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TURÁN-TYPE L^r-INEQUALITIES FOR POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$, and $r \geq 1$, Aziz [1] proved

$$\left\{\int_0^{2\pi} \left|1 + k^n e^{i\theta}\right|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)| \geq n \left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}.$$

In this paper, we obtain an improved extension of the above inequality into polar derivative. Further, we also extend an inequality on polar derivative recently proved by Rather et al. [20] into L^r-norm. Our results not only extend some known polynomial inequalities, but also reduce to some interesting results as particular cases.

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1. Introduction

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and for each $r > 0$, we define

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}.$$

If we take limit as $r \to \infty$ and make use of the well-known fact from analysis [23, 25] that

$$\lim_{r\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Let $p(z)$ be a polynomial of degree n, it was shown by Turán [26] that if $p(z)$ has all its zeros in $|z| \le 1$, then

$$\|p'\|_\infty \ge \frac{n}{2} \|p\|_\infty. \tag{1.1}$$

Inequality (1.1) is sharp and equality holds for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. Inequality (1.1) was refined by Aziz and Dawood [2] in the form

$$\|p'\|_\infty \ge \frac{n}{2} \left\{ \|p\|_\infty + \min_{|z|=1} |p(z)| \right\}. \tag{1.2}$$

Inequality (1.1) of Turán [26] has been of considerable interest and applications and it would be of interest to seek its generalization for polynomials having all their zeros in $|z| \le k, k > 0$. The case when $0 < k \le 1$ was settled by Malik [16] and proved

$$\|p'\|_\infty \ge \frac{n}{1+k} \|p\|_\infty. \tag{1.3}$$

While for the case $k \ge 1$, Govil [10] proved

$$\|p'\|_\infty \ge \frac{n}{1+k^n} \|p\|_\infty. \tag{1.4}$$

Equality in (1.4) holds for $p(z) = z^n + k^n, k \ge 1$.

As a refinement of inequality (1.3), Govil [11] proved

$$\|p'\|_\infty \ge \frac{n}{1+k} \left(\|p\|_\infty + \frac{1}{k^{n-1}} \min_{|z|=k} |p(z)| \right). \tag{1.5}$$

Equality in (1.5) holds for $p(z) = (z+k)^n$.

Aziz and Shah [4] generalized (1.3) by considering the class of polynomials having all their zeros in $|z| \leq k,\ k \leq 1$, with s- fold zeros at the origin and proved

$$\|p'\|_\infty \geq \frac{n+sk}{1+k}\|p\|_\infty. \tag{1.6}$$

The result is sharp and the extremal polynomial is $p(z) = z^s\,(z+k)^{n-s}$, $0 \leq s \leq n$.

Again, under the same hypothesis, it was Govil [11] who improved upon (1.4) by proving

$$\|p'\|_\infty \geq \frac{n}{1+k^n}\left\{\|p\|_\infty + \min_{|z|=k}|p(z)|\right\}. \tag{1.7}$$

Equality in (1.7) holds for $p(z) = z^n + k^n, k \geq 1$.

For the first time in 1984, Malik [15] extended inequality (1.1) proved by Turán [26] into L^r-norm and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $r > 0$,

$$\|1+z\|_r\|p'\|_\infty \geq n\|p\|_r. \tag{1.8}$$

The result is sharp and equality holds for $p(z) = (z+1)^n$ (see [21]).

In 1988, Aziz [1] obtained the L^r-norm extension of inequality (1.4) by proving the following result.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for $r \geq 1$,*

$$\|1+k^n z\|_r\|p'\|_\infty \geq n\|p\|_r. \tag{1.9}$$

The result is sharp and equality holds for $p(z) = \alpha z^n + \beta k^n, |\alpha| = |\beta|$.

For a polynomial $p(z)$ of degree n and a complex number α, let

$$D_\alpha p(z) = np(z) + (\alpha - z)p^{'}(z)$$

denote the polar derivative of the polynomial $p(z)$ with respect to α.

Note that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p^{'}(z).$$

Aziz and Rather [3] first extended inequality (1.3) to the polar derivative version and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then for every complex number α with $|\alpha| \geq k$,

$$\|D_\alpha p\|_\infty \geq n\left(\frac{|\alpha|-k}{1+k}\right)\|p\|_\infty. \tag{1.10}$$

Further, in the same paper [3], they also extended (1.4) to polar derivative and obtained

$$\|D_\alpha p\|_\infty \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\|p\|_\infty, \tag{1.11}$$

where α is any complex number with $|\alpha| \geq k$.

The corresponding polar derivative analogue of (1.7) and a refinement of (1.11) was given by Dewan et al. [7]. They proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$, then for any complex number α with $|\alpha| \geq k$,

$$\|D_\alpha p\|_\infty \geq \frac{n}{1+k^n}\left\{(|\alpha|-k)\|p\|_\infty + \left(|\alpha| + \frac{1}{k^{n-1}}\right)\min_{|z|=k}|p(z)|\right\}. \tag{1.12}$$

Recently, Govil and Kumar [12] proved a generalization and improvement of inequality (1.11), incorporating the leading coefficient and constant term of the polynomial (see [6], [22]).

Theorem 1.2. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$*, then for every complex number* α *with* $|\alpha| \geq k$*,*

$$\|D_\alpha p\|_\infty \geq \frac{|\alpha|-k}{1+k^n}\left\{n+s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right\}\|p\|_\infty. \tag{1.13}$$

Also, Rather et al. [20] proved a generalization and improvement of inequality (1.10) by involving leading coefficient and constant term of the polynomial.

