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ψ-TYPE CONTRACTION AND JAGGI TYPE HYBRID CONTRACTION IN BIPOLAR METRIC SPACES

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Abstract. In this paper, we will introduce the notion of ψ-type and Jaggi type hybrid contraction in a bipolar metric space and show the existence and uniqueness of fixed point for such type of contractions. In the end, we will provide some corollaries and support our theorems by examples.

1. Introduction

To obtain new fixed point theorems, the researchers have two options either they generalize the metric space introduced by Fréchet [3] in 1906 or Banach contraction principle introduced by Banach [1] in 1922. Since then, a number of generalization ([2],[4],[5],[6],[9]) in Banach contraction principle are done by weakening the Banach hypothesis like Meir-Keeler contraction [9], $\alpha - \psi$ contraction by Samet ([14],[15],[16]) *et al.* etc. In 1977, Jaggi [4] was the first

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who introduced rational contraction in a metric space to obtain fixed point theorems.

Recently in 2019, Karapinar and Fulga [6] introduced Jaggi type hybrid contraction in metric spaces. On the other hand, researchers also introduced new generalized metric spaces like G-metric space by Mustafa and Sims [10] in 2006 etc. In 2016, Mutlu and Gürdal [11] introduced bipolar metric space. Then a number of fixed point theorems were proved in bipolar metric space ([7],[8],[12],[13]) by many authors till now.

In this paper also, we will prove fixed point theorems for ψ-type contraction and Jaggi type hybrid contraction in bipolar metric spaces.

We need some basic notions and definitions from the literature to prove the results as follows:

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(1) ψ is non-decreasing.
(2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iteration of ψ.

These functions are known as (c)-comparison functions. It can be easily verified that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

In 2016, Mutlu and Gurdal introduced the bipolar metric space as follows:

Definition 1.1. ([11]) Let X and Y be two nonempty sets and $d : X \times Y \to [0,\infty)$ be a map satisfying the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$ for all $(x,y) \in X \times Y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X \cap Y$;
(3) $d(x_1,y_2) \leq d(x_1,y_1) + d(x_2,y_1) + d(x_2,y_2)$, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then d is called a bipolar metric and (X,Y,d) is called a bipolar metric space.

If $X \cap Y = \phi$ then space is called disjoint otherwise joint. The set X is called left pole and the set Y is called right pole of (X,Y,d). The elements of X, Y and $X \cap Y$ are called left, right and central elements, respectively.

Definition 1.2. ([11]) Let (X,Y,d) be a bipolar metric space. Then any sequence $\{x_n\} \subseteq X$ is called left sequence and is said to be convergent to right element say y if $d(x_n,y) \to 0$ as $n \to \infty$. Similiarly, a right sequence $\{y_n\} \subseteq Y$ is said to be convergent to a left element say x if $d(x,y_n) \to 0$ as $n \to \infty$.

Definition 1.3. ([11]) Let (X,Y,d) be a bipolar metric space.

(1) A sequence $\{(x_n,y_n)\}$ on $X \times Y$ is called a bisequence on (X,Y,d).
(2) If both the sequences $\{x_n\}$ and $\{y_n\}$ converge, then the bisequence $\{(x_n,y_n)\}$ is said to be convergent. If both the sequences $\{x_n\}$ and $\{y_n\}$

converge to the same point $u \in X \cap Y$ then the bisequence $\{(x_n, y_n)\}$ is called biconvergent.

(3) A bisequence $\{(x_n, y_n)\}$ on (X, Y, d) is said to be Cauchy bisequence, if for each $\epsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, y_m) < \epsilon$ for all $n, m \geq N$.

(4) A bipolar metric space is said to be complete, if every Cauchy bisequence is convergent in this space.

Definition 1.4. ([11]) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces and $T : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function:

(1) If $TX_1 \subseteq X_2$ and $TY_1 \subseteq Y_2$, then T is called covariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

(2) If $TX_1 \subseteq Y_2$ and $TY_1 \subseteq X_2$, then T is called contravariant mapping and is denoted by $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$.

