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FIXED POINT RESULTS IN SOFT RECTANGULAR b-METRIC SPACE

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Abstract. The fundamental aim of the proposed work is to introduce the concept of soft rectangular b-metric spaces, which involves generalizing the notions of rectangular metric spaces and b-metric spaces. Furthermore, an investigation into specific characteristics and topological aspects of the underlying generalization of metric spaces is conducted. Moreover, the research establishes fixed point theorems for mappings that satisfy essential criteria within soft rectangular b-metric spaces. These theorems offer a broader perspective on established results in fixed point theory. Additionally, several congruous examples are presented to enhance the understanding of the introduced spatial framework.

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1. Introduction

It is standard practice to adopt mathematical tools to analyze different characteristics of the structure and its many divisions performance. As a result, dealing with uncertainty and erroneous data in a variety of situations is very natural. Managing ambiguities and vagueness, whether they are inherent in the statistics or brought about by the mathematical procedures used to obtain the solution made up of various scenarios, can be difficult in many ways.

L.A. Zadeh [28] pioneered fuzzy set theory, which has evolved into a useful tool for dealing with ambiguities and uncertainties. To tackle the unpredictability linked with actual-world information scenarios, Molodtsov [21] designed soft sets as a mathematical concept. Maji et al. [18] introduced the basic operations on soft sets. In [23], Pei and Miao reformulated some operations, such as intersection and the concept of soft subsets. Soft functions have been explored from a variety of perspectives by many researchers [5, 17, 19, 26, 27].

Aras et al. [2] talked about the continuity of soft mappings. The soft topology was proposed differently in [9, 13, 16, 24]. The concepts of real sets as well as real numbers in a soft way were proposed and also analyzed in [11] by Das and Samanta. They established soft metric conceptual frameworks in [12]. Abbas et al. [1] proposed a conception of contraction function in the soft sense. Later, fixed points in soft contraction mapping were explored by many researchers. The very famous Banach contraction principle [6], developed by Banach, boosted functional analysis research. In [3, 4, 7, 8, 10, 14, 15, 20, 22, 25] authors added the concept of various generalizations of metric spaces along with several fixed point consequences within these spaces.

The conceptual frame of soft rectangle b-metric space is proposed in this study, which is demonstrated not to be Hausdorff and extends the ideas of soft, rectangular, and b-metric spaces. The Banach contraction theorem, the Ciric-type contraction theorem, and the Kannan-fixed point theorem are established equivalently in soft rectangular b-metric spaces. Further, the importance of the above results is demonstrated using appropriate examples.

2. Preliminaries

The following are some preliminary considerations for the writing of the article. The notations $\mathcal{K}, E$ and $\mathcal{P}(\mathcal{K})$ are used to indicate the initial universe, the collection of all parameters, and the collection of all subsets of $\mathcal{K}$, respectively. For a more comprehensive analysis, references such as [1, 2, 7, 11, 12, 13, 16, 18, 20, 21] can be viewed.

Definition 2.1. $(\mathfrak{F}, A)$ is said to be a soft set on the universal set $\mathcal{K}$ with reference to the parameter set $A \subseteq E$ if $\mathfrak{F}$ is a set based function on A that takes the values into $\mathcal{P}(\mathcal{K})$.

Definition 2.2. Any soft set $(\mathfrak{F}, A)$ on the initial universe $\mathcal{K}$ is said to be an absolute soft set of $\mathcal{K}$ if $\mathfrak{F}(\lambda) = \mathcal{K}$ for all $\lambda \in A$. It is signified with $\tilde{\mathcal{K}}_A$.

Definition 2.3. Let $(\mathfrak{F}, A)$, $(\mathfrak{I}, B)$ be two soft sets. The soft union of $(\mathfrak{F}, A)$ and $(\mathfrak{I}, B)$ is $(\mathfrak{D}, C)$ written as $(\mathfrak{F}, A) \tilde{\cup} (\mathfrak{I}, B) = (\mathfrak{D}, C)$, wherein $C = A \cup B$ and the mapping $\mathfrak{D}$ is defined by:

$$\mathfrak{D}(\lambda) = \begin{cases} \mathfrak{F}(\lambda), & \lambda \in A \setminus B, \\ \mathfrak{I}(\lambda), & \lambda \in B \setminus A, \\ \mathfrak{F}(\lambda) \cup \mathfrak{I}(\lambda), & \lambda \in A \cap B. \end{cases}$$

Definition 2.4. Let $(\mathfrak{F}, A)$, $(\mathfrak{I}, B)$ be two soft sets. The soft intersection $(\mathfrak{D}, C)$ of $(\mathfrak{F}, A)$ and $(\mathfrak{I}, B)$ is written as $(\mathfrak{F}, A) \tilde{\cap} (\mathfrak{I}, B) = (\mathfrak{D}, C)$, wherein $C = A \cap B$ and the mapping $\mathfrak{D}$ is defined by, $\mathfrak{D}(\lambda) = \mathfrak{F}(\lambda) \cap \mathfrak{I}(\lambda),\ \forall \lambda \in C$.

Definition 2.5. Let $(\mathfrak{F}, A)$ and $(\mathfrak{I}, B)$ be two soft sets such that $A \cap B \neq \phi$. Then, the soft set difference (restricted difference) is another soft set $(\mathfrak{D}, C) = (\mathfrak{F}, A) \setminus_s (\mathfrak{I}, B)$ wherein $C = A \cap B$ defined as, $\mathcal{D}(\lambda) = \mathfrak{F}(\lambda) \setminus_s \mathfrak{I}(\lambda),\ \forall \lambda \in C$.

Consider $\mathbb{R}$ as the collection of all real numbers, signifying the collection of all non-empty bounded subsets on the set $\mathbb{R}$ with $\mathfrak{B}(\mathbb{R})$.

Definition 2.6. A soft real set is a mapping $\mathfrak{F}\colon A \to \mathfrak{B}(\mathbb{R})$, where $\mathfrak{B}(\mathbb{R})$ is a collection of all non-empty bounded subsets of $\mathbb{R}$. To a soft real set $(\mathfrak{F}, A)$, if $\mathfrak{F}(\lambda) = \{x\}$, for every $\lambda \in A$ and for some $x \in \mathcal{K}$ then it is claimed to be a soft real number. It is represented as $\tilde{x}$.

In other words, for a soft real set $(\mathfrak{F}, A)$, if $\mathfrak{F}(\lambda)$ is a singleton member of $\mathcal{P}(\mathcal{K})$ for each $\lambda \in A$, then it is claimed as a soft real number.

Definition 2.7. A soft point is a soft set on the initial universe $\mathcal{K}$ for which $\mathfrak{F}(\lambda) = \{x\}$ for exactly one parameter $\lambda \in A$ and for some $x \in \mathcal{K}$ and $\mathfrak{F}(\mu) = \phi$, $\forall\ \mu \in A \setminus \{\lambda\}$. A soft point is signified by $\tilde{x}_\lambda$.
A soft point $\tilde{x}_\lambda$ is claimed to belong a soft set $(\mathfrak{F}, A)$ on the condition that $\tilde{x}_\lambda = x \subseteq \mathfrak{F}(\lambda)$, also shown as $\tilde{x}_\lambda \tilde{\in} (\mathfrak{F}, A)$. Collectively, a group of soft points of $(\mathfrak{F}, A)$ is signified with $SP(\mathfrak{F}, A)$.

The assembly of all soft real numbers and non-negative soft real numbers upon a parameter set E are denoted by $\mathbb{R}(E)$ and $\mathbb{R}(E)^*$ respectively. For a soft real number $\tilde{x}$, if $\tilde{x}(\lambda) = \{t\}$ for some $t \in \mathbb{R}$, then it is denoted by $\bar{t}$.

Remark 2.8. For any two soft real numbers $\tilde{p}$ and $\tilde{q}$ and $A \subseteq E$, we say

(1) $\tilde{p} \tilde{\leq} \tilde{q} \iff \tilde{p}(\mu) \leq \tilde{q}(\mu), \quad \forall \mu \in A;$
(2) $\tilde{p} \tilde{<} \tilde{q} \iff \tilde{x}(\mu) < \tilde{q}(\mu), \quad \forall \lambda \in A;$
(3) $\tilde{p} \tilde{\geq} \tilde{q} \iff \tilde{p}(\mu) \geq \tilde{q}(\mu), \quad \forall \lambda \in A;$
(4) $\tilde{p} \tilde{>} \tilde{q} \iff \tilde{p}(\mu) > \tilde{q}(\mu), \quad \forall \lambda \in A.$

Definition 2.9. For an absolute soft set $\tilde{\mathcal{K}}_E$ over a non-empty initial universe for a parameter set E, a mapping $\breve{\mathfrak{D}}_{\mathsf{sm}} \colon SP(\tilde{\mathcal{K}}_E) \times SP(\tilde{\mathcal{K}}_E) \to \mathbb{R}(E)^*$ is purported to be a soft metric on $\tilde{\mathcal{K}}_E$ if the following conditions satisfies:

(SM1) $\breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) \tilde{\geq} \bar{0}, \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SM2) $\breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) = \bar{0} \iff \tilde{p}_{a_i} = \tilde{q}_{a_j}, \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SM3) $\breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) = \breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{q}_{a_j}, \tilde{p}_{a_i}), \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SM4) $\breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{p}_{a_i}, \tilde{r}_{a_k}) \tilde{\leq} \breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) + \breve{\mathfrak{D}}_{\mathsf{sm}}(\tilde{q}_{a_j}, \tilde{r}_{a_k}), \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j}, \tilde{r}_{a_k} \tilde{\in} \tilde{\mathcal{K}}_E.$

The soft metric $\breve{\mathfrak{D}}_{\mathsf{sm}}$ together with the soft set $\tilde{\mathcal{K}}$ is known as a soft metric space. It is denoted as $(\tilde{\mathcal{K}}, \breve{\mathfrak{D}}_{\mathsf{sm}})$ or $(\tilde{\mathcal{K}}, \breve{\mathfrak{D}}_{\mathsf{sm}}, E)$.