Theorem 1.3. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \leq 1$*, then for every complex number* α *with* $|\alpha| \geq k$*,*

$$\|D_\alpha p\|_\infty \geq \frac{n(|\alpha|-t)}{1+k}\left\{1+\frac{k}{n}\left(s+\frac{k^{n-s}|c_{n-s}|-|c_0|}{k^{n-s}|c_{n-s}|+|c_0|}\right)\right\}\|p\|_\infty, \tag{1.14}$$

where

$$t = \frac{(n-s)k^2|c_{n-s}|+|c_{n-s-1}|}{(n-s)|c_{n-s}|+|c_{n-s-1}|}. \tag{1.15}$$

2. Lemmas

The following lemmas are needed for the proof of theorems and the corollaries. For a polynomial $p(z)$ of degree n, we will use $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

Lemma 2.1. ([16]) *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \leq 1$, then for $|z| = 1$,*

$$|q^{'}(z)| \leq k|p^{'}(z)|.$$

Lemma 2.2. ([24]) *Let $p(z) = c_n z^n + \sum\limits_{\nu=\mu}^{n} c_{n-\nu}z^{n-\nu}$, $1 \leq \mu \leq n$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$. Then for $|z| = 1$,*

$$|p'(z)| \geq \frac{n|c_n|k^{\mu-1} + \mu|c_{n-\mu}|}{n|c_n|k^{2\mu} + \mu|c_{n-\mu}|k^{\mu-1}}|q'(z)|, \tag{2.1}$$

where

$$q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Lemma 2.3. *If $p(z)$ is a polynomial of degree n, then for every $R \geq 1$ and $r > 0$,*

$$\left\{\int_0^{2\pi} \left|p\left(Re^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq R^n \left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{2.2}$$

As far as Lemma 2.3 is concerned, it is difficult to trace its origin. It was deduced from a well-known result of Hardy [13], according to which for every function $f(z)$ analytic in $|z| < t_0$, and for every $r > 0$,

$$\left\{\int_0^{2\pi} \left|f\left(te^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}$$

is a non-decreasing function of t for $0 < t < t_0$. If $p(z)$ is a polynomial of degree n, then $f(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$ is again a polynomial of degree at most n, that is, an entire function and by Hardy's result for $r > 0$,

$$\left\{\int_0^{2\pi} \left|f\left(te^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}},$$

for $t = \frac{1}{R} < 1$. This is equivalent to (2.2).

Lemma 2.4. ([8]) *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq 1$*, then for* $|z| = 1$*,*

$$\left|p^{'}(z)\right| \geq \frac{1}{2}\left\{n + s + \frac{|c_{n-s}| - |c_0|}{|c_{n-s}| + |c_0|}\right\}|p(z)|.$$

Lemma 2.5. ([14], [18]) *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z| < 1$*, then for every* $R \geq 1$ *and* $r > 0$*,*

$$\left\{\int_0^{2\pi}\left|p(Re^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.3}$$

where

$$E_r = \frac{\left\{\int_0^{2\pi}|1 + R^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1 + e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}}. \tag{2.4}$$

This lemma was proved by Boas and Rahman [14] for $r \geq 1$. Later, Rahman and Schmeisser [18] showed the validity for $0 < r < 1$ as well.

Lemma 2.6. ([19]) *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$*,* $k \leq 1$*, then for* $|z| = 1$*,*

$$\left|p^{'}(z)\right| \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(s + \frac{k^{n-s}|c_{n-s}| - |c_0|}{k^{n-s}|c_{n-s}| + |c_0|}\right)\right\}|p(z)|.$$

Lemma 2.7. ([9]) *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z| < k$*,* $k > 0$*, then for* $|z| < k$*,*

$$|p(z)| > m, \tag{2.5}$$

where

$$m = \min_{|z|=k}|p(z)|.$$

Lemma 2.8. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$*,* $k > 0$*, then for any complex number* λ *with* $|\lambda| < 1$ *and* $m = \min_{|z|=k}|p(z)|$*,*

$$k^n|c_{n-s}| - |\lambda|m - k^s|c_0| \geq 0. \tag{2.6}$$

Proof. By hypothesis, $p(z) = z^s h(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$. Then, the polynomial $P(z) = e^{-i\arg c_{n-s}} h(z)$ has the same zeros as $h(z)$. Now,

$$\begin{aligned} P(z) &= e^{-i\arg c_{n-s}} \left\{c_0 + c_1 z + \cdots + c_{n-s-1} z^{n-s-1} + |c_{n-s}| e^{i\arg c_{n-s}} z^{n-s}\right\} \\ &= e^{-i\arg c_{n-s}} \left\{c_0 + c_1 z + \cdots + c_{n-s-1} z^{n-s-1}\right\} + |c_{n-s}| z^{n-s}. \end{aligned} \tag{2.7}$$

Now, on $|z| = k$ for any complex number λ with $|\lambda| < 1$ and $m = \min_{|z|=k} p(z) \neq 0$, we have

$$\left|\frac{m\lambda}{k^n} z^{n-s}\right| < \frac{m}{k^s} = \min_{|z|=k} |h(z)| = \min_{|z|=k} |P(z)| \le |P(z)|.$$

Then by Rouche's theorem, $R(z) = P(z) - \frac{m|\lambda|}{k^n} z^{n-s}$ has all its zeros in $|z| < k$. Applying Vieta's formula to the polynomial $R(z)$, we get

$$\frac{|c_0|}{\left||c_{n-s}| - \frac{m|\lambda|}{k^n}\right|} < k^{n-s}. \tag{2.8}$$

Since $P(z)$ is a polynomial of degree $n - s$ having all its zeros in $|z| \le k$, $Q(z) = z^{n-s}\overline{P\left(\frac{1}{\bar z}\right)}$ is a polynomial of degree at most $n - s$ having no zero in $|z| < \frac{1}{k}$. Applying Lemma 2.7 to $Q(z)$, we have

$$|c_{n-s}| = |Q(0)| > \min_{|z|=\frac{1}{k}} |Q(z)| = \frac{1}{k^{n-s}} \min_{|z|=k} |P(z)| = \frac{m}{k^n},$$

that is,

$$|c_{n-s}| > \frac{m}{k^n}. \tag{2.9}$$

Using (2.9) to (2.8), we have

$$k^n |c_{n-s}| - |\lambda| m - k^s |c_0| > 0. \tag{2.10}$$

For $m = \min_{|z|=k} |p(z)| = 0$, the result follows trivially, simply on applying the similar argument of inequality (2.8) to the polynomial $h(z) = \sum_{j=0}^{n-s} c_j z^j$, that is,

$$k^n |c_{n-s}| - k^s |c_0| \ge 0. \tag{2.11}$$

This completes the proof. □

The following lemma was proved by [20]. However, we present an alternative proof of this lemma.