Definition 1.5. ([11]) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be two bipolar metric spaces.

(1) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx_0, Ty) < \epsilon$ whenever $d_1(x_0, y) < \delta$.

(2) A map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right continuous at a point $y_0 \in Y$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_2(Tx, Ty_0) < \epsilon$ whenever $d_1(x, y_0) < \delta$.

(3) A map T is called continuous if it is left continuous at each $x_0 \in X$ and right continuous at each $y_0 \in Y$.

(4) A contravariant map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if T is continuous as covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$.

2. Main Results

Here, we will prove our main results by introducing the notions of generalized contractive mappings in bipolar metric spaces.

Definition 2.1. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be a covariant mapping. Then T is said to be ψ-contractive covariant mapping if there exists $\psi \in \Psi$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)) \tag{2.1}$$

for all $(x, y) \in X \times Y$.

Definition 2.2. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping. Then T is said to be ψ-contractive contravariant mapping if there exists $\psi \in \Psi$ such that

$$d(Ty, Tx) \leq \psi(d(x, y)) \tag{2.2}$$

for all $(x, y) \in X \times Y$.

Theorem 2.3. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightrightarrows (X, Y)$ be a contractive covariant mapping satisfying equation* (2.1). *Then T has a unique fixed point.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$. Construct two sequences $\{x_n\} \in X$ and $\{y_n\} \in Y$ such that $x_n = Tx_{n-1}$ and $y_n = Ty_{n-1}$. Then $\{(x_n, y_n)\}$ is a bisequence in $X \times Y$.

Now putting $x = x_n$ and $y = y_n$ in equation (2.1), we obtain that

$$d(x_{n+1}, y_{n+1}) = d(Tx_n, Ty_n) \leq \psi(d(x_{n-1}, y_{n-1})).$$

By induction, we say that

$$d(x_{n+1}, y_{n+1}) \leq \psi^{n+1}(d(x_0, y_0)). \tag{2.3}$$

Again, putting $x = x_{n-1}$ and $y = y_n$ in equation (2.1), we obtain that

$$d(x_n, y_{n+1}) = d(Tx_{n-1}, Ty_n) \leq \psi(d(x_{n-1}, y_n)).$$

By induction, we have

$$d(x_n, y_{n+1}) \leq \psi^n(d(x_0, y_1)). \tag{2.4}$$

Since $\sum_{n=1}^{\infty} \psi^n(a) < \infty$ for each $a > 0$, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(d(x_0, y_1)) < \frac{\epsilon}{2} \text{ and } \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}. \tag{2.5}$$

Now, for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_{n+1}) + d(x_{n+1}, y_{n+1}) + d(x_{n+1}, y_{n+2}) \\ &\quad + \cdots + d(x_{m-1}, y_{m-1}) + d(x_{m-1}, y_m) \\ &= \sum_{k=n}^{m-1} d(x_k, y_{k+1}) + \sum_{k=n}^{m-2} d(x_{k+1}, y_{k+1}). \end{aligned}$$

Now, using equation (2.3) and (2.4), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m-1} \psi^k d(x_0, y_1) + \sum_{k=n}^{m-2} \psi^{k+1} d(x_0, y_0), \\ &\leq \sum_{k=N}^{m-1} \psi^k d(x_0, y_1) + \sum_{k=n}^{m-2} \psi^{k+1} d(x_0, y_0), \\ &\leq \sum_{n \geq N} \psi^n d(x_0, y_1) + \sum_{n \geq N} \psi^{n+1} d(x_0, y_0). \end{aligned} \tag{2.6}$$

Using equation (2.5) in (2.6), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.7}$$

Similarly, one can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{2.8}$$

From equations (2.7) and (2.8), we can say that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, $\{(x_n, y_n)\}$ is convergent and thus biconverges to a point $v \in X \cap Y$ and $Tx_n = x_{n+1} \to v \in X \cap Y$ guarantees that $\{x_{n+1}\}$ has unique limit.

