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SOME NEW SEQUENCE SPACES DEFINED BY MODULUS FUNCTION

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Abstract. The main object of this paper is to introduce the sequence spaces $[\hat{V}, \lambda, f, p]_0$ (Δ^r, E), $[\hat{V}, \lambda, f, p]_1(\Delta^r, E)$, $[\hat{V}, \lambda, f, p]_\infty(\Delta^r, E)$, $\hat{S}_\lambda(\Delta^r, E)$ and $\hat{S}_{\lambda_0}(\Delta^r, E)$ where E is a Banach space which arise from the notion of generalized de la Vellèe Poussin means and the concept of modulus function, examine them and give various properties and inclusion relations on these spaces.

1. Introduction

Let ω denote the set of all sequences (real or complex); l_∞, c and c_0 be respectively the Banach spaces of all bounded, convergent and null sequences with the usual norm $\|x\| = \sup_k |x_k|$, where $k \in \mathbb{N} = 1, 2, ...$, the set of positive integers. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. The generalized de la Vallèe Poussin means of a sequence x is defined as:

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \quad \text{where} \quad I_n = [n - \lambda_n + 1, n], \quad \text{for} \quad n \in \mathbb{N}.$$

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number l [7] if $t_n(x) \to l \; as \; n \to \infty$. If $\lambda_n = n$, then (V, λ)-summability and strong (V, λ)-summability are reduced to $(C, 1)$-summability and $[C, 1]$-summability.

Kizmaz [5] defined the difference sequence spaces,

$$X(\Delta) = \{ x = (x_k) : \Delta x \in X \},$$

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where $X = l_\infty, c$ and c_0 and $\Delta x = (x_k - x_{k+1})$. Then Et and Colak [2] generalized the above sequence spaces as:

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\},$$

for $X = l_\infty, c$ and c_0 where $r \in \mathbb{N}$ and $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1}), \Delta^r x = (\Delta^r x_k - \Delta^r x_{k+1})$, and $\Delta^r x_k = \sum_{\mu=0}^{k} \begin{pmatrix} n \\ r \end{pmatrix} x_{k+\mu}$.

Later on the difference sequence spaces have been studied by Malkowsky and Parashar [11], Et and Basarir [1] and others.

The concept of paranorm is related to linear metric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $p : x \to R$ is called a paranorm, if

(p.1) $p(0) \geq 0$,
(p.2) $p(x) \geq 0, \quad \forall\, x \in X$,
(p.3) $p(-x) = p(x), \quad \forall\, x \in X$,
(p.4) $p(x + y) \leq p(x) + p(y), \quad \forall\, x,\ y \in X$ (triangle inequality),
(p.5) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ $(n \to \infty)$, then $p(x_n\lambda_n - x\lambda) \to 0$ $(n \to \infty)$,(continuity of multiplication of vectors).

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total. It is well known that the metric of any linear metric space is given by same total paranorm ([13], Theorem 10.4.2).

Following Wilansky [13] and Maddox [10], a modulus function f is a function from $[0,\ \infty)$ to $[0,\ \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ $\forall x, y \geq 0$,
(iii) f is increasing,
(iv) f if continuous from right at $x = 0$.

Maddox [9] introduced and studied the sets:

$$[\hat{c}]_0 = \left\{x \in \omega : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}| = 0 \quad \text{uniformly in } m\right\},$$

$$[\hat{c}] = \{x \in \omega : x - le \in [\hat{c}] \quad \text{for some in } l \in C\}$$

of sequences that are strongly almost convergent to zero and strongly almost convergent.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let

$D = \max\{2,\ 2^{H-1}\}$. For $a_k,\ b_k \in C$ the set of complex numbers. We have (see, [8]) that

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \tag{1.1}$$

2. Some new sequence spaces defined by modulus function

In this section, we prove some results involving the sequence spaces $[\hat{V}, \lambda, f, p]_0(\Delta^r, E)$, $[\hat{V}, \lambda, f, p]_1(\Delta^r, E)$, $[\hat{V}, \lambda, f, p]_\infty(\Delta^r, E)$.

Definition 2.1. Let E be a Banach space. We define $\omega(E)$ to be the vector space of all E-valued sequences that is $\omega(E) = \{x \in \omega(E) : x_k \in E\}$. Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets:

$$[\hat{V}, \lambda, f, p]_1(\Delta^r, E) = \left\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f\|\Delta^r x_{k+m} - l\|]^{p_k} = 0 \text{ for some } l, \text{ uniformly in } m\right\}.$$

$$[\hat{V}, \lambda, f, p]_0(\Delta^r, E) = \left\{x \in \omega(E) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} [f\|\Delta^r x_{k+m}\|]^{p_k} = 0 \text{ uniformly in } m\right\}.$$

$$[\hat{V}, \lambda, f, p]_\infty(\Delta^r, E) = \left\{x \in \omega(E) : \sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} [f\|\Delta^r x_{k+m}\|]^{p_k} < \infty\right\}.$$

If $x \in [\hat{V}, \lambda, f, p]_1(\Delta^r, E)$ then we write $x_k \to l[\hat{V}, \lambda, f, p]_1(\Delta^r, E)$ and l will be called λ_E- difference limit of x with respect to the modulus f.