Lemma 2.9. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \leq 1$*, then for every complex number* λ *with* $|\lambda| < 1$ *and* $m = \min_{|z|=k} |p(z)|$,

$$t_m \leq k, \tag{2.12}$$

where $t_m = \dfrac{(n-s)k^2\left(|c_{n-s}| - \frac{|\lambda|m}{k^n}\right) + |c_{n-s-1}|}{(n-s)\left(|c_{n-s}| - \frac{|\lambda|m}{k^n}\right) + |c_{n-s-1}|}$.

Proof. Following the same argument as in the beginning of Lemma 2.8, it follows that $R(z) = P(z) - \frac{|\lambda|m}{k^n} z^{n-s}$ has all its zeros in $|z| < k$. Applying Vieta's formula to $R(z)$, we have

$$\left| \frac{c_{n-s-1}}{|c_{n-s}| - \frac{|\lambda|m}{k^n}} \right| < k(n-s). \tag{2.13}$$

Using (2.9), (2.13) becomes

$$\frac{|c_{n-s-1}|}{|c_{n-s}| - \frac{|\lambda|m}{k^n}} < k(n-s),$$

which implies

$$(1-k)|c_{n-s-1}| \leq k(1-k)(n-s)\left(|c_{n-s}| - \frac{|\lambda|m}{k^n}\right),$$

which gives

$$t_m \leq k. \tag{2.14}$$

Similarly, for $m = 0$, we get

$$t \leq k, \tag{2.15}$$

where t is as defined in Theorem 1.3. □

3. Main results

For the last more than 30 years, there is no generalizations and improvements of Theorem 1.1 due to Aziz [1] concerning polar derivative of a polynomial. In this direction, we are able to prove the following generalized L^r-norm extension of Theorem 1.2, which further gives an improved and a generalized L^r-norm analogue in polar derivative of Theorem 1.1. More precisely, we prove:

Theorem 3.1. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$*, then for every complex number* α *with* $|\alpha| \geq k$ *and* λ *with* $|\lambda| < 1$ *and* $r > 0$,

$$\left\| D_\alpha \left\{ p\left(e^{i\theta}\right) - \frac{m}{k^n} \lambda e^{in\theta} \right\} \right\|_r \geq \frac{|\alpha| - k}{2E_r} A \left\| p\left(e^{i\theta}\right) - \frac{m}{k^n} \lambda e^{in\theta} \right\|_r, \tag{3.1}$$

where

$$m = \min_{|z|=k} |p(z)|,$$

$$A = \left\{ n + s + \frac{k^n |c_{n-s}| - |\lambda| m - |c_0| k^s}{k^n |c_{n-s}| - |\lambda| m + |c_0| k^s} \right\}$$

and

$$E_r = \frac{\left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}.$$

Proof. By hypothesis, $p(z)$ has all its zeros in $|z| \leq k, k \geq 1$. For $m = \min_{|z|=k} |p(z)| \neq 0$, consider a polynomial $R(z) = p(z) - \frac{m}{k^n} \lambda z^n$, where λ is a complex number with $|\lambda| < 1$.

Now, on $|z| = k$

$$\left|\frac{m}{k^n} \lambda z^n\right| < \frac{m}{k^n} k^n \leq |p(z)|.$$

Then by Rouche's theorem, it follows that $R(z)$ has all its zeros in $|z| < k$ and in case $m = 0$, $R(z) = p(z)$. Thus, in any case, $R(z)$ has all its zeros in $|z| \leq k$. Then, the polynomial $P(z) = R(kz)$ has all its zeros in $|z| \leq 1$. It is easy to verify that for $|z| = 1$,

$$|Q'(z)| = \left| nP(z) - zP'(z) \right|, \tag{3.2}$$

where

$$Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}.$$

Applying Lemma 2.1 to $P(z)$, we have for $|z| = 1$

$$\left| Q'(z) \right| \leq \left| P'(z) \right|. \tag{3.3}$$

Using (3.2) and (3.3), we have for $\left|\frac{\alpha}{k}\right| \geq 1$ and $|z| = 1$,

$$\begin{aligned}\left|D_{\frac{\alpha}{k}}P(z)\right| &= \left|nP(z) + \left(\frac{\alpha}{k} - z\right) P^{'}(z)\right| \\ &\geq \left|\frac{\alpha}{k}\right| |P^{'}(z)| - \left|nP\left(z\right) - zP^{'}\left(z\right)\right| \\ &= \left|\frac{\alpha}{k}\right| |P^{'}(z)| - |Q^{'}\left(z\right)| \\ &\geq \left(\left|\frac{\alpha}{k}\right| - 1\right) |P^{'}(z)|. \end{aligned} \tag{3.4}$$

Applying Lemma 2.4 to $P(z)$, we have for $|z| = 1$

$$\left|P^{'}(z)\right| \geq \frac{1}{2}\left\{n + s + \frac{k^{n-s}|c_{n-s} - \frac{m}{k^n}\lambda| - |c_0|}{k^{n-s}|c_{n-s} - \frac{m}{k^n}\lambda| + |c_0|}\right\} |P(z)|. \tag{3.5}$$

Now, using the fact that the fuction $f(x) = \frac{x-|a|}{x+|a|}$ is a non-decreasing function of x and in view of (2.9), we get

$$\left|P^{'}(z)\right| \geq \frac{1}{2}\left\{n + s + \frac{k^n|c_{n-s}| - |\lambda m| - |c_0|k^s}{k^n|c_{n-s}| - |\lambda m| + |c_0|k^s}\right\} |P(z)|. \tag{3.6}$$

Combining (3.6) and (3.4), we get

$$\left|D_{\frac{\alpha}{k}}P\left(z\right)\right| \geq \frac{|\alpha| - k}{2k}\left\{n + s + \frac{k^n|c_{n-s}| - |\lambda m| - |c_0|k^s}{k^n|c_{n-s}| - |\lambda m| + |c_0|k^s}\right\} |P(z)|.$$

