Nonlinear Functional Analysis and Applications
Vol. 28, No. 3 (2023), pp. 775-791
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.03.12
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON COUPLED COINCIDENCE POINTS IN MULTIPLICATIVE METRIC SPACES WITH AN APPLICATION

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Abstract. In this manuscript, we prove the existence of the coupled coincidence point by using g-couplings in multiplicative metric spaces (MMS). Further we show that existence of a fixed point in ordered MMS having t-property. Finally, some examples and application are presented for attesting to the credibility of our results.

[0]Received December 2, 2022. Revised January 21, 2023. Accepted January 24, 2023.
[0]2020 Mathematics Subject Classification: 47H10, 54H25, 46J10.
[0]Keywords: Multiplicative metric space, binary relations, contraction mappings.
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1. Introduction and preliminaries

Banach [8] in 1922 laid the foundation stone of fixed point theory in metric spaces. Later, several authors generalized the Banach contraction principle. Turinici [25] was the first who examined fixed points in ordered sets. In 2004, Ran and Reurings [22] worked on Banach contraction principle in ordered sets and assumed the contractive condition only to hold on the comparable elements instead of the whole space as in Banach contraction principle. There is vast literature on fixed point in ordered metric spaces, see [1, 4, 5, 7, 14, 18].

Guo et al. [12] introduced the concept of coupled fixed point and after that several authors gave results on coupled fixed and coupled coincidence point, see [2, 19]. Recently, Choudhury et al. [11] introduced the idea of couplings. The main results of Choudhury et al. [11] have been generalized by many researchers, see [6, 7]. Later, the idea of coupling was further studied by Rashid et al. [23] and introduced the idea of g-couplings.

The idea of multiplicative metric space(MMS), which is a generalization of metric space, was first introduced by Bashirov et al. [9] in 2008. The main idea behind introducing MMS was to replace usual triangular inequality by the multiplicative triangle inequality. Later, many research papers were reported on fixed points in MMS, see [3, 13, 15, 16, 17].

Now, in this article, we have shown the existence of coupled coincidence points and uniqueness of strong coupled coincidence points for g-couplings in MMS. We illustrate our results by examples and also provide an application in finding a solution to integral equations.

Definition 1.1. ([9]) For any non-empty set $\mathcal{G}$, multiplicative metric is a function $s : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ satisfying the following axioms:

(M1) $s(\xi, \eta) \geq 1$ for all $\xi, \eta \in \mathcal{G}$ and $s(\xi, \eta) = 1$ iff $\xi = \eta$;
(M2) $s(\xi, \eta) = s(\eta, \xi)$ for all $\xi, \eta \in \mathcal{G}$;
(M3) $s(\xi, \eta) \leq s(\xi, z) \cdot s(z, \eta)$ for all $\xi, \eta, z \in \mathcal{G}$.

Example 1.2. ([9]) Let $\mathbb{R}_+^n$ be the set of all n-tuples of positive real numbers and we define $s : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}$ by

$$s(\xi, \eta) = h(\frac{\xi_1}{\eta_1}) \cdot h(\frac{\xi_2}{\eta_2}) \cdots h(\frac{\xi_n}{\eta_n}),$$

where $\xi = (\xi_1, \xi_2, ..., \xi_n),\ \eta = (\eta_1, \eta_2, ..., \eta_n) \in \mathbb{R}_+^n$ and $h:\ \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$h(l) = \begin{cases} l, & l \geq 1, \\ \frac{1}{l}, & l < 1. \end{cases}$$

Then $(\mathcal{G}, s)$ is a multiplicative metric space.

Definition 1.3. ([21]) Let $(\mathcal{G}, s)$ be a MMS and $\{\xi_n\}$ be a sequence in $\mathcal{G}$. For $\xi \in \mathcal{G}$ and $\epsilon > 1$, a subset $B_\epsilon(\xi) = \{\eta \in \mathcal{G} : s(\xi, \eta) < \epsilon\}$ of $\mathcal{G}$ is called a multiplicative open ball centered at ξ with radius ϵ. Analogously one can define multiplicative closed ball as $B_\epsilon(\xi) = \{\eta \in \mathcal{G} : s(\xi, \eta) \leq \epsilon\}$.

Definition 1.4. ([21]) Let $(\mathcal{G}, s)$ be a MMS, $\{\xi_n\}$ be a sequence in $\mathcal{G}$ and $\xi \in \mathcal{G}$. If for every multiplicative open ball $B_\epsilon(\xi) = \{\eta \in \mathcal{G} : s(\xi, \eta) < \epsilon\}$, $\epsilon > 1$, there exists a natural number N such that for $n \geq N$, $\xi_n \in B_\epsilon(\xi)$, then the sequence $\{\xi_n\}$ is said to be multiplicative converging to ξ. We denote as $\xi_n \to \xi$ $(n \to \infty)$.

