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REICH-TYPE CONTRACTION ON EXTENDED METRIC SPACE OF TYPE (ϕ, ρ) AND SOME FIXED POINT RESULTS

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Abstract. In this article, we present a new notion called "extended metric spaces of type (ϕ, ρ)" as a generalization of extended b-metric spaces. Also, we establish a fixed point result of a Reich-type contraction on an extended metric space of type (ϕ, ρ). We also provide several examples to demonstrate the significance of the established results.

1. Introduction

The concept of fixed points is one of the most investigated topics in scientific studies. It is commonly developed by applying it in different spaces and with various conditions, namely, metric space, and b-metric space. For some results in fixed point theory (see [1]-[13], [15], [17]-[29]).

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In 2017 Kamran et al. [14] extended the notion of $b-$metric spaces to the notion of extended $b-$metric spaces and they proved a Banach contraction theorem in such spaces. Recently, Mlaiki et al.[16] presented another new metric space called "controlled metric type space" as a generalization of $b-$metric spaces and proved the Banach contraction theorem in such spaces.

In this paper, we establish a new fixed point result on extended metric space of type (ϕ, ρ) of a Reich-type contraction[8]. Some examples are also provided to illustrate the significance of the presented results in terms of the existence and uniqueness of fixed point.

Kamran et al. [14] introduced the concept of extended b-metric space as a generalization of b-metric spaces as follows:

Definition 1.1. ([14]) Let $\alpha : D \times D \to [1, \infty)$. The function $\iota : D \times D \to [0, \infty)$ is said to be an extended b-metric if for all $\tau, \varphi, \omega \in D$, we have

(1) $\iota(\tau, \varphi) = 0$ if and only if $\tau = \varphi$,
(2) $\iota(\tau, \varphi) = \iota(\varphi, \tau)$,
(3) $\iota(\tau, \varphi) \leq \alpha(\tau, \varphi)[\iota(\tau, \omega) + \iota(\omega, \varphi)]$.

Then the pair (D, ι) is called an extended b-metric space. In this current paper, we will say (D, ι) is an extended b-metric space through α.

2. Main results

We start our work by given the notion of extended metric spaces of type (ϕ, ρ) as a generalization of extended $b-$metric spaces as follows:

Definition 2.1. On a nonempty set D, consider $\phi, \rho : D \times D \to [1, \infty)$. An extended metric of type (ϕ, ρ) is a function $\Lambda : D \times D \to [1, \infty)$ that achieves:

(1) $\Lambda(\tau, \varphi) = 0$ if and only if $\tau = \varphi$,
(2) $\Lambda(\tau, \varphi) = \Lambda(\varphi, \tau)$,
(3) $\Lambda(\tau, \varphi) \leq \phi(\tau, \varphi)\ \Lambda(\tau, \omega) + \rho(\tau, \varphi)\ \Lambda(\omega, \varphi)\ \ \forall \tau, \varphi, \omega \in D$.

Henceforth (D, Λ) is refereed as an extended metric space of type (ϕ, ρ).

Remark 2.2. An extended metric space of type (ψ, ρ) in general is not an extended b-metric through ψ or ρ.

Example 2.3. Put $D = \{0, 1, 2\}$, set the distance function Λ: $D \times D \to [0, \infty)$ as

$\Lambda(\tau, \varphi)$	0	1	2
0	$\Lambda(0,0) = 0$	$\Lambda(1,0) = 1$	$\Lambda(2,0) = \frac{2}{5}$
1	$\Lambda(0,1) = 1$	$\Lambda(1,1) = 0$	$\Lambda(2,1) = \frac{6}{25}$
2	$\Lambda(0,2) = \frac{2}{5}$	$\Lambda(1,2) = \frac{6}{25}$	$\Lambda(2,2) = 0$