Replacing $P(z)$ by $R(kz)$ in the above inequality, we obtain

$$\left|nR(kz) + \left(\frac{\alpha}{k} - z\right) kR^{'}(kz)\right| \geq \frac{|\alpha| - k}{2k} A|R(kz)|, \tag{3.7}$$

where

$$A = \left\{n + s + \frac{k^n|c_{n-s}| - |\lambda m| - |c_0|k^s}{k^n|c_{n-s}| - |\lambda m| + |c_0|k^s}\right\}.$$

Inequality (3.7) is equivalent to

$$\left|nR(kz) + (\alpha - kz)\, R'(kz)\right| \geq \frac{|\alpha| - k}{2k} A|R(kz)|,$$

therefore for any $r > 0$, we have

$$\left|D_\alpha R\left(ke^{i\theta}\right)\right|^r \geq \left(\frac{|\alpha| - k}{2k} A\right)^r \left|R(ke^{i\theta})\right|^r, \quad 0 \leq \theta < 2\pi,$$

and hence

$$\left\{\int_0^{2\pi} \left|D_\alpha R\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha| - k}{2k} A\left\{\int_0^{2\pi} \left|R(ke^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.8}$$

Since $P(z)$ has all its zeros in $|z| \leq 1$, $Q(z)$ is a polynomial of degree at most n having all its zeros in $|z| \geq 1$. Applying Lemma 2.5 to $Q(z)$, we get

$$\left\{\int_0^{2\pi} \left|Q\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r \left\{\int_0^{2\pi} \left|Q(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.9}$$

Now, it can be easily obtained that

$$\left|Q(ke^{i\theta})\right| = k^n \left|R\left(e^{i\theta}\right)\right|$$

and

$$\left|Q\left(e^{i\theta}\right)\right| = \left|R\left(ke^{i\theta}\right)\right|.$$

With the above two relations, (3.9) gives

$$k^n \left\{\int_0^{2\pi} \left|R\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq E_r \left\{\int_0^{2\pi} \left|R\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.10}$$

Since $D_\alpha R(z)$ is a polynomial of degree at most $(n-1)$, applying Lemma 2.3 to $D_\alpha R(z)$ with $R = k \geq 1$, we have

$$\frac{1}{k^{n-1}} \left\{\int_0^{2\pi} \left|D_\alpha R\left(ke^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} \left|D_\alpha R\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.11}$$

Using (3.11) to (3.8), we get

$$k^{n-1} \left\{\int_0^{2\pi} \left|D_\alpha R\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha| - k}{2k} A \left\{\int_0^{2\pi} \left|R(ke^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$\left\{\int_0^{2\pi} \left|D_\alpha R\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha| - k}{2E_r} A \left\{\int_0^{2\pi} \left|R\left(e^{i\theta}\right)\right|^r d\theta\right\}^{\frac{1}{r}},$$

which is equivalent to

$$\left\{\int_0^{2\pi} \left|D_\alpha \left\{p\left(e^{i\theta}\right) - \frac{m}{k^n}\lambda e^{in\theta}\right\}\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{|\alpha| - k}{2E_r} A \left\{\int_0^{2\pi} \left|p\left(e^{i\theta}\right) - \frac{m}{k^n}\lambda e^{in\theta}\right|^r d\theta\right\}^{\frac{1}{r}}.$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Suppose $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k, k \geq 1$.

Now,

$$m = \min_{|z|=k} |p(z)| \leq \max_{|z|=k} |p(z)|. \tag{3.13}$$

By a simple deduction from Maximum Modulus Principle, we have

$$\max_{|z|=k} |p(z)| \leq k^n \max_{|z|=1} |p(z)|. \tag{3.14}$$

Using (3.14) to (3.13), we get

$$m \leq k^n \max_{|z|=1} |p(z)|,$$

that is,

$$\frac{m}{k^n} \leq \max_{|z|=1} |p(z)|. \tag{3.15}$$

For any complex number λ with $|\lambda| < 1$, we have

$$\frac{|\lambda| m}{k^n} < \max_{|z|=1} |p(z)|. \tag{3.16}$$

Remark 3.3. Letting $r \to \infty$ in (3.1) and noting the fact that $E_r \to \frac{1+k^n}{2}$ as limit $r \to \infty$, then we obtain

$$\max_{|z|=1} \left| D_\alpha \left\{ p(z) - \frac{m\lambda}{k^n} z^n \right\} \right| \geq \frac{|\alpha| - k}{1 + k^n} A \max_{|z|=1} \left| p(z) - \frac{m\lambda}{k^n} z^n \right|, \tag{3.17}$$

that is,

$$\max_{|z|=1} \left| D_\alpha p(z) - \frac{|\alpha| m n \lambda}{k^n} z^{n-1} \right| \geq \frac{|\alpha| - k}{1 + k^n} A \max_{|z|=1} \left| p(z) - \frac{m\lambda}{k^n} z^n \right|. \tag{3.18}$$

Let z_0 on $|z| = 1$ be such that

$$\max_{|z|=1} \left| D_\alpha p(z) - \frac{|\alpha| m n \lambda}{k^n} z^{n-1} \right| = \left| D_\alpha p(z_0) - \frac{|\alpha| m n \lambda}{k^n} z_0^{n-1} \right|. \tag{3.19}$$

In the right hand side of (3.19), if we choose the argument of λ such that

$$\left| D_\alpha p(z_0) - \frac{|\alpha| m n \lambda}{k^n} z_0^{n-1} \right| = |D_\alpha p(z_0)| - \frac{n|\alpha||\lambda|}{k^n} m. \tag{3.20}$$

From (3.19) and (3.20), (3.18) becomes

$$|D_\alpha p(z_0)| - \frac{n|\alpha||\lambda|}{k^n} m \geq \frac{|\alpha| - k}{1 + k^n} A \max_{|z|=1} \left| p(z) - \frac{m\lambda}{k^n} z^n \right|. \tag{3.21}$$