Now, by using equation (2.1) and $\psi(t) < t$, we get

$$d(x_{n+1}, Tv) \leq \psi(d(x_n, v)) < d(x_n, v). \tag{2.9}$$

Taking limit $n \to \infty$ in equation (2.9), we have

$$d(v, Tv) \leq 0.$$

This implies that $d(v, Tv) = 0$, that is, $Tv = v$. Hence v is the fixed point.

Next, we have to prove the uniqueness of the fixed point. Let us suppose, if possible that u is also the fixed point of T and $u \neq v$. Then, equation (2.1) implies that

$$d(u, v) = d(Tu, Tv) \leq \psi(d(u, v)) < d(u, v),$$

which is a contradiction. Hence we have $u = v$. □

Theorem 2.4. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contractive contravariant mapping satisfying equation (2.2). Then T has a unique fixed point.*

Proof. Let $x_0 \in X$. We define a bisequence $\{(x_n, y_n)\}$ as $x_{n+1} = Ty_n$ and $y_n = Tx_n$ for all $n \in \mathbb{N}$.

Now putting $x = x_n$ and $y = y_{n-1}$ in equation (2.2), we obtain that

$$d(x_n, y_n) = d(Ty_{n-1}, Tx_n) \leq \psi(d(x_n, y_{n-1})).$$

By induction, we obtain

$$d(x_n, y_n) \leq \psi^n(d(x_0, y_0)). \tag{2.10}$$

Again, putting $x = x_{n+1}$ and $y = y_n$ in equation (2.2), we obtain that

$$d(x_{n+1}, y_n) = d(Ty_n, Tx_n) \leq \psi(d(x_n, y_n)).$$

By induction, we have

$$d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0)). \tag{2.11}$$

Since $\sum_{n=1}^{\infty} \psi^n(a) < \infty$ for each $a > 0$, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(d(x_0, y_0)) < \frac{\epsilon}{2} \quad \text{and} \quad \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}. \tag{2.12}$$

Now, for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &= \sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k). \end{aligned}$$

Now, using equation (2.10) and (2.11), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{k=N}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{n \geq N} \psi^n d(x_0, y_0) + \sum_{n \geq N} \psi^{n+1} d(x_0, y_0). \end{aligned}$$

Using equation (2.12), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.13}$$

Similarly, we can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{2.14}$$

From equations (2.13) and (2.14), we can say that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, $\{(x_n, y_n)\}$ is convergent and thus biconverges to a point $v \in X \cap Y$, so $Ty_n = x_{n+1} \to v$ and $Tx_n = y_n \to v \in X \cap Y$ guarantees that $\{x_{n+1}\}$ has unique limit.

Now, by using equation (2.2) and $\psi(t) < t$, we get

$$d(x_{n+1}, Tv) = d(Ty_n, Tv) \leq \psi(d(v, y_n)) < d(v, y_n). \tag{2.15}$$

Taking limit $n \to \infty$ in equation (2.15), we have

$$d(v, Tv) \leq 0.$$

This implies that $d(v, Tv) = 0$, that is, $Tv = v$. Hence v is the fixed point of T. Next, we have to prove the uniqueness. Let us suppose, if possible that u is also the fixed point of T and $u \neq v$. Then, equation (2.2) implies that

$$d(u, v) = d(Tu, Tv) \leq \psi(d(u, v)) < d(u, v),$$

which is a contradiction. Hence, we haves $u = v$. □

Definition 2.5. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping. Then T is said to be generalized ψ-contractive contravariant mapping if there exists $\psi \in \Psi$ such that

$$d(Ty, Tx) \leq \psi(M(x, y)), \tag{2.16}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(Ty, y), \frac{d(x, Tx) + d(Ty, y)}{2}\right\}$$

for all $(x, y) \in X \times Y$.

Theorem 2.6. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a generalized ψ-contractive contravariant mapping satisfying equation* (2.16). *Then T has a unique fixed point.*

Proof. Let $x_0 \in X$. Construct a bisequence $\{(x_n, y_n)\}$ as $x_{n+1} = Ty_n$ and $y_n = Tx_n$ for all $n \in \mathbb{N}$.