And define the functions ρ, ϕ: $D \times D \to [1, \infty)$ by

$\phi(\tau, \varphi)$	0	1	2
0	$\phi(0,0) = 1$	$\phi(1,0) = \frac{11}{10}$	$\phi(2,0) = 1$
1	$\phi(0,1) = \frac{11}{10}$	$\phi(1,1) = 1$	$\phi(2,1) = 1$
2	$\phi(0,2) = 1$	$\phi(1,2) = 1$	$\phi(2,2) = 1$

and

$\rho(\tau, \varphi)$	0	1	2
0	$\rho(0,0) = 1$	$\rho(1,0) = \frac{6}{5}$	$\rho(2,0) = \frac{151}{100}$
1	$\rho(0,1) = \frac{6}{5}$	$\rho(1,1) = 1$	$\rho(2,1) = \frac{8}{5}$
2	$\rho(0,2) = \frac{151}{100}$	$\rho(1,2) = \frac{8}{5}$	$\rho(2,2) = 1$

Then, it is clear that (D, Λ) is an extended metric space of type (ϕ, ρ).
Since

$$1 = \Lambda(0,1) > \phi(0,1)\Lambda(0,2) + \phi(0,1)\Lambda(2,1) = 0.704$$

and

$$1 = \Lambda(0,1) > \rho(0,1)\Lambda(0,2) + \rho(0,1)\Lambda(2,1) = 0.768,$$

(D, Λ) is not an extended b-metric space through the use of function ϕ or function ρ.

Now, we will present the topological definitions for the extended metric space of type (ϕ, ρ), which will be applied to our current paper.

Definition 2.4. Let (D, Λ) be an extended metric space of type (ϕ, ρ) and let $\{\xi_n\}_{n\in N}$ be a sequence on D. Then:

(1) $\{\xi_n\}_{n\in N}$ converges to some $\xi \in D$ if for each $\varepsilon > 0$, there exists an integer n_ε such that $\nu(\xi_n, \xi) < \varepsilon$ for each $n > n_\varepsilon$. Mathematically it is written as $\lim_{n\to\infty} \nu(\xi_n, \xi) = 0$.

(2) $\{\xi_n\}_{n\in N}$ is a Cauchy sequence if for each positive number ε, $\Lambda(\xi_m, \xi_n) < \varepsilon$ for all $m > n > n_\varepsilon$, where n_ε is a positive integer. Mathematically it is written as $\lim_{n,m\to\infty} \nu(\xi_n, \xi_m) = 0$.

(3) The space (D, Λ) is said to be a complete extended metric space of type (ϕ, ρ) if every Cauchy sequence converges to D.

Definition 2.5. Let (D, Λ) be an extended metric space of type (ϕ, ρ). For $\xi \in D$ and $c > 0$.

(1) The open set $\Omega(\xi, c)$ is defined as

$$\Omega(\xi, c) = \{\eta \in D, \Lambda(\xi, \eta) < c\}.$$

(2) The map $T : D \to D$ is called continuous at $\xi \in D$ if for all $\varepsilon > 0$, there exists $m > 0$ such that $T(\Omega(\xi, m)) \subseteq \Omega(T\xi, \varepsilon)$.

Remark 2.6. If T is continuous at $\xi \in D$, $\{\xi_n\}$ is a sequence in D and $\xi_n \to \xi$, then $T\xi_n \to T\xi$ when n approaches to ∞.

Now, we state and prove our main result:

Theorem 2.7. *Let (D, Λ) be a complete extended metric space of type (ϕ, ρ) and $T : D \to D$ be a self -mapping. Assume there exist $d, h, r \in (0, 1)$ that satisfy $k = \frac{d+h}{1-r} < 1$, and*

$$\Lambda(T\tau, T\varphi) \leq d\Lambda(\tau, \varphi) + h\Lambda(\tau, T\tau) + r\Lambda(\varphi, T\varphi) \tag{2.1}$$

for all $\tau, \varphi \in D$. For $\eta_0 \in D$, put $\eta_n = T^n\eta_0$. Now, assume that

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\phi(\eta_{i+1}, \eta_m)}{\phi(\eta_i, \eta_m)} \rho(\eta_i, \eta_m) < \frac{1}{k}. \tag{2.2}$$

Also, suppose that

$$\rho(\eta, T\eta) < \frac{d + h}{r(1 - r)}. \tag{2.3}$$

Then, T has a unique fixed point in D.