Since $|D_\alpha p(z_0)| \leq \max_{|z|=1} |D_\alpha p(z)|$, (3.21) gives

$$\max_{|z|=1} |D_\alpha p(z)| - \frac{n|\alpha||\lambda|}{k^n} m \geq \frac{|\alpha| - k}{1 + k^n} A \max_{|z|=1} \left| p(z) - \frac{m\lambda}{k^n} z^n \right|. \tag{3.22}$$

Let z_1 on $|z| = 1$ be such that $\max_{|z|=1} |p(z)| = |p(z_1)|$. Then

$$\begin{aligned} \max_{|z|=1} \left| p(z) - \frac{m\lambda}{k^n} z^n \right| &\geq \left| p(z_1) - \frac{m\lambda}{k^n} z^n \right| \\ &\geq \left| |p(z_1)| - \frac{m|\lambda|}{k^n} \right|. \end{aligned} \tag{3.23}$$

Using (3.16) to (3.23), we get

$$\max_{|z|=1}\left|p(z)-\frac{m\lambda}{k^n}z^n\right| \geq \max_{|z|=1}|p(z)|-\frac{m|\lambda|}{k^n}. \tag{3.24}$$

Using (3.24) to (3.22), we obtain

$$\max_{|z|=1}|D_\alpha p(z)|-\frac{n|\alpha||\lambda|}{k^n}m \geq \frac{|\alpha|-k}{1+k^n}A\left(\max_{|z|=1}|p(z)|-\frac{|\lambda|}{k^n}m\right). \tag{3.25}$$

Setting $|\lambda|=l$ in (3.25), we have

$$\max_{|z|=1}|D_\alpha p(z)|-\frac{n|\alpha|l}{k^n}m \geq \frac{|\alpha|-k}{1+k^n}A\left(\max_{|z|=1}|p(z)|-\frac{l}{k^n}m\right), \tag{3.26}$$

which on simplification and letting the limit as $l \to 1$ gives the following result recently proved by Abdullah Mir [17, Theorem 1].

Corollary 3.4. *If* $p(z)=z^s\left(\sum_{j=0}^{n-s}c_jz^j\right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$*, then for every complex number* α *with* $|\alpha| \geq k$*,*

$$\begin{aligned}\|D_\alpha p\|_\infty \geq& \frac{n}{1+k^n}\left\{(|\alpha|-k)\|p\|_\infty+\left(|\alpha|+\frac{1}{k^{n-1}}\right)m\right\}\\ &+\frac{|\alpha|-k}{1+k^n}\left\{s+\frac{k^n|c_{n-s}|-m-|c_0|k^s}{k^n|c_{n-s}|-m+|c_0|k^s}\right\}\left\{\|p\|_\infty-\frac{m}{k^n}\right\}, \end{aligned}\tag{3.27}$$

where $m=\min_{|z|=k}|p(z)|$.

Remark 3.5. Using the three facts (3.15), (2.6) and (2.9) in (3.27), it is clear that Corollary 3.4 gives the improvement of (1.12).

Remark 3.6. Dividing both sides of (3.27) of Corollary 3.4 by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following result due to Abdullah Mir [17, Corollary 2] which gives an improvement of (1.7) because of the same three facts (3.15), (2.6) and (2.9).

Corollary 3.7. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k, k \ge 1$, *then*

$$\left\|p'\right\|_\infty \ge \frac{n}{1+k^n}\left(\|p\|_\infty + m\right) + \frac{1}{1+k^n}\left\{s + \frac{k^n|c_{n-s}| - m - |c_0|k^s}{k^n|c_{n-s}| - m + |c_0|k^s}\right\}\left\{\|p\|_\infty - \frac{m}{k^n}\right\}, \qquad (3.28)$$

where $m = \min_{|z|=k} |p(z)|$. *The result is sharp and equality in* (3.28) *holds for* $p(z) = z^n + k^n$.

Remark 3.8. Dividing both sides of (3.1) of Theorem 3.1 by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following generalized L^r-norm extension of Corollary 3.7.

Corollary 3.9. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k, k \ge 1$, *then for every complex number* λ *with* $|\lambda| < 1$ *and* $r > 0$,

$$\|p'(e^{i\theta}) - \frac{mn}{k^n}\lambda e^{i(n-1)\theta}\|_r \ge \frac{A}{2E_r}\|p(e^{i\theta}) - \frac{m}{k^n}\lambda e^{in\theta}\|_r, \qquad (3.29)$$

where m, A *and* E_r *are as defined in Theorem* 3.1.

Remark 3.10. Putting $\lambda = 0$ in (3.1) of Theorem 3.1, we get the following L^r-norm extension of Theorem 1.2 which gives an improved and a generalized L^r-norm analogue in polar derivative of Theorem 1.1.

Corollary 3.11. *If* $p(z) = z^s\left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k, k \ge 1$, *then for every complex number* α *with* $|\alpha| \ge k$ *and* $r > 0$,

$$\|D_\alpha p\|_r \ge \frac{|\alpha| - k}{2E_r}\left\{n + s + \frac{k^{n-s}|c_{n-s}| - |c_0|}{k^{n-s}|c_{n-s}| + |c_0|}\right\}\|p\|_r\,, \qquad (3.30)$$

where E_r *is as defined in Theorem* 3.1.

Remark 3.12. If we let $r \to \infty$ in (3.30), Corollary 3.11 reduces to Theorem 1.2. Further, dividing both sides of it by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following improvement of (1.4) which in fact is a result obtained by Govil and Kumar([12], Corollary 1.2).

Corollary 3.13. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \geq 1$*, then*

$$\|p'\|_\infty \geq \frac{1}{1+k^n} \left\{ n + s + \frac{k^{n-s}|c_{n-s}| - |c_0|}{k^{n-s}|c_{n-s}| + |c_0|} \right\} \|p\|_\infty. \tag{3.31}$$

The result is sharp and equality in (3.31) *holds for* $p(z) = z^n + k^n$.