Definition 2.10. For an absolute soft set $\tilde{\mathcal{K}}_E$ over a non-null initial universal set for a parameter set E, a mapping $\breve{\mathfrak{D}}_{\mathsf{sbm}} \colon SP(\tilde{\mathcal{K}}_E) \times SP(\tilde{\mathcal{K}}_E) \to \mathbb{R}(E)^*$ is purported to be a soft b-metric on $\tilde{\mathcal{K}}_E$ with a soft coefficient $\tilde{s}$ if the following conditions satisfies for $\breve{\mathfrak{D}}_{\mathsf{sbm}}$:

(SBM1) $\breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) \tilde{\geq} \bar{0}, \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SBM2) $\breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) = \bar{0} \iff \tilde{p}_{a_i} = \tilde{q}_{a_j}, \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SBM3) $\breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) = \breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{q}_{a_j}, \tilde{p}_{a_i}), \quad \forall\, \tilde{p}_{a_i}, \tilde{q}_{a_j} \tilde{\in} \tilde{\mathcal{K}}_E;$
(SBM4) $\breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{p}_{a_i}, \tilde{r}_{a_k}) \tilde{\leq} \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{p}_{a_i}, \tilde{q}_{a_j}) + \breve{\mathfrak{D}}_{\mathsf{sbm}}(\tilde{q}_{a_j}, \tilde{r}_{a_k})], \forall \tilde{p}_{a_i}, \tilde{q}_{a_j}, \tilde{r}_{a_k} \tilde{\in} \tilde{\mathcal{K}}_E.$

The soft b-metric $\breve{\mathfrak{D}}_{\mathsf{sbm}}$ together with the soft set $\tilde{\mathcal{K}}$ is known as a soft b-metric space with a soft coefficient $\tilde{s}$, denoted as $(\tilde{\mathcal{K}}, \breve{\mathfrak{D}}_{\mathsf{sbm}}, \tilde{s})$.

Definition 2.11. Let $(\tilde{\mathcal{K}}, \breve{\mathfrak{D}}_{\mathsf{sm}}, A)$ and $(\tilde{\mathcal{M}}, \breve{\mathfrak{D}}_{\mathsf{sm}}, B)$ be soft metric spaces. Then, a map $(\mathfrak{f}, \phi) : (\tilde{\mathcal{K}}, \breve{\mathfrak{D}}_{\mathsf{sm}}, A) \to (\tilde{\mathcal{M}}, \breve{\mathfrak{D}}_{\mathsf{sm}}, B)$ is called a soft mapping if $\mathfrak{f} \colon \tilde{\mathcal{K}} \to \tilde{\mathcal{M}}$ and $\phi \colon A \to B$.

3. Main results

Prior to actually establishing the main result, we must first introduce certain modified definitions, which are as below:

Definition 3.1. Consider a non-empty set X and a non-empty collection of parameters E. Consider $\tilde{X}_E$ as an absolute soft set, and the collection of all soft points of $\tilde{X}_E$ be signified as $SP(\tilde{X}_E)$. Also, the collection of all non-negative

soft real numbers is signified as $\mathbb{R}(E)^*$, and $[0, \infty)E$ indicates all soft real numbers in the interval $[0, \infty)$. Upon this foundation, the soft rectangular b-metric space using the concept of soft points is defined as below:
A map $\breve{\mathfrak{D}}_{\mathsf{srb}} : SP(\tilde{X}_E) \times SP(\tilde{X}_E) \to \mathbb{R}(E)^*$ is claimed as a soft rectangular b-metric (or soft b generalized metric) over a soft set $\tilde{X}_E$ with a soft coefficient $\tilde{s}$ if the following conditions are fulfilled:

(SRBM1) $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \tilde{\geq} \bar{0}$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}_E$;
(SRBM2) $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \bar{0}$ if and only if $\tilde{x}_{e_1} = \tilde{y}_{e_2}$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}_E$;
(SRBM3) $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{e_2}, \tilde{x}_{e_1})$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}_E$;
(SRBM4) $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) \tilde{\leq} \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{e_1}, \tilde{u}_{e_3}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{u}_{e_3}, \tilde{v}_{e_4}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{v}_{e_4}, \tilde{y}_{e_2})]$ for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}_E$ and for distinct soft points $\tilde{u}_{e_2}, \tilde{v}_{e_4} \tilde{\in} \tilde{X}_E \backslash \{\tilde{x}_{e_1}, \tilde{y}_{e_2}\}$.

The absolute soft set $\tilde{X}_E$ together with the soft rectangular b-metric $\breve{\mathfrak{D}}_{\mathsf{srb}}$ is purported to be a soft rectangular b-metric space or soft b-generalized metric space with a soft coefficient $\tilde{s}$ and it is signified by $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$.

Example 3.2. Let $X = \{x, y, z\}$, $E = \{a_1, a_2\}$ are set of parameters. Then, $SP(\tilde{X}_E) = \{\tilde{x}_{a_1}, \tilde{y}_{a_1}, \tilde{z}_{a_1}, \tilde{x}_{a_2}, \tilde{y}_{a_2}, \tilde{z}_{a_2}\}$.
Define a mapping $\breve{\mathfrak{D}}_{\mathsf{srb}} : SP(\tilde{X}_E) \times SP(\tilde{X}_E) \to \mathbb{R}(E)^*$ by

$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{u}_{a_i}, \tilde{v}_{a_j}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{v}_{a_j}, \tilde{u}_{a_i})$ for all distinct $\tilde{u}_{a_i}, \tilde{v}_{a_j} \in SP(\tilde{X}_E)$, $i, j \in \{1, 2\}$.
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{w}_{a_i}, \tilde{w}_{a_i}) = \bar{0}, \ \forall \ \tilde{w}_{a_i} \in SP(\tilde{X}_E)$.
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_1}, \tilde{x}_{a_2}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_2}, \tilde{y}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_2}, \tilde{z}_{a_2}) = \overline{40}$;
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_1}, \tilde{y}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_2}, \tilde{z}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_2}, \tilde{y}_{a_1}) = \overline{43}$;
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_2}, \tilde{z}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{z}_{a_2}, \tilde{z}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_1}, \tilde{z}_{a_1}) = \overline{63}$;
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_2}, \tilde{x}_{a_1}) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_2}, \tilde{y}_{a_2}) = \overline{109}$;
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_1}, \tilde{z}_{a_2}) = \overline{1543}$; $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_1}, \tilde{z}_{a_2}) = \overline{120}$;
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_2}, \tilde{z}_{a_2}) = \overline{196}$; $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{z}_{a_1}, \tilde{x}_{a_1}) = \overline{259}$.

Here, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{z}_{a_2}, \tilde{x}_{a_1}) = \overline{1543}$ and
$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{z}_{a_2}, \tilde{x}_{a_2}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_{a_2}, \tilde{y}_{a_1}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_{a_1}, \tilde{x}_{a_1}) = \overline{196} + \overline{40} + \overline{43} = \overline{279}$. Then, clearly $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ is a soft rectangular b-metric spaces with a soft coefficient $\tilde{s} = \overline{5.54}$.

Definition 3.3. For a soft point $\tilde{x}_\lambda \tilde{\in} \tilde{X}_E$, a soft open ball with center $\tilde{x}_\lambda$ and radius $\tilde{r}$, where $\tilde{r}$ is a positive soft real number is defined as,

$$\mathbb{B}_{\tilde{r}}(\tilde{x}_\lambda) = \{\tilde{y}_\mu \in \tilde{X}_E \ : \ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) \tilde{<} \tilde{r}\}.$$

In soft rectangular b-metric spaces, soft open balls aren't necessarily open. This is exemplified in the Example 3.10. Consider $\mathfrak{U}$ as the collection of all subsets $\mathfrak{C}$ of X fulfilling the criteria that there exists $\tilde{r} \tilde{>} \bar{0}$ such that $\mathbb{B}_{\tilde{r}}(\tilde{x}_\lambda) \tilde{\subseteq} \tilde{\mathfrak{C}}_E$ for every $\tilde{x}_\lambda \tilde{\in} \tilde{\mathfrak{C}}_E$. Then, for a soft rectangular b-metric space $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$, $\mathfrak{U}$

yields a soft topology that is not necessarily Hausdorff (See Example 3.10). Every soft metric space is a soft rectangular metric space, and each soft rectangular metric space is a soft rectangular b-metric space (with soft coefficient $\bar{1}$). Example 3.6 demonstrates that the converse is not always true. Every soft metric space turns out to be a soft b-metric space, and each soft b-metric space becomes a soft rectangular b-metric spaces. Lemma 3.5 is provided in this regard.