Proof. For $\eta_0 \in D$, construct a sequence $\{\eta_n\}$ in D by putting $\eta_n = T\eta_{n-1}$. Then the condition (2.1) implies that

$$\begin{aligned}\Lambda(\eta_n, \eta_{n+1}) &= \Lambda(T\eta_{n-1}, T\eta_n)\\ &\leq d\Lambda(\eta_{n-1}, \eta_n) + h\Lambda(\eta_{n-1}, T\eta_{n-1}) + r\Lambda(\eta_n, T\eta_n)\\ &= d\Lambda(\eta_{n-1}, \eta_n) + h\Lambda(\eta_{n-1}, \eta_n) + r\Lambda(\eta_n, \eta_{n+1}).\end{aligned}$$

Thus, we conclude

$$\Lambda(\eta_n, \eta_{n+1}) \leq \frac{d + h}{1 - r}\Lambda(\eta_{n-1}, \eta_n) = k\Lambda(\eta_{n-1}, \eta_n). \tag{2.4}$$

It follows from (2.4) that

$$\Lambda(\eta_n, \eta_{n+1}) \leq k^n\Lambda(\eta_0, \eta_1). \tag{2.5}$$

Now, for all n , $m \in N$ with $m > n$, and by using $\phi(a,b) \geq 1$, $\rho(a,b) \geq 1$, we find

$$\begin{aligned}
\Lambda(\eta_n,\eta_m) &\leq \phi(\eta_n,\eta_m)\Lambda(\eta_n,\eta_{n+1}) + \rho(\eta_n,\eta_m)\Lambda(\eta_{n+1},\eta_m)\\
&\leq \phi(\eta_n,\eta_m)\Lambda(\eta_n,\eta_{n+1}) + \rho(\eta_n,\eta_m)\phi(\eta_{n+1},\eta_m)\Lambda(\eta_{n+1},\eta_{n+2})\\
&\quad + \rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\Lambda(\eta_{n+2},\eta_m)\\
&\leq \phi(\eta_n,\eta_m)\Lambda(\eta_n,\eta_{n+1}) + \rho(\eta_n,\eta_m)\phi(\eta_{n+1},\eta_m)\Lambda(\eta_{n+1},\eta_{n+2})\\
&\quad + \rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\phi(\eta_{n+2},\eta_m)\Lambda(\eta_{n+2},\eta_{n+3})\\
&\quad + \rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\rho(\eta_{n+2},\eta_m)\Lambda(\eta_{n+3},\eta_m)\\
&\leq \ \vdots\\
&\leq \phi(\eta_n,\eta_m)\Lambda(\eta_n,\eta_{n+1}) + \rho(\eta_n,\eta_m)\phi(\eta_{n+1},\eta_m)\Lambda(\eta_{n+1},\eta_{n+2})\\
&\quad + \rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\phi(\eta_{n+2},\eta_m)\Lambda(\eta_{n+2},\eta_{n+3})\\
&\quad \vdots \qquad .\\
&\quad + \rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\ldots\rho(\eta_{m-2},\eta_m)\Lambda(\eta_{m-1},\eta_m)\\
&\leq \rho(\eta_{n-1},\eta_m)\phi(\eta_n,\eta_m)\Lambda(\eta_n,\eta_{n+1})\\
&\quad + \rho(\eta_{n-1},\eta_m)\rho(\eta_n,\eta_m)\phi(\eta_{n+1},\eta_m)\Lambda(\eta_{n+1},\eta_{n+2})\\
&\quad + \rho(\eta_{n-1},\eta_m)\rho(\eta_n,\eta_m)\rho(\eta_{n+1},\eta_m)\phi(\eta_{n+2},\eta_m)\Lambda(\eta_{n+2},\eta_{n+3})\\
&\quad \vdots\\
&\quad + \rho(\eta_{n-1},\eta_m)\rho(\eta_n,\eta_m)\ldots\rho(\eta_{m-2},\eta_m)\phi(\eta_{m-1},\eta_m)\Lambda(\eta_{m-1},\eta_m)\\
&= \sum_{i=n}^{m-1}\phi(\eta_i,\eta_m)\prod_{j=n}^{i}\rho(\eta_{j-1},\eta_m)\Lambda(\eta_i,\eta_{i+1}).
\end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we get