Throughout the paper Z will denote any of the notation $0,\ 1,\ \infty$. In case $f(x) = x$, $p_k = 1$ for all $k \in\ N$, we shall write $[\hat{V}, \lambda]_z(\Delta^r, E)$ and $[\hat{V}, \lambda, f]_z(\Delta^r, E)$ instead of $[\hat{V}, \lambda, f, p]_z(\Delta^r, E)$.

Theorem 2.2. *Let the sequence (p_k) be bounded. Then the sequence spaces $[\hat{V}, \lambda, f, p]_z(\Delta^r, E)$ are linear.*

Proof. We shall prove it for $[\hat{V},\lambda,f,p]_0(\Delta^r,E)$. The others can be proved in the same manner. Let $x,y\in[\hat{V},\lambda,f,p]_z(\Delta^r,E)$ and $\beta,\ \mu\in C$. Then there exists positive numbers M_β and M_μ such that $|\beta|\le M_\beta$ and $|\mu|\le N_\mu$. Since f is sub additive and Δ^r is linear we have

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r(\beta x_{k+m}+\mu y_{k+m}\|]^{p_k}\\&\le\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r(\beta x_{k+m}\|]^{p_k}+\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r(\mu y_{k+m}\|]^{p_k}\\&\le D(M_\beta)^H\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r x_{k+m}\|]^{p_k}+D\frac{1}{\lambda_n}(N_\mu)^H\sum_{k\in I_n}[f\|\Delta^r y_{k+m}\|]^{p_k}\to 0,\end{aligned}$$

as $n\to\infty$, uniformly in m. This proves that $[\hat{V},\lambda,f,p]_z(\Delta^r,E)$ is a linear space. □

Theorem 2.3. *Let f be a modulus function, then*

$$[\hat{V},\lambda,f,p]_0(\Delta^r,E)\subset[\hat{V},\lambda,f,p]_1(\Delta^r,E)\subset[\hat{V},\lambda,f,p]_\infty(\Delta^r,E).$$

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x\in[\hat{V},\lambda,f,p]_1(\Delta^r,E)$. By definition of f we have for all $m\in\mathbf{N}$,

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r\ x_{k+m}\|]^{p_k}&=\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r\ x_{k+m}-l+l\|]^{p_k},\\&\le D\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r\ x_{k+m}-l\|]^{p_k}+D\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|l\|]^{p_k}.\end{aligned}$$

There exists a positive integer K_l such that $\|l\|\le K_l$. Hence we have

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r\ x_{k+m}\|]^{p_k}\\&\le\frac{D}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r\ x_{k+m}-l\|]^{p_k}+\frac{D}{\lambda_n}\sum_{k\in I_n}[K_l f(1)]^H\lambda_n.\end{aligned}$$

Since $x\in[\hat{V},\lambda,f,p]_1(\Delta^r,E)$ we have $x\in[\hat{V},\lambda,f,p]_\infty(\Delta^r,E)$ and this completes the proof. □

Theorem 2.4. *$[\hat{V},\lambda,f,p]_0(\Delta^r,E)$ is a paranormed (need not be total paranorm) with*

$$g_\Delta(x)=\sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r(\ x_{k+m}\|]^{p_k}\right)^{\frac{1}{M}},$$

where $M = \max\{1, \sup_k p_k\}$.

Proof. From Theorem 2.3, for each $x \in [\hat{V}, \lambda, f, p]_0(\Delta^r, E)$, $g_\Delta(x)$ exists. Clearly, $g_\Delta(x) = g_\Delta(-x)$. Furthermore $g_\Delta(x) = 0$ implies $\Delta^r x_k = 0$ implies $x = 0$. Since $f(x) = 0$, we get $g_\Delta(x) = 0$ for $x = 0$. Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using the Minkowski's inequality and definition of f, for each n we have

$$\begin{aligned} g_\Delta(x+y) &= \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^r x_{k+m} + \Delta^r y_{k+m}\|)]^{p_k} \right)^{\frac{1}{M}} \\ &\leq \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^r x_{k+m}\|) + f(\|\Delta^r y_{k+m})\|]^{p_k} \right)^{\frac{1}{M}} \\ &\leq \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^r x_{k+m}\|)]^{p_k} \right)^{\frac{1}{M}} \\ &\quad + \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^r y_{k+m}\|)]^{p_k} \right)^{\frac{1}{M}} \\ &\leq g_\Delta(x) + g_\Delta(y). \end{aligned}$$