Now putting $x = x_n$ and $y = y_{n-1}$ in equation (2.16), we obtain that

$$d(x_n, y_n) = d(Ty_{n-1}, Tx_n) \leq \psi(M(x_n, y_{n-1})), \tag{2.17}$$

where

$$\begin{aligned} &M(x_n, y_{n-1}) \\ &= \max\left\{d(x_n, y_{n-1}), d(x_n, Tx_n), d(Ty_{n-1}, y_{n-1}), \frac{d(x_n, Tx_n) + d(Ty_{n-1}, y_{n-1})}{2}\right\} \\ &= \max\left\{d(x_n, y_{n-1}), d(x_n, y_n), d(x_n, y_{n-1}), \frac{d(x_n, y_n) + d(x_n, y_{n-1})}{2}\right\} \\ &\leq \max\{d(x_n, y_{n-1}), d(x_n, y_n)\}. \end{aligned} \tag{2.18}$$

Suppose that $d(x_n, y_{n-1}) < d(x_n, y_n)$. Then equation (2.18) becomes

$$M(x_n, y_{n-1}) < d(x_n, y_n).$$

Putting this in equation (2.17), we get

$$d(x_n, y_n) \leq \psi(d(x_n, y_n)) < d(x_n, y_n),$$

which is a contradiction. So, $d(x_n, y_{n-1}) > d(x_n, y_n)$ and equation (2.18) becomes

$$M(x_n, y_{n-1}) < d(x_n, y_{n-1}). \tag{2.19}$$

Using equation (2.19) in (2.17), we obtain that

$$d(x_n, y_n) \leq d(x_n, y_{n-1}).$$

By induction, we obtain

$$d(x_n, y_n) \leq \psi^n(d(x_0, y_0)). \tag{2.20}$$

Similarly, putting $x = x_{n+1}$ and $y = y_n$ in equation (2.16), we obtain that

$$d(x_{n+1}, y_n) = d(Ty_n, Tx_n) \leq \psi(M(x_n, y_n)), \tag{2.21}$$

where

$$\begin{aligned} M(x_n, y_n) &= \max\left\{d(x_n, y_n), d(x_n, Tx_n), d(Ty_n, y_n), \frac{d(x_n, Tx_n)+d(Ty_n, y_n)}{2}\right\} \\ &= \max\left\{d(x_n, y_n), d(x_n, y_n), d(x_{n+1}, y_n), \frac{d(x_n, y_n) + d(x_{n+1}, y_n)}{2}\right\} \\ &\leq \max\{d(x_{n+1}, y_n), d(x_n, y_n)\}. \end{aligned} \tag{2.22}$$

Suppose that $d(x_n, y_n) < d(x_{n+1}, y_n)$. Then equation (2.22) becomes $M(x_n, y_n) < d(x_{n+1}, y_n)$, Then equation (2.21) becomes

$$d(x_{n+1}, y_n) \leq \psi(d(x_{n+1}, y_n)) < d(x_{n+1}, y_n),$$

which is a contradiction. So, $d(x_{n+1}, y_n) < d(x_n, y_n)$ and equation (2.22) becomes

$$M(x_n, y_n) < d(x_n, y_n). \tag{2.23}$$

Using equation (2.23) in (2.21), we obtain that

$$d(x_{n+1}, y_n) \leq d(x_n, y_n). \tag{2.24}$$

By induction, we obtain

$$d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0)). \tag{2.25}$$

Since, $\sum_{n=1}^{\infty} \psi^n(a) < \infty$ for each $a > 0$, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(d(x_0, y_0)) < \frac{\epsilon}{2} \quad \text{and} \quad \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}. \tag{2.26}$$

Now, for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &= \sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k). \end{aligned}$$

Now, using equation (2.20) and (2.25), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{k=N}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{n \geq N} \psi^n d(x_0, y_0) + \sum_{n \geq N} \psi^{n+1} d(x_0, y_0). \end{aligned}$$

Using equation (2.26), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.27}$$

Similarly, we can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{2.28}$$

From equations (2.27) and (2.28), we can say that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, $\{(x_n, y_n)\}$ is convergent and thus biconverges to a point $v \in X \cap Y$, so $Ty_n = x_{n+1} \to v$ and $Tx_n = y_n \to v \in X \cap Y$ guarantees that $\{x_{n+1}\}$ has unique limit.