Figure 3.1 below depicts some of the natural consequences of some of the previously created generalizations of metric spaces.

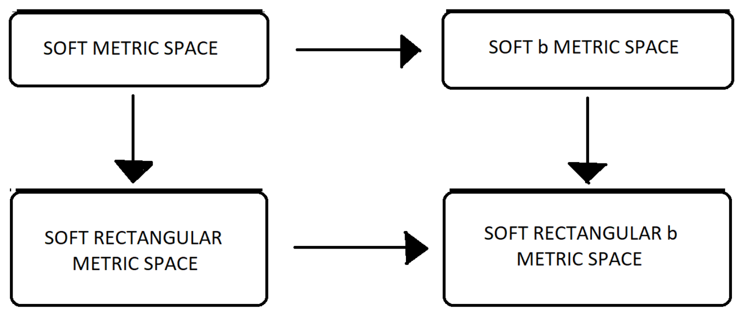

Figure 3.1

Remark 3.4. When the parameter set E is a singleton set, the definition of a soft rectangular b-metric space and rectangular b-metric space coincides.

Lemma 3.5. *Every soft b-metric space with a soft coefficient $\tilde{s}$ is a soft rectangular b-metric space with a soft coefficient $\tilde{s}^2$.*

Proof. Assume $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{sb}})$ as a soft b-metric space with a soft coefficient $\tilde{s}$. Then, for all $\tilde{x}_{e_1}, \tilde{y}_{e_2} \tilde{\in} \tilde{X}_E$ and for distinct soft points $\tilde{u}_{e_2}, \tilde{v}_{e_4} \tilde{\in} \tilde{X}_E \backslash \{\tilde{x}_{e_1}, \tilde{y}_{e_2}\}$, it follows from the triangle inequatilty that

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{x}_{e_1}, \tilde{y}_{e_2}) &\tilde{\leq} \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{x}_{e_1}, \tilde{u}_{e_3}) + \breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{u}_{e_3}, \tilde{y}_{e_2})] \\
&\tilde{\leq} \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{x}_{e_1}, \tilde{u}_{e_3}) + \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{u}_{e_3}, \tilde{v}_{e_4}) + \breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{v}_{e_4}, \tilde{y}_{e_2})]] \\
&= \tilde{s}\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{x}_{e_1}, \tilde{u}_{e_3}) + \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{u}_{e_3}, \tilde{v}_{e_4}) + \breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{v}_{e_4}, \tilde{y}_{e_2})] \\
&\tilde{\leq} \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{x}_{e_1}, \tilde{u}_{e_3}) + \breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{u}_{e_3}, \tilde{v}_{e_4}) + \breve{\mathfrak{D}}_{\mathsf{sb}}(\tilde{v}_{e_4}, \tilde{y}_{e_2})].
\end{aligned}$$

So, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{sb}})$ is a soft rectangular b-metric space with a soft coefficient $\tilde{s}^2$. □

Example 3.6. Consider the set $X = \mathbb{N}$ and $A = \{1, 3\}$ with parameter set $E \subseteq \mathbb{R}$. Define a function $\breve{\mathfrak{D}}_{\mathsf{s}} : SP(\tilde{X}_E) \times \; SP(\tilde{X}_E) \to \mathbb{R}(E)^*$ by

$\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{y}_{e_2}) = \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{y}_{e_2},\tilde{x}_{e_1})$ for all $\tilde{x}_{e_1},\tilde{y}_{e_2}\tilde{\in}\tilde{X}_E$.

$$\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{y}_{e_2}) = \begin{cases} \bar{0} & \text{if } \ \tilde{x}_{e_1} = \tilde{y}_{e_2}, \\ \overline{15}\bar{\gamma} & \text{if } \ \tilde{x}_{e_1},\tilde{y}_{e_2} \in SP(\tilde{A}_E), \tilde{x}_{e_1} \neq \tilde{y}_{e_2}, \\ \bar{3}\bar{\gamma} & \text{if } \ \text{either } \tilde{x}_{e_1} \text{or } \tilde{y}_{e_2} \notin SP(\tilde{A}_E), \tilde{x}_{e_1} \neq \tilde{y}_{e_2}, \\ \bar{\gamma} & \text{if } \ \tilde{x}_{e_1},\tilde{y}_{e_2} \in SP(\tilde{X}_E)\backslash SP(\tilde{A}_E), \tilde{x}_{e_1} \neq \tilde{y}_{e_2}, \end{cases}$$

where $\bar{\gamma}\tilde{>}\bar{0}$ is a constant soft real number. If $\tilde{x}_{e_1},\tilde{y}_{e_2} \in SP(\tilde{A}_E)$ then, $\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{y}_{e_2}) = \overline{15}\bar{\gamma}$. For $\tilde{u}_{e_3},\tilde{v}_{e_4} \in SP(\tilde{X}_E)\backslash SP(\tilde{A}_E)$, we have

$$\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{u}_{e_3}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{u}_{e_3},\tilde{v}_{e_4}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{v}_{e_4},\tilde{y}_{e_2}) = \bar{3}\bar{\gamma} + \bar{\gamma} + \bar{3}\bar{\gamma} = \bar{7}\bar{\gamma}.$$

Since $\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{y}_{e_2}) = \overline{15}\bar{\gamma}\tilde{>}\bar{7}\bar{\gamma} = \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_{e_1},\tilde{u}_{e_3})\tilde{+}\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{u}_{e_3},\tilde{v}_{e_4})\tilde{+}\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{v}_{e_4},\tilde{y}_{e_2})$, $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{s}})$ is an soft rectangular b-metric space with a soft coefficient $\tilde{s} = \overline{2.15} \tilde{>} \bar{1}$. However, $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{s}})$ is not an soft rectangular metric space.

We denote a sequence of soft points in a soft rectangular b-metric space $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ by $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$.

Definition 3.7. A sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in a soft rectangular b-metric space $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ is said to be convergent to a point $\tilde{y}_\mu$ in $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ if for any positive soft real number $\tilde{\epsilon}\tilde{>}\bar{0}$, chosen arbitrarily, there exists $k \in \mathbb{Z}^+$ such that $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n},\tilde{y}_\mu)\tilde{<}\tilde{\epsilon}$ whenever $n \geq k$.
Equivalently, $\lim\limits_{n\to\infty}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n},\tilde{y}_\mu) = \bar{0}$ or $\lim\limits_{n\to\infty}\tilde{x}^n_{\lambda_n} = \tilde{y}_\mu$.

Definition 3.8. A sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in a soft rectangular b-metric space $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ is purported to be a Cauchy sequence of $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ if for any positive soft real number $\tilde{\epsilon}\tilde{>}\bar{0}$, chosen arbitrarily, there exists $k \in \mathbb{Z}^+$ such that $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n},\tilde{x}^m_{\lambda_m})\tilde{<}\tilde{\epsilon}$ whenever $n,m \geq k,\ m > n$.
Equivalently, $\lim\limits_{n\to\infty}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n},\tilde{x}^m_{\lambda_m}) = \bar{0}$ for $m > n$.

Definition 3.9. A soft rectangular b-metric space $(\tilde{X}_E,\breve{\mathfrak{D}}_{\mathsf{srb}})$ is purported to be complete if each soft Cauchy sequence $\{\tilde{x}^n_{\lambda_n}\}$ in $\tilde{X}_E$ converges in $\tilde{X}_E$.

