v_k = x^*$. Hence either $T_1 x^*$ is closed or bounded. Therefore $\lim_{k\to\infty} v_k = x^* \in T_1 x^*$. Similarly $x^* \in T_i x^*$, for $i = 1, 2, ..., k$, that is, $x^* \in F(T)$. From Lemma 2.11, we have $p = x^*$. This implies that $\{x_k\}$ is $\Delta-$convergent to $p \in F(T)$. □

4. Conclusion

Started with iteration scheme (1.1) introduced by Alagoz et al., we obtain strong convergence and $\Delta-$convergence of a sequence defined by (1.1) for a family of multi-valued quasi-nonexpansive mappings and multi-valued Osilike-Berinde nonexoansive mappings in the framework of complete hyperbolic spaces. Our results are new and generalizes several results.

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