Now, by using equation (2.16) and $\psi(t) < t$, we get

$$d(x_{n+1}, Tv) = d(Ty_n, Tv) \leq \psi(M(v, y_n)) < M(v, y_n), \tag{2.29}$$

where

$$\begin{aligned} M(v, y_n) &= \max\left\{ d(v, y_n), d(v, Tv), d(Ty_n, y_n) \frac{d(v, Tv) + d(Ty_n, y_n)}{2} \right\} \\ &\leq \max\{d(v, y_n), d(v, Tv), d(Ty_n, y_n)\}. \end{aligned} \tag{2.30}$$

Using equation (2.30) in (2.29) and taking limit $n \to \infty$, we get

$$d(v, Tv) \leq 0,$$

this implies that $d(v, Tv) = 0$, that is, $Tv = v$. So v is the fixed point of T.

Next, we have to prove the uniqueness. Let us suppose, if possible that u is also the fixed point of T and $u \neq v$. Then, equation (2.16) implies that

$$d(u, v) = d(Tu, Tv) \leq \psi(M(v, u)) < M(v, u),$$

where

$$\begin{aligned} M(v,u) &= \max\Big\{d(v,u), d(v,Tv), d(Tu,u), \frac{d(v,Tv), d(Tu,u)}{2}\Big\}, \\ &\leq d(u,v). \end{aligned}$$

This is a contradiction, so we have $u = v$. Hence T has a unique fixed point. □

Definition 2.7. Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant mapping. Then T is said to be Jaggi type hybrid contravariant mapping if there exists $\psi \in \Psi$ such that

$$d(Ty, Tx) \leq \psi(\mathfrak{J}_s^T(x, y)), \tag{2.31}$$

for all distinct $(x, y) \in X \times Y$, where $s \geq 0$ and $\alpha_i \geq 0$ for $i = 1, 2$ such that $\alpha_1 + \alpha_2 = 1$ and

$$\mathfrak{J}_s^T(x,y) = \begin{cases} \Big[\alpha_1(\frac{d(x,Tx)d(Ty,y)}{d(x,y)})^s + \alpha_2(d(x,y))^s\Big]^{\frac{1}{s}}, & \text{if } s > 0 \\ (d(x,Tx))^{\alpha_1}(d(Ty,y))^{\alpha_2}, & \text{if } s = 0. \end{cases}$$

If x and y are different elements in X and Y, $x, y \notin F_T(X \cup Y)$, where

$$F_T(X \cup Y) = \{x \in X \cup Y : Tx = x\}.$$

Theorem 2.8. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a Jaggi type hybrid contravariant continuous mapping satisfying equation (2.31). Then T has a fixed point.*

Proof. Let $x_0 \in X$. Construct a bisequence $\{(x_n, y_n)\}$ as $x_{n+1} = Ty_n$ and $y_n = Tx_n$ for all $n \in \mathbb{N}$.

We will prove the theorem in two cases (i) when $s > 0$, (ii) when $s = 0$.

Case (i): When $s > 0$. Now putting $x = x_n$ and $y = y_{n-1}$ in equation (2.31), then we get

$$d(x_n, y_n) = (d(Ty_{n-1}, Tx_n)) \leq \psi(\mathfrak{J}_s^T(x_n, y_{n-1})), \tag{2.32}$$

where

$$\begin{aligned} \mathfrak{J}_s^T(x_n, y_{n-1}) &= Big\alpha_1(\frac{d(x_n, Tx_n)d(Ty_{n-1}, y_{n-1})}{d(x_n, y_{n-1})})^s + \alpha_2(d(x_n, y_{n-1}))^s\Big]^{\frac{1}{s}} \\ &= \Big[\alpha_1(\frac{d(x_n, y_n)d(x_n, y_{n-1})}{d(x_n, y_{n-1})})^s + \alpha_2(d(x_n, y_{n-1}))^s\Big]^{\frac{1}{s}} \\ &= [\alpha_1(d(x_n, y_n))^s + \alpha_2(d(x_n, y_{n-1}))^s]^{\frac{1}{s}}. \end{aligned} \tag{2.33}$$