Remark 3.14. It may be noted that as the polynomial $p(z)$ is of degree $n \geq 1$, the leading coefficient c_n can not be zero and using the fact (2.6), it is clear that inequality (3.31) always gives improved bounds over the bound given by inequality (1.4). Also, for $k = 1$, in (3.28) and (3.31), the corresponding results sharpen (1.2) and (1.1) respectively.

Next, we prove the following L^r-norm inequality, which not only yields the L^r analog of (1.14) of Theorem 1.3 as a particular case, but also reduces to a rich number of interesting inequalities as special cases.

Theorem 3.15. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \leq 1$*, then for every complex number* α *with* $|\alpha| \geq k$ *and* λ *with* $|\lambda| < 1$ *and* $r > 0$,

$$\left\| D_\alpha \left\{ p(e^{i\theta}) - \frac{m}{k^n} \lambda e^{in\theta} \right\} \right\|_r \geq (|\alpha| - t_m) B_m \left\| p(e^{i\theta}) - \frac{m}{k^n} \lambda e^{in\theta} \right\|_r, \tag{3.32}$$

where

$$m = \min_{|z|=k} |p(z)|,$$

$$t_m = \frac{(n-s)k^2 \left(|c_{n-s}| - \frac{|\lambda| m}{k^n} \right) + |c_{n-s-1}|}{(n-s) \left(|c_{n-s}| - \frac{|\lambda| m}{k^n} \right) + |c_{n-s-1}|} \tag{3.33}$$

and

$$B_m = \frac{n}{1+k} \left\{ 1 + \frac{k}{n} \left(s + \frac{k^n|c_{n-s}| - |\lambda| m - |c_0| k^s}{k^n|c_{n-s}| - |\lambda| m + |c_0| k^s} \right) \right\}. \tag{3.34}$$

Proof. By hypothesis, $p(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, with s-fold zero at the origin. Now, consider a polynomial $P(z) = p(z) - \frac{m}{k^n} \lambda z^n$, where λ is a complex number with $|\lambda| < 1$, $m = \min_{|z|=k} |p(z)|$. Following the similar argument in the beginning of the proof of Theorem 3.1, it follows that $P(z)$ has all its zeros in $|z| \leq k$. Since $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, therefore, by Lemma 2.2 for $\mu = 1$, we have

$$\delta|P'(z)| \geq |Q'(z)| \quad \text{for } |z| = 1, \tag{3.35}$$

where $\delta = \dfrac{(n-s)k^2|c_{n-s} - \frac{\lambda m}{k^n}| + |c_{n-s-1}|}{(n-s)|c_{n-s} - \frac{\lambda m}{k^n}| + |c_{n-s-1}|}$ and $Q(z) = z^n \overline{P\left(\frac{1}{\overline{z}}\right)}$.

Since $k \leq 1$, it follows by derivative test that $\dfrac{(n-s)k^2 x + |c_{n-s-1}|}{(n-s)x + |c_{n-s-1}|}$ is a non-increasing function of x. Therefore, in view of (2.9), it follows that

$$\begin{aligned} \delta &= \frac{(n-s)k^2|c_{n-s} - \frac{\lambda m}{k^n}| + |c_{n-s-1}|}{(n-s)|c_{n-s} - \frac{\lambda m}{k^n}| + |c_{n-s-1}|} \\ &\leq \frac{(n-s)k^2\left(|c_{n-s}| - \frac{|\lambda| m}{k^n}\right) + |c_{n-s-1}|}{(n-s)\left(|c_{n-s}| - \frac{|\lambda| m}{k^n}\right) + |c_{n-s-1}|} \\ &= t_m. \end{aligned}$$

Therefore, from (3.35), we get

$$t_m|P'(z)| \geq |Q'(z)| \quad \text{for } |z| = 1. \tag{3.36}$$

Also, for every real or complex number α with $|\alpha| \geq k$, since, $|Q'(z)| = |nP(z) - zP'(z)|$ for $|z| = 1$, we have

$$\begin{aligned} |D_\alpha P(z)| &= \left|nP(z) + (\alpha - z)P'(z)\right| \\ &\geq \left||\alpha||P'(z)| - |nP(z) - zP'(z)|\right| \\ &= \left||\alpha||P'(z)| - |Q'(z)|\right|. \end{aligned} \tag{3.37}$$

Using (3.36) and Lemma 2.9, we have for $|z| = 1$

$$\begin{aligned} |\alpha||P'(z)| - |Q'(z)| &\geq \left||\alpha||P'(z)| - t_m|P'(z)|\right| \\ &= (|\alpha| - t_m)|P'(z)|. \end{aligned} \tag{3.38}$$

(3.37) on using (3.38), we have

$$|D_\alpha P(z)| \geq (|\alpha| - t_m)|P'(z)| \quad \text{for } |z| = 1. \tag{3.39}$$

Applying Lemma 2.6 to $P(z)$, we have

$$\left|P'(z)\right| \geq \frac{n}{1+k}\left\{1 + \frac{k}{n}\left(s + \frac{k^{n-s}|c_{n-s} - \frac{\lambda m}{k^n}| - |c_0|}{k^{n-s}|c_{n-s} - \frac{\lambda m}{k^n}| + |c_0|}\right)\right\}|P(z)|. \tag{3.40}$$

Now, using the fact that $\frac{y-|c|}{y+|c|}$, is a non-decreasing function of y and (2.9), we get

$$\left|P^{'}(z)\right| \geq \frac{n}{1+k}\left\{1+\frac{k}{n}\left(s+\frac{k^n|c_{n-s}|-|\lambda m|-|c_0|k^s}{k^n|c_{n-s}|-|\lambda m|+|c_0|k^s}\right)\right\}|P(z)|. \tag{3.41}$$

Combining (3.38) and (3.41), we obtain

$$|D_\alpha P(z)| \geq (|\alpha|-t_m)B_m|P(z)| \quad \text{for } |z|=1, \tag{3.42}$$

where

$$B_m = \frac{n}{1+k}\left\{1+\frac{k}{n}\left(s+\frac{k^n|c_{n-s}|-|\lambda m|-|c_0|k^s}{k^n|c_{n-s}|-|\lambda m|+|c_0|k^s}\right)\right\}.$$