Example 3.10. Consider two sets A and B, where $A = \{1/t, t \in \mathbb{N}\}\backslash\{1\}$ and $B = \{1,2,3\}$. Let $X = A \cup B$ and $E \subseteq \mathbb{R}$. Consider a map $\breve{\mathfrak{D}}_{\mathsf{s}} : SP(\tilde{X}_E) \times\ SP(\tilde{X}_E) \to [0,\ \infty)E$ defined by: $\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_\lambda,\tilde{y}_\mu) = \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{y}_\mu,\tilde{x}_\lambda)$ for each $\tilde{x}_\lambda,\tilde{y}_\mu\tilde{\in}\tilde{X}_E$.

$$\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_\lambda,\tilde{y}_\mu) = \begin{cases} \bar{0} & \text{if } \ \tilde{x}_\lambda = \tilde{y}_\mu, \\ \bar{6}\bar{\beta} & \text{if } \ \tilde{x}_\lambda,\tilde{y}_\mu \in SP(\tilde{A}_E), \tilde{x}_\lambda \neq \tilde{y}_\mu, \\ \frac{\bar{\beta}}{\bar{3}\bar{t}} & \text{if } \ \tilde{x}_\lambda \in SP(\tilde{A}_E) \text{ and } \tilde{y}_\mu \notin SP(\tilde{A}_E), \\ \frac{\bar{\beta}}{\bar{2}} & \text{if } \ \textit{otherwise}, \end{cases}$$

where $\bar{\beta}\tilde{>}\bar{0}$ is a constant soft real number. Then, the following observations subsequently arise:

(1) $\breve{\mathfrak{D}}_{\mathsf{s}}(\bar{\frac{1}{2}}, \bar{\frac{1}{3}}) = \bar{6}\bar{\beta}$, $\breve{\mathfrak{D}}_{\mathsf{s}}(\bar{\frac{1}{2}}, \bar{1}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\bar{1}, \bar{2}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\bar{2}, \bar{\frac{1}{3}}) = \frac{\bar{\beta}}{\bar{6}} + \frac{\bar{\beta}}{\bar{2}} + \frac{\bar{\beta}}{\bar{9}} = \frac{\bar{7}\bar{\beta}}{\bar{9}}$. Thus, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$ is a soft rectangular b-metric space with a soft coefficient $\tilde{s} = \bar{8} \tilde{>} \bar{1}$. However, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$ is not a soft rectangular metric space. Henceforth, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$ is not a soft metric space.

(2) $\breve{\mathfrak{D}}_{\mathsf{s}}(1, \frac{1}{p}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\frac{1}{p}, 2) = \frac{\bar{\beta}}{\bar{3}.\bar{p}} + \frac{\bar{\beta}}{\bar{3}.\bar{p}} = \frac{\bar{2}\bar{\beta}}{\bar{3}.\bar{p}} \to \bar{0}$ as $p \to \infty$. So, there doesn't exist any soft real number $\tilde{s}\tilde{>}\bar{1}$ satisfying $\breve{\mathfrak{D}}_{\mathsf{s}}(1,2)\tilde{\leq}\breve{\mathfrak{D}}_{\mathsf{s}}(1, \frac{1}{p}) + \breve{\mathfrak{D}}_{\mathsf{s}}(\frac{1}{p}, 2)$. Hence, there doesn't exist any soft real number $\tilde{s}\tilde{>}\bar{1}$ so that the condition $\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_\lambda, \tilde{z}_\nu)\tilde{\leq}\breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{x}_\lambda, \tilde{y}_\mu) + \breve{\mathfrak{D}}_{\mathsf{s}}(\tilde{y}_\mu, \tilde{z}_\nu)$ holds for all $\tilde{x}_\lambda, \tilde{y}_\mu, \tilde{z}_\nu \in SP(\tilde{X}_E)$. Thus, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$ is not a soft b-metric space.

(3) $\mathbb{B}_{\frac{\bar{\beta}}{\bar{3}}}((\tilde{\frac{1}{2}})_6) = \{(\tilde{\frac{1}{2}})_\lambda : \lambda \in E\} \tilde{\cup} SP(\tilde{B}_E)$. For a soft point $\tilde{2}_5$ in $\tilde{X}_E$, there doesn't exist any open ball $\mathbb{B}_{\tilde{r}}((\tilde{2})_5)$ about the soft point $\tilde{2}_5$ contained in $\mathbb{B}_{\frac{\bar{\beta}}{\bar{3}}}((\tilde{\frac{1}{2}})_6)$. Therefore, $\mathbb{B}_{\frac{\bar{\beta}}{\bar{3}}}((\tilde{\frac{1}{2}})_6)$ is not open in $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$.

(4) The sequence $\{(\tilde{\frac{1}{n}})^n_{\lambda_n}\}_{n\in\mathbb{N}}$ converges to all the soft points in $SP(\tilde{B}_E)$. Henceforth, limit of the sequence $\{(\tilde{\frac{1}{n}})^n_{\lambda_n}\}_{n\in\mathbb{N}}$ is not unique.
Also, $\breve{\mathfrak{D}}_{\mathsf{s}}((\tilde{\frac{1}{n}})^n_{\lambda_n}, (\tilde{\frac{1}{n+q}})^{n+q}_{\lambda_{n+q}}) = \bar{6}.\bar{\beta} \not\to \bar{0}$ as $n \to \infty$ for any possible value of $q \in \mathbb{N}$. So, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{s}})$ is not Hausdorff.

Remark 3.11. This is exemplified in the above example that a limit of a sequence in a soft rectangular b-metric space is not necessarily unique and also every convergent sequence in a soft rectangular b-metric space is not necessarily a Cauchy sequence.

Definition 3.12. In a soft rectangular b-metric space $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$, a soft mapping $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ is continuous at a soft point $\tilde{x}_\lambda\tilde{\in}\tilde{X}_E$ if for a soft sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in $\tilde{X}_E$ such that $\underset{n\to\infty}{lim}\, \tilde{x}^n_{\lambda_n} = \tilde{x}_\lambda$ satisfies $\underset{n\to\infty}{lim}(\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n} = (\mathfrak{f}, \phi_E)\tilde{x}_\lambda$.

The following two theorems together provide the counterpart of the Banach contraction principle in a soft rectangular b-metric space.

Theorem 3.13. *Let $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ be a complete soft rectangular b-metric space with a soft coefficient $\tilde{s}\tilde{>}\bar{1}$. Assume that a map $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ satisfies*

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu) \tilde{\leq} \tilde{\mathfrak{K}}\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) \tag{3.1}$$

for some $\tilde{\mathfrak{K}} \,\tilde{\in}[\bar{0}, \frac{\bar{1}}{\bar{s}})$*, and for every two distinct soft points* $\tilde{x}_\lambda, \tilde{y}_\mu \tilde{\in} \tilde{X}_E$*. Then,* $(\mathfrak{f}, \phi_E)$ *has a unique fixed point.*

Proof. Consider a soft point $\tilde{x}^0_\lambda$ in $\tilde{X}_E$. Frame a soft sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in $\tilde{X}_E$ inductively as $\tilde{x}^{n+1}_{\lambda_{n+1}} = (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n} = (\mathfrak{f}^{n+1}(\tilde{x}^0_\lambda))_{{\phi_E}^{n+1}(\lambda)}$.
Assume that there is an integer $n > 0$ in a way that $\tilde{x}^n_{\lambda_n} = \tilde{x}^{n+1}_{\lambda_{n+1}}$. Then clearly the element $\tilde{x}^n_{\lambda_n}$ is fixed point of $(\mathfrak{f}, \phi_E)$. So, we assume that $\tilde{x}^n_{\lambda_n} \neq \tilde{x}^{n+1}_{\lambda_{n+1}}$ for all $n \in \mathbb{N}\mathrm{U}\{0\}$.

Denote $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})$ by $\breve{\mathfrak{D}}_{\mathsf{srb}^n}$. It follows from the condition (3.1) that

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}) \\
&\tilde{\leq} \tilde{\mathfrak{K}}\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}) \\
&= \tilde{\mathfrak{K}}\, \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-2}_{\lambda_{n-2}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}) \\
&\tilde{\leq} \tilde{\mathfrak{K}}^2\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-2}_{\lambda_{n-2}}, \tilde{x}^{n-1}_{\lambda_{n-1}}) \\
&\;\;\vdots \\
&\tilde{\leq} \tilde{\mathfrak{K}}^n\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1})
\end{aligned}$$

or

$$\breve{\mathfrak{D}}_{\mathsf{srb}^n} \tilde{\leq}\, \tilde{\mathfrak{K}}^n\, \breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.2}$$

Assuming $\tilde{x}^0_\lambda$ as a non-periodic soft point of $(\mathfrak{f}, \phi_E)$. If $\tilde{x}^0_\lambda = \tilde{x}^n_{\lambda_n}$, then from the above inequality, for any $n \geq 2$, we obtain,

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}^0_\lambda) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}),$$

it implies that

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \\
&\tilde{\leq}\, \tilde{\mathfrak{K}}^n\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1}).
\end{aligned}$$

So, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1}) \tilde{\leq}\, \tilde{\mathfrak{K}}^n\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1})$, which is a contradiction. Thus, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, \tilde{x}^1_{\lambda_1}) = \bar{0}$ which implies $\tilde{x}^0_\lambda = \tilde{x}^1_{\lambda_1}$ and thus, $\tilde{x}^0_\lambda$ is a soft fixed point.