Suppose that, $d(x_n, y_{n-1}) < d(x_n, y_n)$. Then equation (2.33) becomes

$$\mathfrak{J}_s^T(x_n, y_{n-1}) \leq d(x_n, y_n).$$

Using this in equation (2.32), we get

$$d(x_n, y_n) \leq \psi(\mathfrak{J}_s^T(d(x_n, y_n))) < d(x_n, y_n),$$

which is a contradiction. So, $d(x_n, y_{n-1}) > d(x_n, y_n)$, by induction, we get

$$d(x_n, y_n) \leq \psi^n(d(x_0, y_0)). \tag{2.34}$$

Similarly, putting $x = x_{n+1}$ and $y = y_n$ in equation (2.32), we obtain that

$$d(x_{n+1}, y_n) = (d(Ty_n, Tx_n)) \leq \psi(\mathfrak{J}_s^T(x_n, y_n)),$$

where

$$\begin{aligned}\mathfrak{J}_s^T(x_n, y_n) &= \Big[\alpha_1(\frac{d(x_n, Tx_n)d(Ty_n, y_n)}{d(x_n, y_n)})^s + \alpha_2(d(x_n, y_n))^s\Big]^{\frac{1}{s}} \\ &= \Big[\alpha_1(\frac{d(x_n, y_n)d(x_{n+1}, y_n)}{d(x_{n+1}, y_n)})^s + \alpha_2(d(x_n, y_n))^s\Big]^{\frac{1}{s}} \\ &= [\alpha_1(d(x_n, y_n))^s + \alpha_2(d(x_{n+1}, y_n))^s]^{\frac{1}{s}}. \end{aligned} \tag{2.35}$$

Suppose that, $d(x_n, y_n) < d(x_{n+1}, y_n)$. Then equation (2.35) becomes

$$\mathfrak{J}_s^T(x_n, y_n) \leq d(x_{n+1}, y_n).$$

Therefore, we have

$$d(x_{n+1}, y_n) \leq \psi(\mathfrak{J}_s^T(x_n, y_n)) < d(x_{n+1}, y_n),$$

which is a contradiction. So, $d(x_{n+1}, y_n) < d(x_n, y_n)$, by induction, we get

$$d(x_{n+1}, y_n) \leq \psi^{n+1}(d(x_0, y_0)). \tag{2.36}$$

Since $\sum_{n=1}^{\infty} \psi^n(a) < \infty$ for each $a > 0$, for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(d(x_0, y_0)) < \frac{\epsilon}{2} \quad \text{and} \quad \psi^{n+1}(d(x_0, y_0)) < \frac{\epsilon}{2}. \tag{2.37}$$

Now, for $n, m \in \mathbb{N}$ with $m > n \geq N$, applying the condition (3) of Definition 1.1, we get

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &= \sum_{k=n}^{m} d(x_k, y_k) + \sum_{k=n}^{m-1} d(x_{k+1}, y_k). \end{aligned}$$

Now, using equation (2.34) and (2.36), we get

$$\begin{aligned} d(x_n, y_m) &\leq \sum_{k=n}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{k=N}^{m} \psi^k d(x_0, y_0) + \sum_{k=n}^{m-1} \psi^{k+1} d(x_0, y_0) \\ &\leq \sum_{n\geq N} \psi^n d(x_0, y_0) + \sum_{n\geq N} \psi^{n+1} d(x_0, y_0). \end{aligned}$$

Using equation (2.37), we get

$$d(x_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{2.38}$$

Similarly, we can prove easily for $n, m \in \mathbb{N}$ with $n > m \geq N$ that

$$d(x_n, y_m) < \epsilon. \tag{2.39}$$

From equations (2.38) and (2.39), we can say that $\{(x_n, y_n)\}$ is a Cauchy bisequence. Since (X, Y, d) is a complete bipolar metric space, $\{(x_n, y_n)\}$ is convergent and thus biconverges to a point $v \in X \cap Y$. As T is continuous so $x_n \to v$ implies $Tx_n = y_n \to Tv$. So v is the fixed point of T.