Hence $g_\Delta(x)$ is sub additive. Finally, to check the continuity of multiplication, let us take any complex number β. By definition of f we have

$$\begin{aligned} g_\Delta(\beta x) &= \sup_{m,n} \left(\frac{1}{\lambda_n} \sum_{k \in I_n} [f(\|\Delta^r(\beta x_{k+m})\|)]^{p_k} \right)^{\frac{1}{M}} \\ &\leq K_\beta^{\frac{H}{M}} g_\Delta(x), \end{aligned}$$

where K_β is a positive integer such that $|\beta| < K_\beta$. Now, let $\beta \to 0$ for any fixed x with $g_\Delta(x) \neq 0$. By definition of f for $|\beta| < 1$, we have

$$\frac{1}{\lambda_n} \sum_{k \in I_n} [f\|\Delta^r(\beta x_{k+m}\|]^{p_k} < \varepsilon, \tag{2.1}$$

for $n > n_0(\varepsilon)$.

Also, for all n with $1 \leq n \leq n_0$ and for all m, taking β small enough, since f is continuous we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f\|\Delta^r(\beta x_{k+m}\|]^{p_k} < \varepsilon. \tag{2.2}$$

(1.1) and (2.1) together imply that $g_\Delta(\beta x) \to 0$ as $\beta \to 0$. □

Theorem 2.5. *If $r \geq 1$, then the inclusion*

$$[\hat{V},\lambda,f]_Z(\Delta^{r-1},E) \subset [\hat{V},\lambda,f,p]_Z(\Delta^r,E),$$

is strict. In general

$$[\hat{V},\lambda,f]_Z(\Delta^i,E) \subset [\hat{V},\lambda,f,p]_Z(\Delta^r,E),\ \forall\ i = 1,2,...,r-1$$

and the inclusion is strict.

Proof. We give the proof for $z = \infty$ only. For $Z = 0$ and $Z = 1$, the proof is similar. Let $x \in [\hat{V},\lambda,f]_\infty(\Delta^{r-1},E)$. Then, we have

$$\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}[f\left(\|\Delta^{r-1}x_{k+m}\|\right)]^{p_k} < \infty.$$

By definition of f, we have

$$\begin{aligned}&\sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}[f\left(\|\Delta^r x_{k+m}\|\right)]\\ &\leq \sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}[f\left(\|\Delta^{r-1}x_{k+m}\|\right)] + \sup_{m,n}\frac{1}{\lambda_n}\sum_{k\in I_n}[f\left(\|\Delta^{r-1}x_{k+m}\|\right)]\\ &< \infty.\end{aligned}$$

Thus,

$$[\hat{V},\lambda,f]_Z(\Delta^{r-1},E) \subset [\hat{V},\lambda,f,p]_Z(\Delta^r,E).$$

Proceeding in this way one can show that

$$[\hat{V},\lambda,f]_Z(\Delta^i,E) \subset [\hat{V},\lambda,f,p]_Z(\Delta^r,E),\ \forall\ i = 1,2,...,r-1.$$

Let $E = C$ and $\lambda_n = n$ for each $n \in N$. Then the sequence $x = (k^r)$, belongs to $[\hat{V},\lambda,f]_Z(\Delta^r,E)$ but does not belong to $[\hat{V},\lambda,f]_Z(\Delta^{r-1},E)$ for $f(x) = x$. (If $x = (k^r)$ then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1}x_k = (-1)^{r+1}r!\left(k + \frac{(r-1)}{2}\right)$ for all $k \in N$). □

The proof of the following result is a routine work.

Theorem 2.6. $[\hat{V},\lambda,f,p]_1(\Delta^{r-1},E) \subset [\hat{V},\lambda,f,p]_0(\Delta^r,E)$.

Theorem 2.7. *Let $f,\ f_1,\ f_2$ be modulus functions. Then we have*

(i) $[\hat{V},\lambda,f_1,p]_Z(\Delta^r,E) \subset [\hat{V},\lambda,f\circ f_1,p]_Z(\Delta^r,E)$.

(ii) $[\hat{V}, \lambda, f_1, p]_Z(\Delta^r, E) \bigcap [\hat{V}, \lambda, f_2, p]_Z(\Delta^r, E) \subset [\hat{V}, \lambda, f_1 + f_2, p]_Z(\Delta^r, E)$.