Suppose that $\tilde{x}^n_{\lambda_n} \neq \tilde{x}^m_{\lambda_m}$, for each $n, m \in \mathbb{N}, n \neq m$. By inequality (3.1), for any positive integer n, we have

$$\begin{aligned}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n},\tilde{x}^{n+2}_{\lambda_{n+2}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f},\phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}},(\mathfrak{f},\phi_E)\tilde{x}^{n+1}_{\lambda_{n+1}})\\ &\tilde{\leq} \tilde{\mathfrak{K}}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}},\tilde{x}^{n+1}_{\lambda_{n+1}})\\ &= \tilde{\mathfrak{K}}\ \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f},\phi_E)\tilde{x}^{n-2}_{\lambda_{n-2}},(\mathfrak{f},\phi_E)\tilde{x}^{n}_{\lambda_n})\\ &\ \ \vdots\\ &\tilde{\leq} \tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{0}_{\lambda},\tilde{x}^{1}_{\lambda_1})\\ &= \tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}.\end{aligned}$$

So,

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n},\tilde{x}^{n+2}_{\lambda_{n+2}}) \tilde{\leq} \tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.3}$$

Taking $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n},\tilde{x}^{n+q}_{\lambda_{n+q}})$ in following listed distinct cases:

In the first case, if q is an odd number, that is, $q=2m+1$, for some $m\in\mathbb{N}$, then by the inequality (3.2), we have

$$\begin{aligned}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n},\tilde{x}^{n+q}_{\lambda_{n+q}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n},\tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})\\ &\tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}+\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}},\tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})]\\ &\tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}+\tilde{s}\ \{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}},\tilde{x}^{n+3}_{\lambda_{n+3}})\\ &\quad +\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}},\tilde{x}^{n+4}_{\lambda_{n+4}})+\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}},\tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})\}]\\ &= \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}]+\tilde{s}^2\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}}\\ &\quad +\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}},\tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})]\\ &\ \ \vdots\\ &\tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}]+\tilde{s}^2\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}}]\\ &\quad +\tilde{s}^3\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+4}}+\breve{\mathfrak{D}}_{\mathsf{srb}^{n+5}}]+\ldots+\tilde{s}^m\ \breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m}}\\ &\tilde{\leq} \tilde{s}\ [\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}+\tilde{\mathfrak{K}}^{n+1}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}]+\tilde{s}^2\ [\tilde{\mathfrak{K}}^{n+2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}+\tilde{\mathfrak{K}}^{n+3}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}]\\ &\quad +\tilde{s}^3\ [\tilde{\mathfrak{K}}^{n+4}\breve{\mathfrak{D}}_{\mathsf{srb}^0}+\tilde{\mathfrak{K}}^{n+5}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}]+\ldots+\tilde{s}^m\tilde{\mathfrak{K}}^{n+2m}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}\\ &\tilde{\leq} \tilde{s}\tilde{\mathfrak{K}}^n\ [\bar{1}+\tilde{s}\tilde{\mathfrak{K}}^2+\tilde{s}^2\tilde{\mathfrak{K}}^4+\ldots]\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}\\ &\quad +\tilde{s}\ \tilde{\mathfrak{K}}^{n+1}\ [\bar{1}+\tilde{s}\ \tilde{\mathfrak{K}}^2+\tilde{s}^2\ \tilde{\mathfrak{K}}^4+\ldots]\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}\\ &= \frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \qquad (\text{as } \tilde{s}\tilde{\mathfrak{K}}^2\tilde{<}\bar{1}).\end{aligned}$$

Hence, the obtained condition is

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}}) \tilde{\leq} \frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2} \tilde{s}\tilde{\mathfrak{K}}^n \breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.4}$$

Since $\tilde{\mathfrak{K}} \tilde{\in} [\bar{0}, \frac{\bar{1}}{\tilde{s}})$, the right hand side of the above inequality (3.4) is,

$$\frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2} \tilde{s}\tilde{\mathfrak{K}}^n \breve{\mathfrak{D}}_{\mathsf{srb}^0} \to \bar{0} \quad \text{as} \quad n \to \infty.$$

Secondly, if q is an even number, that is, $q = 2m$, for some $m \in \mathbb{N}$ then by the inequality (3.2) and (3.3), we have

$$\begin{aligned}
&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) \\
&\quad = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\quad \tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})] \\
&\quad \tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \tilde{s}\ \{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\qquad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}}, \tilde{x}^{n+4}_{\lambda_{n+4}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})\}] \\
&\quad = \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}} + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})] \\
&\qquad \vdots \\
&\quad \tilde{\leq} \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}}] \\
&\qquad + \tilde{s}^3\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+4}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+5}}] + \ldots + \tilde{s}^{m-1}\ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m-4}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m-3}}] \\
&\qquad + \tilde{s}^{m-1}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2m-2}_{\lambda_{n+2m-2}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\quad \tilde{\leq} \tilde{s}\ [\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n+1}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}] + \tilde{s}^2\ [\tilde{\mathfrak{K}}^{n+2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n+3}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&\qquad + \tilde{s}^3\ [\tilde{\mathfrak{K}}^{n+4}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n+5}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&\qquad + \cdots + \tilde{s}^{m-1}\ [\tilde{\mathfrak{K}}^{n+2m-4}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n+2m-3}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}] + \tilde{s}^{m-1}\tilde{\mathfrak{K}}^{n+2m-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad \tilde{\leq} \tilde{s}\tilde{\mathfrak{K}}^n\ [\bar{1} + \tilde{s}\tilde{\mathfrak{K}}^2 + \tilde{s}^2\tilde{\mathfrak{K}}^4 + \ldots]\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\qquad + \tilde{s}\tilde{\mathfrak{K}}^{n+1}\ [\bar{1} + \tilde{s}\tilde{\mathfrak{K}}^2 + \tilde{s}^2\tilde{\mathfrak{K}}^4 + \ldots]\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}^{m-1}\tilde{\mathfrak{K}}^{n+2m-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad = \frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}^{m-1}\tilde{\mathfrak{K}}^{n+2m-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad \tilde{<} \frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}^{2m}\tilde{\mathfrak{K}}^{n+2m-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad \tilde{\leq} \frac{\bar{1}+\tilde{\mathfrak{K}}}{\bar{1}-\tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}.
\end{aligned}$$

Hence, the obtained condition is

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \tilde{\leq} \frac{\bar{1} + \tilde{\mathfrak{K}}}{\bar{1} - \tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.5}$$

Since $\tilde{\mathfrak{K}}\ \tilde{\in}[\bar{0}, \frac{\bar{1}}{\tilde{s}})$, the right hand side of (3.5) is

$$\frac{\bar{1} + \tilde{\mathfrak{K}}}{\bar{1} - \tilde{s}\tilde{\mathfrak{K}}^2}\ \tilde{s}\tilde{\mathfrak{K}}^n\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}^{n-2}\ \breve{\mathfrak{D}}_{\mathsf{srb}^0} \to \bar{0} \quad \text{as} \quad n \to \infty.$$

Using the obtained inequalities (3.4) and (3.5), we have

$$\lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) = \bar{0} \quad \text{for all values of}\ \ q \geq 1. \tag{3.6}$$

Therefore, $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ is a soft Cauchy sequence in $\tilde{X}_E$. By the completeness property of $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$, there exists $\tilde{x}_\lambda \tilde{\in} \tilde{X}_E$ such that

$$\lim_{n\to\infty} \tilde{x}^{n}_{\lambda_n} = \tilde{x}_\lambda. \tag{3.7}$$

Next, we show that the limiting soft point $\tilde{x}_\lambda$ is a fixed point of $(\mathfrak{f}, \phi_E)$. For $n \in \mathbb{N}$,

$$\begin{aligned}
&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda)\\
&\quad \tilde{\leq}\ \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda)]\\
&\quad =\ \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda)]\\
&\quad \tilde{\leq}\ \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \tilde{\mathfrak{K}}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}_\lambda)].
\end{aligned}$$

According to the above inequality, (3.6) and (3.7), we get $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) = \bar{0}$. Thus, $\tilde{x}_\lambda$ is a fixed point of $(\mathfrak{f}, \phi_E)$ in $\tilde{X}_E$.