Case (ii): When $s = 0$. Putting $x = x_n$ and $y = y_{n-1}$ in equation (2.31), we get

$$d(x_n, y_n) = (d(Ty_{n-1}, Tx_n)) \leq \psi(\mathfrak{J}_s^T(x_n, y_{n-1})), \tag{2.40}$$

where

$$\begin{aligned} \mathfrak{J}_s^T(x_n, y_{n-1}) &= (d(x_n, Tx_n))^{\alpha_1} (d(Ty_{n-1}, y_{n-1}))^{\alpha_2} \\ &= (d(x_n, y_n))^{\alpha_1} (d(x_n, y_{n-1}))^{\alpha_2}. \end{aligned}$$

Using this in equation (3.40), we have

$$d(x_n, y_n) \leq \psi((d(x_n, y_n))^{\alpha_1} (d(x_n, y_{n-1}))^{\alpha_2}) < (d(x_n, y_n))^{\alpha_1} (d(x_n, y_{n-1}))^{\alpha_2}.$$

This implies that $(d(x_n, y_n))^{1-\alpha_1} < d(x_n, y_{n-1}))^{\alpha_2}$.

Clearly,

$$d(x_n, y_n) < d(x_n, y_{n-1}). \tag{2.41}$$

Similarly, we can obtain

$$d(x_{n+1}, y_n) < d(x_n, y_n). \tag{2.42}$$

Now by using the same tool as in the case $s > 0$, we can easily show that T has a fixed point. □

Example 2.9. Let $X = (-\infty, 0]$, $Y = [0, \infty)$ and $d : (-\infty, 0] \times [0, \infty) \to [0, \infty)$ as $d(x, y) = |x - y|$. Then, clearly, (X, Y, d) is a complete bipolar metric space. Define $T : \mathbb{R} \to \mathbb{R}$ as $Tx = \frac{-x}{3}$. Clearly, T is contravariant continuous map. Taking $\psi(z) = \frac{z}{2}$.

$$\begin{aligned} d(Ty, Tx) &= \Big|\frac{-y}{3} - \frac{-x}{3}\Big| \\ &= \Big|\frac{-y}{3} + \frac{x}{3}\Big| \\ &= \Big|\frac{y}{3} + \frac{a}{3}\Big|, \end{aligned} \tag{2.43}$$

where $x = -a$ for $a \geq 0$. For all $(x, y) \in X \times Y$.

$$M(x, y) = \max\Big\{d(x, y), d(x, Tx), d(Ty, y), \frac{d(x, Tx) + d(Ty, y)}{2}\Big\}.$$

Since $d(x, y) = |x - y| = |a + y|$, $M(x, y) \geq |a + y|$. Clearly, equation (2.16) holds. So, all the conditions of Theorem 2.6 are satisfied. Hence T has a unique fixed point and it is clear that 0 is the fixed point of T.

Example 2.10. Let $X = (-\infty, 0]$, $Y = [0, \infty)$ and $d : (-\infty, 0] \times [0, \infty) \to [0, \infty)$ as $d(x, y) = |x - y|$. Then, clearly (X, Y, d) is a complete bipolar metric space. Define $T : \mathbb{R} \to \mathbb{R}$ as $Tx = \frac{-x}{8}$, then we know that T is a contravariant continuous map.

$$\begin{aligned} d(Ty, Tx) &= \Big|\frac{-y}{8} - \frac{-x}{8}\Big| \\ &= \Big|\frac{-y}{8} + \frac{x}{8}\Big| \\ &= \Big|\frac{y}{8} + \frac{a}{8}\Big|, \end{aligned} \tag{2.44}$$

where $x = -a$ for $a \geq 0$, for all $(x, y) \in X \times Y$.