$$\Lambda(\eta_n,\eta_m) \leq \sum_{i=n}^{m-1}\phi(\eta_i,\eta_m)\prod_{j=n}^{i}\rho(\eta_{j-1},\eta_m)k^i\Lambda(\eta_0,\eta_1). \tag{2.7}$$

Let

$$\Delta_i = \phi(\eta_i,\eta_m)\prod_{j=n}^{i}\rho(\eta_{j-1},\eta_m)k^i\Lambda(\eta_0,\eta_1).$$

Then

$$\Delta_{i+1} = \phi(\eta_{i+1},\eta_m)\prod_{j=n}^{i+1}\rho(\eta_{j-1},\eta_m)k^{i+1}\Lambda(\eta_0,\eta_1).$$

Therefore, we have

$$\sup_{m>1} \lim_{i\to+\infty} \frac{\Delta_{i+1}}{\Delta i} = \sup_{m>1} \lim_{i\to+\infty} \frac{\phi(\eta_{i+1},\eta_m)}{\phi(\eta_i,\eta_m)} \rho(\eta_i,\eta_m) k < 1.$$

The ratio test proves to us that the sequence

$$\left\{ \sum_{i=n}^{m-1} \phi(\eta_i,\eta_m) \prod_{j=n}^{i} \rho(\eta_{j-1},\eta_m) k^i \Lambda(\eta_0,\eta_1) \right\}$$

is Cauchy in a real line, and as a result, the sequence $\{\eta_n\}$ is Cauchy in (D,Λ). Since the space (D,Λ) is complete, the sequence $\{\eta_n\}$ is convergent. This gives us

$$\lim_{n\to\infty} \Lambda(\eta_n,\zeta) = 0 \text{ for some } \zeta \in D. \tag{2.8}$$

If $T\zeta \neq \zeta$, the triangular inequality, (2.1), (2.3) and (2.8) imply that

$$\begin{aligned}
0 &< \Lambda(\zeta,T\zeta) \\
&\leq \phi(\zeta,T\zeta)\Lambda(\zeta,\eta_{n+1}) + \rho(\zeta,T\zeta)\Lambda(\eta_{n+1},T\zeta) \\
&= \phi(\zeta,T\zeta)\Lambda(\zeta,\eta_{n+1}) + \rho(\zeta,T\zeta)\Lambda(T\eta_n,T\zeta). \\
&\leq \phi(\zeta,T\zeta)\Lambda(\zeta,\eta_{n+1}) + \rho(\zeta,T\zeta)[d\Lambda(\eta_n,\zeta) + h\Lambda(\eta_n,T\eta_n) + r\Lambda(\zeta,T\zeta)] \\
&\leq (\frac{d+h}{1-r} = k < 1)\Lambda(\zeta,T\zeta) \\
&< \Lambda(\zeta,T\zeta).
\end{aligned}$$

From the above inequalities, we arrive at

$$0 < \Lambda(\zeta,T\zeta) < \Lambda(\zeta,T\zeta),$$

which is a contradiction. So, we conclude $T\zeta = \zeta$.