Suppose two distinct soft points $\tilde{x}_\lambda$ and $\tilde{y}_\mu$ are fixed points of $(\mathfrak{f}, \phi_E)$. Then, we have

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu)\\
&\tilde{\leq}\ \tilde{\mathfrak{K}}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu)\\
&\tilde{<}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu),
\end{aligned}$$

which is a contradiction. Therefore, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) = \bar{0}$ that is, $\tilde{x}_\lambda = \tilde{y}_\mu$. Hence, the obtained fixed point is unique. This completes the proof. □

Theorem 3.14. *Let* $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ *be a complete soft rectangular b-metric space with a soft coefficient* $\tilde{s}\tilde{>}\bar{1}$. *Consider* $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ *satisfying the condition:*

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu)\ \tilde{\leq}\ \tilde{\mathfrak{K}}\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu)$$

for some $\tilde{\mathfrak{K}} \,\tilde{\in} [\bar{0}, \bar{1})$, *and for every two distinct soft points* $\tilde{x}_\lambda, \tilde{y}_\mu \tilde{\in} \tilde{X}_E$. *Then,* $(\mathfrak{f}, \phi_E)$ *has a unique fixed point.*

Proof. Since the soft real number $\tilde{\mathfrak{K}}$ belongs to the soft interval $[\bar{0}, \bar{1})$, we get

$$\lim_{n\to\infty} \tilde{\mathfrak{K}}^n = \bar{0}.$$

This means that there is a number $p \in \mathbb{N}$ such that

$$\bar{0} \,\tilde{\leq}\, \tilde{\mathfrak{K}}^p \tilde{s} \,\tilde{<}\, \bar{1}. \tag{3.8}$$

Define a soft mapping $(\mathfrak{g}, \eta_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ as

$$(\mathfrak{g}, \eta_E)\tilde{x}_\lambda = (\mathfrak{f}, \phi_E)^p \tilde{x}_\lambda.$$

Then, we obtain

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{g}, \eta_E)\tilde{x}_\lambda, (\mathfrak{g}, \eta_E)\tilde{y}_\mu) \,\tilde{\leq}\, \tilde{\mathfrak{t}}\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu),$$

for every $\tilde{x}_\lambda, \tilde{y}_\mu \tilde{\in} \tilde{X}_E$, where $\tilde{\mathfrak{t}} = \tilde{\mathfrak{K}}^p$. And from condition (3.8), $\bar{0} \,\tilde{\leq}\, \tilde{\mathfrak{t}} \,\tilde{<}\, \frac{\bar{1}}{\tilde{s}}$. It follows from Theorem 3.13 that $(\mathfrak{g}, \eta_E)$ has a unique fixed point $\tilde{y}_\mu$(say). Then, $(\mathfrak{g}, \eta_E)\tilde{y}_\mu = \tilde{y}_\mu$ which implies $(\mathfrak{f}, \phi_E)^p \tilde{y}_\mu = \tilde{y}_\mu$ which implies $(\mathfrak{f}, \phi_E)^{p+1}\tilde{y}_\mu = (\mathfrak{f}, \phi_E)\tilde{y}_\mu$. This means $(\mathfrak{f}, \phi_E)\tilde{y}_\mu$ is a fixed point of $(\mathfrak{g}, \eta_E)$. And since $(\mathfrak{g}, \eta_E)$ has a unique fixed point, $(\mathfrak{f}, \phi_E)\tilde{y}_\mu = \tilde{y}_\mu$. □

Example 3.15. Consider $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}\}, B = [4, 5]$ and $E = \{1, 2\}$, a parameter set. Define $\breve{\mathfrak{D}}_{\mathsf{srb}} : SP(\tilde{X}_E) \times\ SP(\tilde{X}_E) \to \mathbb{R}(E)^*$ by

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{p}_\lambda, \tilde{q}_\mu) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{q}_\mu, \tilde{p}_\lambda) \text{ for all distinct } \tilde{p}_\lambda, \tilde{q}_\mu \tilde{\in} SP(\tilde{X}_E);$$
$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{p}_\lambda, \tilde{p}_\lambda) = \bar{0} \text{ for each distinct } \tilde{p}_\lambda \tilde{\in} SP(\tilde{X}_E);$$
$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{2}})_1, (\tilde{\frac{1}{3}})_1) = \breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{2}})_2, (\tilde{\frac{1}{3}})_2) = \overline{0.7};$$
$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{2}})_1, (\tilde{\frac{1}{2}})_2) = \breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{2}})_1, (\tilde{\frac{1}{3}})_2) = \overline{0.49};$$
$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{3}})_1, (\tilde{\frac{1}{3}})_2) = \breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{2}})_2, (\tilde{\frac{1}{3}})_1) = \overline{0.24};$$
$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{p}_\lambda, \tilde{q}_\mu) = \overline{7.2}[|\bar{p} - \bar{q}| + |\lambda - \mu|], \text{ otherwise.}$$

Here,

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{3}})_1, (4)_1) = \overline{7.2}[|\frac{1}{3} - 4| + |1 - 1|] = \overline{26.4}$$

and

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{3}})_1, (\tilde{\frac{1}{3}})_2) + \breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{3}})_2, (\tilde{\frac{1}{2}})_1) + \breve{\mathfrak{D}}_{\mathsf{srb}}((\tilde{\frac{1}{3}})_1, (4)_1) = \overline{25.93}.$$

Then, $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ is a soft rectangular metric space with a soft coefficient $\tilde{s} = \overline{2}$.

However, it is not a soft rectangular metric space. Hence, it is not a soft metric space.

Now, consider a map $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ defined by

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{p}_\lambda) = \begin{cases} (\tilde{\frac{1}{2}})_1, & \text{if } \tilde{p}_\lambda \in SP(\tilde{B}_E), \\ (\tilde{\frac{1}{3}})_2, & \text{otherwise.} \end{cases}$$

Then, $(\mathfrak{f}, \phi_E)$ fulfils the condition stated in the above theorems and hence has a unique fixed point that is, $\tilde{p}_\lambda = (\tilde{\frac{1}{3}})_2$.

Theorem 3.16. *Let $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ be a complete soft rectangular b-metric space with a soft coefficient $\tilde{s}\tilde{>}\bar{1}$. Consider $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ satisfying the condition:*

$$\breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu) \tilde{\leq} \tilde{\mathfrak{K}} \, [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu)] \quad (3.9)$$

for some $\tilde{\mathfrak{K}} \,\tilde{\in}[\bar{0}, \frac{\bar{1}}{\tilde{s}+\bar{1}})$, and for every two distinct soft points $\tilde{x}_\lambda$ and $\tilde{y}_\mu$ in $\tilde{X}_E$. Then, $(\mathfrak{f}, \phi_E)$ has a unique fixed point.

Proof. Assume $\tilde{x}^0_\lambda$ in $\tilde{X}_E$ is chosen arbitrarily. Construct a soft sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in $\tilde{X}_E$ inductively as $\tilde{x}^{n+1}_{\lambda_{n+1}} = (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n} = (\mathfrak{f}^{n+1}(\tilde{x}^0_\lambda))_{{\phi_E}^{n+1}(\lambda)}$.

If there is an integer $n > 0$ such that $\tilde{x}^n_{\lambda_n} = \tilde{x}^{n+1}_{\lambda_{n+1}}$, then clearly the element $\tilde{x}^n_{\lambda_n}$ is a fixed point of $(\mathfrak{f}, \phi_E)$. So, we assume that $\tilde{x}^n_{\lambda_n} \neq \tilde{x}^{n+1}_{\lambda_{n+1}}$ for each $n \in \mathbb{N} \cup \{0\}$. Denote $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})$ by $\breve{\mathfrak{D}}_{\mathsf{srb}^n}$. It follows from the condition (3.9) that

$$\begin{aligned} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}) \\ &\tilde{\leq} \tilde{\mathfrak{K}} \, [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n})] \\ &= \tilde{\mathfrak{K}} \, [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})]. \end{aligned}$$

So, we have

$$\breve{\mathfrak{D}}_{\mathsf{srb}^n} \tilde{\leq} \tilde{\mathfrak{K}}^n \, [\breve{\mathfrak{D}}_{\mathsf{srb}^{n-1}} + \breve{\mathfrak{D}}_{\mathsf{srb}^n}],$$

this implies that

$$\breve{\mathfrak{D}}_{\mathsf{srb}^n} \tilde{\leq} \tilde{\delta} \, \breve{\mathfrak{D}}_{\mathsf{srb}^{n-1}},$$

where $\tilde{\delta} = \frac{\tilde{\mathfrak{K}}}{\bar{1}-\tilde{\mathfrak{K}}} \tilde{<} \frac{\bar{1}}{\tilde{s}}$.

By repeating this process, we obtain

$$\breve{\mathfrak{D}}_{\mathsf{srb}^n} \tilde{\leq} \tilde{\delta}^n \, \breve{\mathfrak{D}}_{\mathsf{srb}^0}. \quad (3.10)$$

Also, assume $\tilde{x}^0_\lambda$ as a non-periodic soft point of $(\mathfrak{f}, \phi_E)$. If $\tilde{x}^0_\lambda = \tilde{x}^n_{\lambda_n}$, then from the above inequality, for any $n \geq 2$, we obtain

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^0_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}^0_\lambda) = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}),$$

this implies that

$$\breve{\mathfrak{D}}_{\mathsf{srb}^0} = \breve{\mathfrak{D}}_{\mathsf{srb}^n} \tilde{\leq} \tilde{\delta}^n \ \breve{\mathfrak{D}}_{\mathsf{srb}^0}.$$

Hence, $\breve{\mathfrak{D}}_{\mathsf{srb}^0} \tilde{\leq} \tilde{\delta}^n \ . \ \breve{\mathfrak{D}}_{\mathsf{srb}^0}$, which is a contradiction. So, $\breve{\mathfrak{D}}_{\mathsf{srb}^0} = \bar{0}$ i.e., $\tilde{x}^0_\lambda = \tilde{x}^1_{\lambda_1}$. Thus, $\tilde{x}^0_\lambda$ is a soft fixed point.