Definition 1.5. ([21]) Let $(\mathcal{G}, s)$ be a MMS, $\{\xi_n\}$ be a sequence in $\mathcal{G}$ and $\xi \in \mathcal{G}$. Then

$$\xi_n \to \xi (n \to \infty) \quad \text{if and only if} \quad s(\xi_n, \xi) \to 1 (n \to \infty).$$

Definition 1.6. ([21]) Let $(\mathcal{G}, s)$ be a MMS and $\{\xi_n\}$ be a sequence in $\mathcal{G}$. Then $\{\xi_n\}$ is said to be a multiplicative Cauchy sequence if for $\epsilon > 1$, there exists a positive integer $N \in \mathbb{N}$ such that $s(\xi_m, \xi_n) < \epsilon$ for all $n, m \geq N$.

Lemma 1.7. ([21]) *Let $(\mathcal{G}, s)$ be a MMS and $\{\xi_n\}$ be a sequence in $\mathcal{G}$. Then $\{\xi_n\}$ is multiplicative Cauchy if and only if $s(\xi_n, \xi_m) \to 1 (n, m \to \infty)$.*

Definition 1.8. ([21]) If every multiplicative Cauchy sequence in $(\mathcal{G}, s)$ is multiplicative convergent in $\mathcal{G}$, then MMS $(\mathcal{G}, s)$ is said to be multiplicative complete.

Definition 1.9. ([10]) For any nonempty set $\mathcal{G}$, let $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a mapping. Then an element $(\xi, \eta) \in \mathcal{G} \times \mathcal{G}$ is said to be a coupled fixed point of $\mathcal{H}$, if $\mathcal{H}(\xi, \eta) = \xi$ and $\mathcal{H}(\eta, \xi) = \eta$.

Definition 1.10. ([10]) For any nonempty set $\mathcal{G}$, let $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{G}$ be mappings. Then an element $(\xi, \eta) \in \mathcal{G} \times \mathcal{G}$ is said to be a coupled coincident point of $\mathcal{H}$ and g, if $\mathcal{H}(\xi, \eta) = g\xi$ and $\mathcal{H}(\eta, \xi) = g\eta$.

Definition 1.11. ([23]) A coupled coincidence point (ξ, η) is said to be a strong coupled coincidence point, if $\xi = \eta$.

Definition 1.12. ([11]) For any nonempty set $\mathcal{G}$, an element $(\xi, \eta) \in \mathcal{G} \times \mathcal{G}$ is a strong coupled fixed point of the mapping $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, if (ξ, η) is a coupled fixed point and $\xi = \eta$, that is, if $\mathcal{H}(\xi, \xi) = \xi$.

Definition 1.13. ([11]) A mapping $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is said to be coupling with respect to two subsets O and P of $\mathcal{G}$, if

$$\mathcal{H}(\xi, \eta) \in P \quad \text{and} \quad \mathcal{H}(\eta, \xi) \in O$$

for $\xi \in O$ and $\eta \in P$.

Definition 1.14. ([23]) For any nonempty set $\mathcal{G}$, let $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{G}$ be two mappings. Then, $\mathcal{H}$ is said to be g-coupling with respect to two subsets O and P of $\mathcal{G}$, if

$$\mathcal{H}(\xi, \eta) \in g(O) \cap P \quad \text{and} \quad \mathcal{H}(\eta, \xi) \in g(P) \cap O$$

for $\xi \in O$ and $\eta \in P$.

2. Main results

First, we give a definition before proceeding the main result.

Definition 2.1. Let $(\mathcal{G}, s)$ be a MMS, $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{G}$ be two mappings and O and P be any two subsets of $\mathcal{G}$. Then the mapping $\mathcal{H}$ is said to be a g-contraction with respect to two subsets O and P of $\mathcal{G}$, if

$$s(\mathcal{H}(\xi, \eta), \mathcal{H}(\varpi, \varrho)) \leq [\max\{s(g\xi, g\varpi), s(g\eta, g\varrho)\}]^{\lambda} \tag{2.1}$$

for all $\xi, \varrho \in O$, $\eta, \varpi \in P$, where $\lambda \in (0, 1)$.

Now, we give an example of Definition 2.1.