That is,

$$\left|D_\alpha\left\{p(z)-\frac{m}{k^n}\lambda z^n\right\}\right| \geq (|\alpha|-t_m)B_m\left|p(z)-\frac{m}{k^n}\lambda z^n\right| \quad \text{for } |z|=1. \tag{3.43}$$

For each $r>0$, and for each θ, $0 \leq \theta < 2\pi$, (3.43) equivalently gives

$$\left|D_\alpha\left\{p(e^{i\theta})-\frac{m}{k^n}\lambda e^{in\theta}\right\}\right|^r \geq (|\alpha|-t_m)^r B_m^r\left|p(e^{i\theta})-\frac{m}{k^n}\lambda e^{in\theta}\right|^r. \tag{3.44}$$

Integrating both sides of (3.44) with respect to θ from 0 to 2π, we obtain

$$\int_0^{2\pi}\left|D_\alpha\left\{p(e^{i\theta})-\frac{m}{k^n}\lambda e^{in\theta}\right\}\right|^r d\theta \geq (|\alpha|-t_m)^r B_m^r \int_0^{2\pi}\left|p(e^{i\theta})-\frac{m}{k^n}\lambda e^{in\theta}\right|^r d\theta,$$

from which the desired conclusion of the theorem follows. □

Remark 3.16. Suppose $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$. Then $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$ has all it zeros in $|z| \geq \frac{1}{k}$, $\frac{1}{k} \geq 1$, that is, has no zero in $|z| < \frac{1}{k}$, $\frac{1}{k} \geq 1$, therefore applying Lemma 2.7 to $q(z)$, we get

$$|q(z)| \geq \min_{|z|=\frac{1}{k}}|q(z)| = \frac{1}{k^n}\min_{|z|=k}|p(z)| \quad \text{for} \quad |z| \leq \frac{1}{k}, \frac{1}{k} \geq 1.$$

Hence, in particular, for $|z|=1$,

$$|q(z)| \geq \frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{3.45}$$

Also, for $|z|=1$, we know

$$|p(z)| = |q(z)|. \tag{3.46}$$

From (3.45) and (3.46), we have

$$\max_{|z|=1}|p(z)| \geq \frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{3.47}$$

Remark 3.17. Letting the limit as $r \to \infty$ in (3.32) of Theorem 3.15 and using the fact (3.47) and following the same argument as in Remark 3.3, we obtain the following generalization of Theorem 1.3 which in fact is a result recently obtained by Rather et. al. [20, Theorem 2].

Corollary 3.18. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k, k \le 1$*, then for every complex number* α *with* $|\alpha| \ge k$ *and for each* l *with* $0 \le l < 1$*,*

$$\|D_\alpha p\|_\infty \ge \left(\frac{n(|\alpha| - t_m)}{1+k}\right)\|p\|_\infty + \frac{nl\,(k|\alpha| + t_m)}{k^n(1+k)}m + \frac{k(|\alpha| - t_m)}{1+k}\left(s + \frac{k^n|c_{n-s}| - lm - k^s|a_0|}{k^n|c_{n-s}| - lm + k^s|a_0|}\right)\left(\|p\|_\infty - \frac{lm}{k^n}\right), \quad (3.48)$$

where m *and* t_m *are as defined in Theorem* 3.15.

Remark 3.19. Dividing both sides of (3.48) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the following generalization of the result due to Rather [20, Corollary 3].

Corollary 3.20. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j\right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k$, $k \le 1$*, then for each* l *with* $0 \le l < 1$*,*

$$\|p'\|_\infty \ge \frac{n}{1+k}\left(\|p\|_\infty + \frac{l}{k^{n-1}}m\right) + \frac{k}{1+k}\left(s + \frac{k^n|c_{n-s}| - lm - k^s|a_0|}{k^n|c_{n-s}| - lm + k^s|a_0|}\right)\left(\|p\|_\infty - \frac{lm}{k^n}\right), \quad (3.49)$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 3.21. Taking limit as $l \to 1$ in (3.49), Corollary 3.20 gives the improvement of (1.5) and a result proved by Aziz and zargar [5, Theorem 1.2] by using the facts (3.47), (2.6) and (2.9).

Remark 3.22. If we take $\lambda = 0$ in (3.32), we get the following L^r extension of Theorem 1.3 which improves a result recently proved by Singh et al. [24, Corollary 1] by using the fact (3.53).

Corollary 3.23. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \leq 1$*, then for every complex number* α *with* $|\alpha| \geq k$*,*

$$\|D_\alpha p\|_r \geq (|\alpha| - t)B \|p\|_r \,, \tag{3.50}$$

where

$$t = \frac{(n-s)k^2|c_{n-s}| + |c_{n-s-1}|}{(n-s)|c_{n-s}| + |c_{n-s-1}|} \tag{3.51}$$

and

$$B = \frac{n}{1+k} \left\{ 1 + \frac{k}{n} \left(s + \frac{k^n|c_{n-s}| - |c_0|k^s}{k^n|c_{n-s}| + |c_0|k^s} \right) \right\}. \tag{3.52}$$

Remark 3.24. We are also interested to show that Corollary 3.23 is improved and generalized L^r-norm extension of inequality (1.10) for this it is sufficient to show $|\alpha| - t \geq |\alpha| - k$, $B \geq \frac{n+sk}{1+k}$, and the first inequality follows readily from inequality (2.12). Now,

$$\begin{aligned} B &= \frac{n}{1+k} \left\{ 1 + \frac{k}{n} \left(s + \frac{k^n|c_{n-s}| - |c_0|k^s}{k^n|c_{n-s}| + |c_0|k^s} \right) \right\} \\ &= \frac{n+sk}{1+k} + \frac{k}{1+k} \left(\frac{k^n|c_{n-s}| - |c_0|k^s}{k^n|c_{n-s}| + |c_0|k^s} \right). \end{aligned}$$

Using (2.11), it follows

$$B \geq \frac{n+sk}{1+k}, \tag{3.53}$$

and hence the claim.