Suppose $\tilde{x}^n_{\lambda_n} \neq \tilde{x}^m_{\lambda_m}$ for all n, m in $\mathbb{N}, n \neq m$. By inequality (3.9) and (3.10), for any positive integer n, we have

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2}_{\lambda_{n+2}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n+1}_{\lambda_{n+1}}) \\
&\tilde{\leq} \ \tilde{\mathfrak{K}} \ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n+1}_{\lambda_{n+1}})] \\
&= \ \tilde{\mathfrak{K}} \ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, \tilde{x}^{n+2}_{\lambda_{n+2}})] \\
&= \ \tilde{\mathfrak{K}} \ [\breve{\mathfrak{D}}_{\mathsf{srb}^{n-1}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] \\
&\tilde{\leq} \ \tilde{\mathfrak{K}} \ [\tilde{\delta}^{n-1}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+1}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&= \ \tilde{\mathfrak{K}} \ \tilde{\delta}^{n-1}[\breve{\mathfrak{D}}_{\mathsf{srb}^0}(\bar{1} + \tilde{\delta}^2)] \\
&= \ \tilde{\gamma} \ \tilde{\delta}^{n-1}\breve{\mathfrak{D}}_{\mathsf{srb}^0},
\end{aligned}$$

where $\tilde{\gamma} = \tilde{\mathfrak{K}}[(\bar{1} + \tilde{\delta}^2)] \tilde{>} \bar{0}$. Therefore,

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2}_{\lambda_{n+2}}) \tilde{\leq} \ \tilde{\gamma} \ \tilde{\delta}^{n-1}\breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.11}$$

Taking $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}})$ into following listed distinct categories:

In the first case, if q is an odd number, that is, $q = 2m + 1$ for some $m \in \mathbb{N}$, then by the inequality (3.10), we have

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}}) \\
&\tilde{\leq} \ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})] \\
&\tilde{\leq} \ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \tilde{s}\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}}, \tilde{x}^{n+4}_{\lambda_{n+4}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})\}] \\
&= \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}} \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})] \\
&\ \ \vdots \\
&\tilde{\leq} \ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}}] \\
&\quad + \tilde{s}^3[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+4}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+5}}] + \ldots + \tilde{s}^m\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m}}
\end{aligned}$$

$$
\begin{aligned}
&\tilde{\leq}\ \tilde{s}[\tilde{\delta}^n \breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+1}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] + \tilde{s}^2[\tilde{\delta}^{n+2}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&\quad + \tilde{s}^3[\tilde{\delta}^{n+4}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+5}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] + ... + \tilde{s}^m\tilde{\delta}^{n+2m}\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\tilde{\leq} \tilde{s}\tilde{\delta}^n[\bar{1}+\tilde{s}\tilde{\delta}^2+\tilde{s}^2\tilde{\delta}^4+...]\breve{\mathfrak{D}}_{\mathsf{srb}^0}+\tilde{s}\tilde{\delta}^{n+1}[\bar{1}+\tilde{s}\tilde{\delta}^2+\tilde{s}^2\tilde{\delta}^4+...]\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&= \ \frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0}.
\end{aligned}
$$

Hence,

$$
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}}) \tilde{\leq} \frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.12}
$$

Since $\tilde{\delta}\ \tilde{\in}[\bar{0}, \frac{\bar{1}}{\tilde{s}})$, the right hand side of the above inequality is

$$
\frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} \to \bar{0} \quad \text{as} \quad n \to \infty.
$$

And if q is an even number, that is, $q = 2m$ for some $m \in \mathbb{N}$, then by the inequality (3.10) and (3.11), we obtain

$$
\begin{aligned}
&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) \\
&\quad = \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\quad \tilde{\leq}\ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})] \\
&\quad \tilde{\leq}\ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}} + \tilde{s}\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}}, \tilde{x}^{n+4}_{\lambda_{n+4}}) \\
&\qquad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})\}] \\
&\quad =\ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}} + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})] \\
&\quad \vdots \\
&\quad \tilde{\leq}\ \tilde{s}[\breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+1}}] + \tilde{s}^2[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+3}}] + \tilde{s}^3[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+4}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+5}}] \\
&\qquad + ... + \tilde{s}^{m-1}[\breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m-4}} + \breve{\mathfrak{D}}_{\mathsf{srb}^{n+2m-3}}] + \tilde{s}^{m-1}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2m-2}_{\lambda_{n+2m-2}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\quad \tilde{\leq}\ \tilde{s}[\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+1}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] + \tilde{s}^2[\tilde{\delta}^{n+2}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&\qquad + \tilde{s}^3[\tilde{\delta}^{n+4}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+5}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] + ... + \tilde{s}^{m-1}[\tilde{\delta}^{n+2m-4}\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\delta}^{n+2m-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}] \\
&\qquad + \tilde{s}^{m-1}\tilde{\gamma}\tilde{\delta}^{n+2m-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad \tilde{\leq}\ \tilde{s}\tilde{\delta}^n[\bar{1} + \tilde{s}\tilde{\delta}^2 + \tilde{s}^2\tilde{\delta}^4 + ...]\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}\tilde{\delta}^{n+1}[\bar{1} + \tilde{s}\tilde{\delta}^2 + \tilde{s}^2\tilde{\delta}^4 + ...]\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\qquad + \tilde{s}^{m-1}\tilde{\gamma}\tilde{\delta}^{n+2m-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0} \\
&\quad = \frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}^{m-1}\tilde{\gamma}\tilde{\delta}^{n+2m-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}
\end{aligned}
$$

$$\tilde{<} \frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{s}^{2m}\tilde{\gamma}\tilde{\delta}^{n+2m-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}$$
$$\tilde{\leq} \frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n.\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\gamma}\tilde{\delta}^{n-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}.$$

Hence, we obtain

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})\tilde{\leq}\frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\gamma}\tilde{\delta}^{n-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0}. \tag{3.13}$$

Since $\tilde{\delta}\ \tilde{\in}[\bar{0}, \frac{\bar{1}}{\tilde{s}})$, the right hand side of (3.13) is

$$\frac{\bar{1}+\tilde{\delta}}{\bar{1}-\tilde{s}\tilde{\delta}^2}\ \tilde{s}\tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\gamma}\tilde{\delta}^{n-3}\breve{\mathfrak{D}}_{\mathsf{srb}^0} \to \bar{0} \quad \text{as} \quad n\to\infty.$$

Using the obtained inequality (3.12) and (3.13), we have

$$\lim_{n\to\infty}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) = \bar{0}, \quad \forall\ q\geq 1. \tag{3.14}$$

Therefore, $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ is a soft Cauchy sequence in $\tilde{X}_E$. By the completeness of $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$, there exists a soft point $\tilde{x}_\lambda\tilde{\in}\tilde{X}_E$ such that

$$\lim_{n\to\infty}\tilde{x}^n_{\lambda_n} = \tilde{x}_\lambda. \tag{3.15}$$

Now, we show that the limiting soft point $\tilde{x}_\lambda$ is a fixed point of $(\mathfrak{f}, \phi_E)$.
For $n\in\mathbb{N}$,

$$\begin{aligned}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f},\phi_E)\tilde{x}_\lambda) &\tilde{\leq}\ \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \\ &\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, (\mathfrak{f},\phi_E)\tilde{x}_\lambda)] \\ &= \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}^n} + \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f},\phi_E)\tilde{x}^n_{\lambda_n}, (\mathfrak{f},\phi_E)\tilde{x}_\lambda)] \\ &\tilde{\leq}\ \tilde{s}\ \ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}^n} + \tilde{\mathfrak{K}}[\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f},\phi_E)\tilde{x}^n_{\lambda_n}) \\ &\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f},\phi_E)\tilde{x}_\lambda)]],\end{aligned}$$

it implies that

$$(1-\tilde{s}\tilde{\mathfrak{K}})\ \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f},\phi_E)\tilde{x}_\lambda)\ \tilde{\leq}\ \tilde{s}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}^n_{\lambda_n}) + \tilde{\delta}^n\breve{\mathfrak{D}}_{\mathsf{srb}^0} + \tilde{\mathfrak{K}}\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}).$$

It follows from the above inequality, the condition $\tilde{\mathfrak{K}}\ \tilde{<}\ \frac{\bar{1}}{\tilde{s}+1}$ and equations (3.14) and (3.15) that $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f},\phi_E)\tilde{x}_\lambda) = \bar{0}$. Thus, $\tilde{x}_\lambda$ is a fixed point of $(\mathfrak{f}, \phi_E)$ in $\tilde{X}_E$.