To prove the uniqueness, we assume $\varphi \in D$ in such manner that $T\varphi = \varphi$ and $\zeta \neq \varphi$. Then, we have

$$\begin{aligned}
0 &< \Lambda(\zeta,\varphi) \\
&= \Lambda(T\zeta,T\varphi) \\
&\leq d\Lambda(\zeta,\varphi) + h\Lambda(\zeta,T\zeta) + r\Lambda(\varphi,T\varphi) \\
&< d\Lambda(\zeta,\varphi),
\end{aligned}$$

which is a contradiction. Thus, $\zeta = \varphi$. This completes the proof. □

Example 2.8. Let $D = \{0,1,2\}$. Set the distance function $\Lambda\colon D\times D \to [0,\infty)$ as

$\Lambda(\tau,\varphi)$	0	1	2
0	$\Lambda(0,0) = 0$	$\Lambda(1,0) = \frac{5}{3}$	$\Lambda(2,0) = \frac{15}{18}$
1	$\Lambda(0,1) = \frac{5}{3}$	$\Lambda(1,1) = 0$	$\Lambda(2,1) = \frac{10}{14}$
2	$\Lambda(0,2) = \frac{15}{18}$	$\Lambda(1,2) = \frac{10}{14}$	$\Lambda(2,2) = 0$

And define the functions ρ, ϕ: $D \times D \to [1, \infty)$ by

$\phi(\tau,\varphi)$	0	1	2
0	$\phi(0,0) = 1$	$\phi(1,0) = 2$	$\phi(2,0) = 2$
1	$\phi(0,1) = 2$	$\phi(1,1) = 1$	$\phi(2,1) = 2$
2	$\phi(0,2) = 2$	$\phi(1,2) = 2$	$\phi(2,2) = 1$

and

$\rho(\tau,\varphi)$	0	1	2
0	$\rho(0,0) = 1$	$\rho(1,0) = \frac{3}{2}$	$\rho(2,0) = \frac{3}{2}$
1	$\rho(0,1) = \frac{3}{2}$	$\rho(1,1) = 1$	$\rho(2,1) = \frac{3}{2}$
2	$\rho(0,2) = \frac{3}{2}$	$\rho(1,2) = \frac{3}{2}$	$\rho(2,2) = 1$

We define $T(0) = 2, T(1) = T(2) = 1$, and take $d = \frac{4}{7},\ h = \frac{1}{8},\ r = \frac{2}{5}$, then it is obvious that the conditions of Theorem 2.7 are hold, so 1 is the only fixed point of T.

Example 2.9. Let $D = [0, 1]$. Define the extended metric space of type (ϕ, ρ) by

$$\Lambda(\tau, \varphi) = |\tau + \varphi|^2.$$

Also, define the two functions ϕ and ρ by $\phi(\tau, \varphi) = 1 + \tau + \varphi$ and $\rho(\tau, \varphi) = 2\{1 + \max(\tau, \varphi)\}$ for all $\tau, \varphi \in D$. Moreover, Define $T : D \to D$ by $Tx = \frac{x^2}{6}$. If we choose $d = \frac{1}{7}, h = \frac{1}{8}, r = \frac{2}{5}$, and $\tau_0 = 0$, then all conditions of Theorem 2.7 are satisfied. Therefore T has a unique fixed point. Here 0 is the unique fixed point of T.

Corollary 2.10. *Let (D, Λ) be a complete extended metric space of type (ϕ, ρ). Take $\lambda \in (0, 1)$ such that $T : D \to D$ satisfies*

$$\Lambda(T\tau, T\varphi) \leq \lambda \Lambda(\tau, \varphi) \tag{2.9}$$

for all $\tau, \varphi \in D$. For $\eta_0 \in D$, put $\eta_n = T^n \eta_0$. Also, assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\phi(\eta_{i+1}, \eta_m)}{\phi(\eta_i, \eta_m)} \rho(\eta_i, \eta_m) < \frac{1}{\lambda}. \tag{2.10}$$

Then, T has a unique fixed point in D.