Proof. We shall only prove (i). Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. Write $y_k = f_1(\|\Delta^r x_{k+m}\|)$ and consider

$$\sum_{k \in I_n} [f(y_{k+m})]^{p_k} = \sum_1 [f(y_{k+m})]^{p_k} + \sum_2 [f(y_{k+m})]^{p_k},$$

where the first summation is over $y_k \le \delta$ and the second summation is over $y_k > \delta$. Since, f is continuous, we have

$$\sum_1 [f(y_{k+m})]^{p_k} < \lambda_n \varepsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that $y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}$. By definition of f we have for $y_k > \delta$, $f(y_k) < 2f(1)\frac{y_k}{\delta}$. Hence,

$$\frac{1}{\lambda_n} \sum_2 [f(y_{k+m})]^{p_k} \le \max\left\{1, (2f(1)\delta^{-1})^H\right\} \frac{1}{\lambda_n} \sum_{k \in I_n} y_{k+m}. \tag{2.4}$$

From (2.2) and (2.3), we obtain $[\hat{V}, \lambda, f, p]_0(\Delta^r, E) \subset [\hat{V}, \lambda, f, p]_0(\Delta^r, E)$.

The proof of (ii) follows from the following inequality:

$$[(f_1 + f_2)(\|\Delta^r x_{k+m}\|)]^{p_k} \le D\,[f_1(\|\Delta^r x_{k+m}\|)]^{p_k} + D\,[f_2(\|\Delta^r x_{k+m}\|)]^{p_k}.$$

□

The following result is a consequence of Theorem 2.7 (i).

Theorem 2.8. *Let f be modulus function. Then*

$$[\hat{V}, \lambda, p]_Z(\Delta^r, E) \subset [\hat{V}, \lambda, f, p]_Z(\Delta^r, E).$$

The idea of statistical convergence was introduced by Fast [3] and studied by various authors ([4], [6], [12]).

Definition 2.9. A sequence $x = (x_k)$ is said to be λ_E^r-statistically convergent to a number l if for every $\varepsilon > 0$,

$$\frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^r x_{k+m} - l\| \ge \varepsilon\}| = 0, \quad \text{uniformly in } m.$$

In this case we write $\hat{S}(\Delta^r, E) - \lim x = l$ or $x_k \to l\hat{S}(\Delta^r, E)$. If $\lambda_n = n$ and $l = 0$ we shall write $\hat{S}(\Delta^r, E)$ and $\hat{S}_{\lambda_0}(\Delta^r, E)$ instead of $\hat{S}_\lambda(\Delta^r, E)$

We establish a relation between the sets $\hat{S}_\lambda(\Delta^r, E)$ and $[\hat{V}, \lambda, f, p]_1(\Delta^r, E)$.

Theorem 2.10. *The inclusion $\hat{S}_\lambda(\Delta^r, E) \subset [\hat{V}, \lambda, f, p]_1(\Delta^r, E)$ holds if and only if f is bounded.*

Proof. We shall assume that f is bounded and $x \in \hat{S}_\lambda(\Delta^r, E)$. Then, there exists a constant M such that $f(x) \leq M$ for all $x \geq 0$. Let $\varepsilon > 0$ be given. We choose η and $\delta > 0$ such that $M\delta + f(\eta) < \varepsilon$. Since, $x \in \hat{S}_\lambda(\Delta^r, E)$, there are $l \in C$ and $n \geq n_0(\eta,\ \gamma) \in N$ such that $\frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^r x_{k+m} - l\| \geq \eta\}| < \delta$, for all $n \geq n_0$ and for all m. Therefore,

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} f(\|\Delta^r x_{k+m} - l\|)^{p_k} \\
&= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - l| \geq \eta}} f(\|\Delta^r x_{k+m} - l\|)^{p_k} \\
&\quad + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - l| < \eta}} f(\|\Delta^r x_{k+m} - l\|)^{p_k} \\
&\leq M \frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^r x_{k+m} - l\| \geq \eta\}| + f(\eta) \\
&< M\delta + f(\eta) \\
&< \varepsilon,
\end{aligned}$$

for all $n \geq n_0$ and m. Hence, $x \in [\hat{V}, \lambda, f, p]_1(\Delta^r, E)$.

Conversely, we assume that f is unbounded. Then there exists a positive sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$f(n) = \begin{cases} t_k, & \text{if } i = k^2, \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$\frac{1}{\lambda_n} |\{k \in I_n : \|\Delta^r x_{k+m} - l\| \geq \varepsilon\}| \leq \frac{\sqrt{\lambda_{n-1}}}{\lambda_n}$$

for all n and m, and so $x \in \hat{S}_\lambda(\Delta^r, E)$ but $x \notin [\hat{V}, \lambda, f, p]_1(\Delta^r, E)$ for $E = C$. □

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