Taking $\psi(z) = \frac{z}{2}$, $s = 1$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$. Since

$$\mathfrak{J}_s^T(x, y) = \Big[\alpha_1\Big(\frac{d(x, Tx)d(Ty, y)}{d(x, y)}\Big)^s + \alpha_2(d(x, y))^s\Big]^{\frac{1}{s}},$$

$$\begin{aligned} \psi(\mathfrak{J}_s^T(x, y)) &= \frac{1}{2}\Big(\frac{1}{2}(\frac{d(x, Tx)d(Ty, y)}{d(x, y)}) + \frac{1}{2}|a + y|\Big) \\ &= \frac{1}{4}\Big|a + y\Big| + B, \\ \Big|\frac{y}{8} + \frac{a}{8}\Big| &\leq \frac{1}{4}\Big|a + y\Big| + B, \end{aligned}$$

where $B = \frac{1}{4}\Big(\frac{d(x,Tx)d(Ty,y)}{d(x,y)}\Big) \geq 0$, equation (2.31) holds. Hence all the conditions of Theorems 2.8 are holds. Thus T has a fixed point. Clearly, 0 is the fixed point of T.

3. Consequences

In this section, we shall discuss about the consequences of Theorem 2.8.

Corollary 3.1. *Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant continuous mapping such that*

$$d(Ty, Tx) \leq \theta\Big[\alpha_1\Big(\frac{d(x, Tx)d(Ty, y)}{d(x, y)}\Big)^s + \alpha_2(d(x, y))^s\Big]^{\frac{1}{s}} \tag{3.1}$$

for all $(x, y) \in X \times Y$ with $x \neq y$ and $s \geq 0$, $\alpha_i \geq 0$, $i = 1, 2$ such that $\alpha_1 + \alpha_2 = 1$ and $\theta \in (0, 1)$. Then T has a fixed point.

Proof. Taking $\psi(z) = \theta z$ in Theorem 2.8, one can get the proof. □

Corollary 3.2. *Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant continuous mapping such that*

$$d(Ty, Tx) \leq \frac{\theta}{2^{\frac{1}{2}}}\Big[\Big(\frac{d(x, Tx)d(Ty, y)}{d(x, y)}\Big)^2 + (d(x, y))^2\Big]^{\frac{1}{2}} \tag{3.2}$$

for all $(x, y) \in X \times Y$ with $x \neq y$ and $\theta \in (0, 1)$. Then T has a fixed point.

Proof. Taking $\psi(z) = \theta z$, $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $s = 2$ in Theorem 2.8. □

Corollary 3.3. *Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant continuous mapping such that*

$$d(Ty, Tx) \leq \alpha\Big(\frac{d(x, Tx)d(Ty, y)}{d(x, y)}\Big) + \beta(d(x, y)) \tag{3.3}$$

for all $(x, y) \in X \times Y$ with $x \neq y$ and $\alpha, \beta \in (0, 1)$. Then T has a fixed point.

Proof. Putting $\theta\alpha_1 = \alpha, \theta\alpha_2 = \beta$ and $s = 1$ in Corollary 3.1. □

Corollary 3.4. *Let (X, Y, d) be a bipolar metric space and $T : (X, Y) \rightleftarrows (X, Y)$ be a contravariant continuous mapping such that*

$$d(Ty, Tx) \leq \theta(d(x, Tx))^{\alpha}(d(Ty, y))^{1-\alpha} \tag{3.4}$$

for all $(x, y) \in X \times Y$ with $x \neq y$ and $\theta, \alpha \in (0, 1)$. Then T has a fixed point.

Proof. Letting $\alpha_1 = \alpha, \alpha_2 = 1 - \alpha, s = 0$ and $\psi(z) = \theta z$ in Theorem 2.8, we get the proof. □

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