Remark 3.25. Since $|\alpha| - t \geq |\alpha| - k$. Using this fact in Corollary 3.23, we get the following L^r-norm extension of a result due to Rather et. al. [19, Theorem 1.3] which is also an improvement and generalization of inequality (1.10).

Corollary 3.26. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \leq s \leq n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k, k \leq 1$*, then for every complex number* α *with* $|\alpha| \geq k$*,*

$$\|D_\alpha p\|_r \geq (|\alpha| - k)B \|p\|_r \,, \tag{3.54}$$

where B *is as defined in Corollary* 3.23.

Taking the limit as $r \to \infty$ in Corollary 3.26 and further dividing both side by $|\alpha|$ and $|\alpha| \to \infty$, we get the following generalization of a result recently proved by Rather et al. [19, Theorem 1.2] and improvement of (1.6) by using the fact (3.53).

Corollary 3.27. *If* $p(z) = z^s \left(\sum_{j=0}^{n-s} c_j z^j \right), 0 \le s \le n$ *is a polynomial of degree* n *having all its zeros in* $|z| \le k, k \le 1$*, then*

$$\left\|p'\right\|_\infty \ge B \left\|p\right\|_\infty, \tag{3.55}$$

where B *is as defined in Corollary* 3.23*.*

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References

[1] A. Aziz, *Integral mean estimates for polynomials with restricted zeros*, J. Approx. Theory, **55** (1988), 232–239.

[2] A. Aziz and Q.M. Dawood, *Inequalities for a polynomial and its derivatives*, J. Approx. Theory, **54** (1988), 306–313.

[3] A. Aziz and N.A. Rather, *A refinement of a theorem of Paul Turán concerning polynomials*, Math. Inequal. Appl., **1** (1998), 231–238.

[4] A. Aziz and W.M. Shah, *Inequalities for a polynomial and its derivative* , Math. Ineq. Appl., **7**(3) (2004), 379–391.

[5] A. Aziz and B.A. Zargar, *Inequalities for the maximum modulus of the derivative of a polynomial*, J. Ineq. Pure Appl. Math., **8**(2) (2007), Art. 37.

[6] Kh.B. Devi, K. Krishnadas and B. Chanam, *Some inequalities on polar derivative of a polynomial*, Nonlinear Funct Anal. Appl., **27**(1) (2022), 141-148.

[7] K.K. Dewan, N. Singh, A. Mir and A. Bhat, *Some inequalities for the polar derivative of a polynomial*, Southeast Asian Bull. Math., **34** (2010), 69–77.

[8] V.N. Dubinin, *Applications of the Schwarz lemma to inequalities for entire functions with constraints on zeros*, J. Math. Sci., **143** (2007), 3069–3076.

[9] R.B. Gardner, N.K. Govil and S.R. Musukula, *Rate of growth of polynomials not vanishing inside a circle*, J. Ineq. Pure Appl. Math., **6**(2) (2005), Art. 53.

[10] N.K. Govil, *On the Derivative of a Polynomial*, Proc. Amer. Math. Soc., **41** (1973), 543–546.

[11] N.K. Govil, *Some inequalities for the derivative of a polynomial*, J. Approx. Theory, **66** (1991), 29–35.

[12] N.K. Govil and P. Kumar, *On Sharpning of an inequality of Turán*, Appl. Anal. Discrete Math., **13** (2019), 711–720.

[13] G.H. Hardy, *The mean value of the modulus of an analytic function*, Proc. Lond. Math. Soc., **14** (1915), 269–277.

[14] R.P. Boas Jr and Q.I. Rahman, *L^p inequalities for polynomials and entire functions*, Arch. Rational Mech. Anal., **11** (1962), 34–39.

[15] M.A. Malik, *An integral mean estimates for polynomials*, Proc. Amer. Math. Soc., **91**(2) (1984), 281–284.
[16] M.A. Malik, *On the derivative of a polynomial*, J. Lond. Math. Soc., **1**(2) (1969), 57–60.
[17] A. Mir, *A note on an inequality of Paul Turán concerning polynomials*, Ramanujan J., **56** (2021), 1061-1071, https://doi.org/10.1007/s11139-021-00494-9.
[18] Q.I. Rahman and G. Schmeisser, *L^p inequalities for polynomials*, J. Approx. Theory, **53** (1988), 26–32.
[19] N.A. Rather, I. Dar and A. Iqbal, *Some inequalities for polynomials with restricted zeros*, Ann. Univ. Ferrara, **67** (2021), 183–189.
[20] N.A. Rather, I. Dar and A. Iqbal, *Some Lower Bound Estimates for the Polar Derivatives of Polynomals with Restricted Zeros* , J. Anal. Num. Theory, **9**(1) (2021), 1–5.
[21] N. Reingachan and B. Chanam, *Bernestien and Turán type inequalities for the polar derivative of a polynomial*, Nonlinear Funct Anal. Appl., **28**(1) (2023), 287-294.
[22] N. Reingachan, R. Soraisam and B. Chanam, *Some inequalities on polar derivative of a polynomial*, Nonlinear Funct Anal. Appl., **27**(4) (2022), 797-805.
[23] W. Rudin, *Real and Complex Analysis*, Tata McGraw-Hill publishing Company, 1977.
[24] T.B. Singh, K.B. Devi, R. Ngamchui, R. Soraisam and B. Chanam, *L^r Inequalities of Generalized Inequalities of Polynomials*, Turk. J. Comput. Math. Edu., **12**(14) (2021), 3525–3531.
[25] A.E. Taylor, *Introduction to Functional Analysis*, John Wiley and Sons, New York, 1958.
[26] P. Turan, *Uber die Ableitung von Polynomen*, Compos. Math., **7** (1939), 89–95.
[27] A. Zireh, *Integral mean estimates for polar derivative of polynomials*, J. Interdiscip. Math., **21**(1) (2018), 29–42.
[28] A. Zireh, E. Khojastehnezhad and S.R. Musawi, *Integral mean estimates for the polar derivative of polynomials whose zeros are within a circle*, J. Inequal. Appl., **2013**:307 (2013).