Proof. Taking h = r =0 in Theorem 2.7. □

Example 2.11. Let $D = \{0, 1, 2\}$. Define the extended metric space $\Lambda : D \times D \to [0, +\infty)$ of type (ϕ, ρ) via

$\Lambda(\tau, \varphi)$	0	1	2
0	$\Lambda(0,0) = 0$	$\Lambda(1,0) = 1$	$\Lambda(2,0) = \frac{2}{5}$
1	$\Lambda(0,1) = 1$	$\Lambda(1,1) = 0$	$\Lambda(2,1) = \frac{6}{25}$
2	$\Lambda(0,2) = \frac{2}{5}$	$\Lambda(1,2) = \frac{6}{25}$	$\Lambda(2,2) = 0$

Also, define the functions $\phi,\ \rho : D \times D \to [1, +\infty)$ by

$\phi(\tau, \varphi)$	0	1	2
0	$\phi(0,0) = 1$	$\phi(1,0) = \frac{6}{5}$	$\phi(2,0) = \frac{151}{100}$
1	$\phi(0,1) = \frac{6}{5}$	$\phi(1,1) = 1$	$\phi(2,1) = \frac{8}{5}$
2	$\phi(0,2) = \frac{151}{100}$	$\phi(1,2) = \frac{8}{5}$	$\phi(2,2) = 1$

and

$\rho(\tau, \varphi)$	0	1	2
0	$\rho(0,0) = 1$	$\rho(1,0) = \frac{6}{5}$	$\rho(2,0) = \frac{8}{5}$
1	$\rho(0,1) = \frac{6}{5}$	$\rho(1,1) = 1$	$\rho(2,1) = \frac{33}{20}$
2	$\rho(0,2) = \frac{8}{5}$	$\rho(1,2) = \frac{33}{20}$	$\rho(2,2) = 1$

Define $T\tau = 1$ for all $\tau \in$ X. Let $\tau_0 = 1$ and $\lambda = \frac{1}{2}$. Then we have

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\phi(\eta_{i+1}, \eta_m)}{\phi(\eta_i, \eta_m)} \rho(\eta_i, \eta_m) = 1 < 2 = \frac{1}{\lambda}.$$

Thus Corollary 2.10 is accomplished.

Corollary 2.12. *On a set D, consider the function $T : D \times D$. Impose (D, Λ) is complete and there exists $a \in (0, \frac{1}{2})$ such that*

$$\Lambda(T\tau, T\varphi) \leq a[\Lambda(\tau, T\tau) + \Lambda(\varphi, T\varphi)] \tag{2.11}$$

for all $\tau, \varphi \ \in D$. Take $\eta_n = T^n \eta_0$. Moreover, assume

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\phi(\eta_{i+1}, \eta_m)}{\phi(\eta_i, \eta_m)} \rho(\xi_i, \eta_m) < \frac{1}{a} - 1. \tag{2.12}$$

Also, for each $\eta \in D$, impose

$$\rho(\eta, T\eta) < \frac{1}{a}. \tag{2.13}$$

Then, T has a unique fixed point.

Proof. The proof follows from Theorem 2.7 by putting $d = 0$ and $h = r$. $\square$

Corollary 2.13. *Assume (D, Λ) is a complete extended $b-$metric space through the function α. Consider the map $T : D \times D$. Let there exists $a \in (0, \frac{1}{2})$ such that*

$$\Lambda(T\tau, T\varphi) \leq a[\Lambda(\tau, T\tau) + \Lambda(\varphi, T\varphi)] \tag{2.14}$$

for all $\tau, \varphi \in D$. For $\eta \in D$, put $\eta_n = T^n\eta_0$. Moreover, assume

$$\sup_{m\geq 1} \lim_{i\to\infty} \alpha(\eta_{i+1}, \eta_m) < \frac{1}{a} - 1. \tag{2.15}$$

Also, for each $\eta \in D$, impose

$$\alpha(\eta, T\eta) < \frac{1}{a}. \tag{2.16}$$

Then, T has a unique fixed point.

Proof. The proof follows from Corollary 2.12 by noting that (D, Λ) is a complete extended $b-$metric space of type (α, α) once we take $\phi = \rho = \alpha$. □

3. Conclusion

We introduced a new definition of metric spaces called extended metric spaces of type (ϕ, ρ) as a generalization of extended b-metric spaces. We also proved the existence and uniqueness of a fixed point for a self-mapping in such spaces that satisfies a set of conditions. In particular, we established and prove a new fixed point result on extended metric spaces of type (ϕ, ρ) of a Reich-type contraction.

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