Suppose two distinct soft points $\tilde{x}_\lambda$ and $\tilde{y}_\mu$ are fixed points of $(\mathfrak{f}, \phi_E)$. Then, by using the condition (3.9), we get

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu) \\
&\tilde{\leq} \ \tilde{\mathfrak{K}}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu)] \\
&= \ \tilde{\mathfrak{K}}\ [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, \tilde{y}_\mu)] \\
&= \bar{0}.
\end{aligned}$$

Thus, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) = \bar{0}$, that is, $\tilde{x}_\lambda = \tilde{y}_\mu$. Hence, the obtained fixed point is unique. □

Theorem 3.17. *Let $(\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ be a complete soft rectangular b-metric space with a soft coefficient $\tilde{s}\tilde{>}\bar{1}$. Consider $(\mathfrak{f}, \phi_E) : (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}}) \to (\tilde{X}_E, \breve{\mathfrak{D}}_{\mathsf{srb}})$ satisfying the condition:*

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu) \ \tilde{\leq}\ \tilde{\mathfrak{K}}\ \max\{&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda), \\
&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu), \\
&\frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu))\}
\end{aligned} \tag{3.16}$$

for some $\tilde{\mathfrak{K}}\ \tilde{\in}(\bar{0}, \bar{1})$ and for every two distinct soft points $\tilde{x}_\lambda$ and $\tilde{y}_\mu$ in $\tilde{X}_E$. Then, $(\mathfrak{f}, \phi_E)$ has a unique fixed point.

Proof. Consider a soft point $\tilde{x}^0_\lambda$ chosen arbitrarily in $\tilde{X}_E$. Set a soft sequence $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ in $\tilde{X}_E$ iteratively as, $\tilde{x}^{n+1}_{\lambda_{n+1}} = (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n} = (\mathfrak{f}^{n+1}(\tilde{x}^0_\lambda))_{{\phi_E}^{n+1}(\lambda)}$.

If there is an integer $n > 0$ such that $\tilde{x}^n_{\lambda_n} = \tilde{x}^{n+1}_{\lambda_{n+1}}$ then clearly the element $\tilde{x}^n_{\lambda_n}$ is a soft fixed point of $(\mathfrak{f}, \phi_E)$. So, assume that $\tilde{x}^n_{\lambda_n} \neq \tilde{x}^{n+1}_{\lambda_{n+1}}$ for all $n \in \mathbb{N} \cup \{0\}$. Then, it follows from the condition (3.16) that,

$$\begin{aligned}
&\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \\
&= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}) \\
&\tilde{\leq}\ \tilde{\mathfrak{K}}\ \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}), \\
&\quad \frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}})\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}))\} \\
&= \ \tilde{\mathfrak{K}}\ \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n})\} \\
&= \ \tilde{\mathfrak{K}}\ \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^n_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})\}.
\end{aligned}$$

If $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n}) \leq \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})$, then from the above condition we get,

$$\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \tilde{\leq} \tilde{\mathfrak{K}} \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}),$$

which is impossible. Thus, $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \leq \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n})$. Hence, $\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})\}$ is a decreasing sequence which is converging to a soft real number L, that is,

$$\lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) = \tilde{L}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) &= \lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n}_{\lambda_n}) \\
&\tilde{\leq} \tilde{\mathfrak{K}} \lim_{n\to\infty} \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^{n}_{\lambda_n}), \\
&\quad \frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, (\mathfrak{f}, \phi_E)\tilde{x}^{n-1}_{\lambda_{n-1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^{n}_{\lambda_n}))\} \\
&= \tilde{\mathfrak{K}} \lim_{n\to\infty} \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n}), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, (\tilde{x}^{n+1}_{\lambda_{n+1}})\} \\
&= \tilde{\mathfrak{K}} \lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n-1}_{\lambda_{n-1}}, \tilde{x}^{n}_{\lambda_n}) \\
&= \tilde{\mathfrak{K}} \tilde{L}.
\end{aligned}$$

Hence, we have

$$\tilde{L} = \bar{0}. \tag{3.17}$$

To show the Cauchyness of the soft sequence, suppose q is an odd number, that is, $q = 2m+1$, $m \in \mathbb{N}$. Utilising the decreasing property of $\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})\}$, we obtain

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}}) \\
&\tilde{\leq} \tilde{s} \cdot [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, \tilde{x}^{n+2}_{\lambda_{n+2}}) \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})] \\
&\tilde{\leq} 2\tilde{s} \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \tilde{s}^2 \cdot \{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}}, \tilde{x}^{n+4}_{\lambda_{n+4}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})\} \\
&\tilde{\leq} 2\tilde{s} \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n}_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + 2\tilde{s}^2 \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\quad + \ldots + 2\tilde{s}^m \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2m-2}_{\lambda_{n+2m-2}}, \tilde{x}^{n+2m-1}_{\lambda_{n+2m-1}}) \\
&\quad + \tilde{s}^m \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2m}_{\lambda_{n+2m}}, \tilde{x}^{n+2m+1}_{\lambda_{n+2m+1}})
\end{aligned}$$

$$\tilde{\leq} (2\tilde{s}\ [\bar{1} + \tilde{s} + \tilde{s}^2 + ... + \tilde{s}^{m-1}] + \tilde{s}^m)\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})$$
$$= (2\tilde{s}\ [\frac{\tilde{s}^m - \bar{1}}{\tilde{s} - \bar{1}}] + \tilde{s}^m)\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}).$$

Hence, from (3.17), we obtain $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) = \bar{0}$.

Now, suppose q is an even number, that is, $q = 2m$, $m \in \mathbb{N}$. Following the decreasing property of $\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}})\}$, we get

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) &= \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\tilde{\leq}\ \tilde{s} \cdot [\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, \tilde{x}^{n+2}_{\lambda_{n+2}}) \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})] \\
&\tilde{\leq}\ 2\tilde{s} \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + \tilde{s}^2 \cdot \{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\quad + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+3}_{\lambda_{n+3}}, \tilde{x}^{n+4}_{\lambda_{n+4}}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+4}_{\lambda_{n+4}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}})\} \\
&\tilde{\leq}\ 2\tilde{s} \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) + 2\tilde{s}^2 \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2}_{\lambda_{n+2}}, \tilde{x}^{n+3}_{\lambda_{n+3}}) \\
&\quad + ... + \tilde{s}^m \cdot \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+2m-1}_{\lambda_{n+2m-1}}, \tilde{x}^{n+2m}_{\lambda_{n+2m}}) \\
&\tilde{\leq}\ 2\tilde{s}\ [\bar{1} + \tilde{s} + \tilde{s}^2 + ... + \tilde{s}^{m-1}]\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}) \\
&= 2\tilde{s}\Big[\frac{\tilde{s}^m - \bar{1}}{\tilde{s} - \bar{1}}\Big]\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+1}_{\lambda_{n+1}}).
\end{aligned}$$

Hence, from (3.17) we obtain $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}^{n+q}_{\lambda_{n+q}}) = \bar{0}$. Thus, $\{\tilde{x}^n_{\lambda_n}\}_{n\in\mathbb{N}}$ is a soft Cauchy sequence. Since the given soft rectangular b-metric space is complete, there exists a soft point $\tilde{x}_\lambda$ such that

$$\lim_{n\to\infty} \tilde{x}^n_{\lambda_n} = \tilde{x}_\lambda.$$

Next, we show that the limiting soft point $\tilde{x}_\lambda$ is fixed point of $(\mathfrak{f}, \phi_E)$.

For $n \in \mathbb{N}$,

$$\begin{aligned}
\lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^{n+1}_{\lambda_{n+1}}, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) \\
&\tilde{\leq}\ \tilde{\mathfrak{K}} \lim_{n\to\infty}\ \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}_\lambda), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda), \\
&\quad \frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, (\mathfrak{f}, \phi_E)\tilde{x}^n_{\lambda_n}) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda))\} \\
&\tilde{\leq}\ \tilde{\mathfrak{K}} \lim_{n\to\infty}\ \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \tilde{x}_\lambda), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}^n_{\lambda_n}, \\
&\quad \tilde{x}^{n+1}_{\lambda_{n+1}}), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda)\} \\
&\tilde{\leq}\ \tilde{\mathfrak{K}} \lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda),
\end{aligned}$$

which gives, $\lim_{n\to\infty} \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) = \bar{0}$. That is, $\tilde{x}_\lambda$ is a soft fixed point of $(\mathfrak{f}, \phi_E)$.

Suppose a soft point $\tilde{y}_\mu$ is another fixed point of $(\mathfrak{f}, \phi_E)$. It follows from the condition (3.16) that

$$\begin{aligned}
\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) &= \breve{\mathfrak{D}}_{\mathsf{srb}}((\mathfrak{f}, \phi_E)\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{y}_\mu) \\
&\tilde{\leq} \tilde{\mathfrak{K}} \quad \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda), \\
&\quad \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu), \frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, (\mathfrak{f}, \phi_E)\tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, (\mathfrak{f}, \phi_E)\tilde{y}_\mu))\} \\
&\tilde{\leq} \tilde{\mathfrak{K}} \quad \max\{\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}_\lambda), \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, \tilde{y}_\mu), \\
&\quad \frac{1}{2}(\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{x}_\lambda) + \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{y}_\mu, \tilde{y}_\mu))\} \\
&= \tilde{\mathfrak{K}}\, \breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu),
\end{aligned}$$

which implies that $\breve{\mathfrak{D}}_{\mathsf{srb}}(\tilde{x}_\lambda, \tilde{y}_\mu) = \bar{0}$, that is, $\tilde{x}_\lambda = \tilde{y}_\mu$. Consequently, the fixed point obtained is unique. $\square$

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