Example 2.2. For $\mathcal{G} = \mathbb{R}$, define $s : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by $s(\xi, \eta) = e^{|\xi - \eta|}$. Clearly, $(\mathcal{G}, s)$ forms a MMS. Let O and P be any two subsets of $\mathcal{G}$. Define $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by $\mathcal{H}(\xi, \eta) = \frac{\lambda}{2} \mid \xi^2 - \eta^2 \mid$ and $g : \mathcal{G} \to \mathcal{G}$ by $g(\xi) = \xi^2$. It is known that for all $l, m \in [0, \infty)$ and $h > 0$, we have $\max\{l^h, m^h\} = (\max\{l, m\})^h$ and for all $l, m \in \mathbb{R}$, $e^{|l-m|} \leq \big\{ \max\{e^{|l|}, e^{|m|}\}\big\}^2$. So, for all $\xi, \varrho \in O$, $\eta, \varpi \in P$ and $\lambda \in (0, 1)$, we have

$$\begin{aligned}
s\big(\mathcal{H}(\xi, \eta), \mathcal{H}(\varpi, \varrho)\big) &= s\Big(\frac{\lambda}{2} \mid \xi^2 - \eta^2 \mid, \frac{\lambda}{2} \mid \varpi^2 - \varrho^2 \mid \Big) \\
&= e^{|\frac{\lambda}{2}|\xi^2 - \eta^2| - \frac{\lambda}{2}|\varpi^2 - \varrho^2||} \\
&\leq e^{\frac{\lambda}{2}|(\xi^2 - \eta^2) - (\varpi^2 - \varrho^2)|} \\
&= e^{\frac{\lambda}{2}|\xi^2 - \eta^2 - \varpi^2 + \varrho^2|} \\
&= e^{\frac{\lambda}{2}|(\xi^2 - \varpi^2) - (\eta^2 - \varrho^2)|} \\
&\leq \big\{ \max\{e^{\frac{\lambda}{2}|(\xi^2 - \varpi^2)|}, e^{\frac{\lambda}{2}|(\eta^2 - \varrho^2)|}\}\big\}^2 \\
&= [\max\{e^{|(\xi^2 - \varpi^2)|}, e^{|(\eta^2 - \varrho^2)|}\}]^{\lambda} \\
&= [\max\{s(g\xi, g\varpi), s(g\eta, g\varrho)\}]^{\lambda}.
\end{aligned}$$

Hence, $\mathcal{H}$ is a g-contraction with respect to two subsets O and P of $\mathcal{G}$.

Theorem 2.3. *Let O and P be any two nonempty subsets of a complete MMS $(\mathcal{G}, s)$. Given $\mathcal{H} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{G}$ such that $\mathcal{H}$ is a g-coupling with respect to two subsets O and P of $\mathcal{G}$ and (2.1) is satisfied. Further, assume that O and P are invariant by g and $g(O)$, $g(P)$ are multiplicative closed subsets of $(\mathcal{G}, s)$. Then:*

(i) *$O \cap P \neq \emptyset$;*
(ii) *$O \times P$ will contain coupled coincidence point of $\mathcal{H}$ and g.*

If in addition, we choose g is injective map on $O \cup P$, then $\mathcal{H}$ and g have a unique strong coupled coincidence point in $O \times P$.

Proof. (i) From (2.1), we have

$$s\Big(\mathcal{H}(\xi, \eta), \mathcal{H}(\varpi, \varrho)\Big) \leq \Big[\max\{s(g\xi, g\varpi), s(g\eta, g\varrho)\}\Big]^{\lambda},$$

where $\xi, \varrho \in O$, $\eta, \varpi \in P$ and $\lambda \in (0, 1)$. Also by given assumption, since O and P are invariant by g and $g(P)$, we have

$$g\xi, g\varrho \in g(O) \subseteq O \text{ and } g\eta, g\varpi \in g(P) \subseteq P.$$

For $\xi_0 \in O$ and $\eta_0 \in P$, in view of definition of a g-coupling, $\mathcal{H}(\xi_0, \eta_0) \in g(O) \cap P$ and $\mathcal{H}(\eta_0, \xi_0) \in g(P) \cap O$. In particular, $\mathcal{H}(\xi_0, \eta_0) \in g(O)$ and $\mathcal{H}(\eta_0, \xi_0) \in g(P)$. So, there exist $\xi_1 \in O$ and $\eta_1 \in P$ we have

$$\mathcal{H}(\xi_0, \eta_0) = g(\xi_1) \text{ and } \mathcal{H}(\eta_0, \xi_0) = g(\eta_1),$$

respectively, in order to get sequences $\{g\xi_n\}$ and $\{g\eta_n\}$ in $g(O)$ and $g(P)$, such that

$$g(\xi_{n+1}) = \mathcal{H}(\xi_n, \eta_n) \text{ and } g(\eta_{n+1}) = \mathcal{H}(\eta_n, \xi_n). \tag{2.2}$$

Now using (2.1) and (2.2), we get

$$\begin{aligned} s\Big(g\xi_1, g\eta_2\Big) &= s\Big(\mathcal{H}(\xi_0, \eta_0), \mathcal{H}(\eta_1, \xi_1)\Big) \\ &\leq \Big[\max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}\Big]^{\lambda} \end{aligned}$$

and

$$\begin{aligned} s\Big(g\eta_1, g\xi_2\Big) &= s\Big(\mathcal{H}(\eta_0, \xi_0), \mathcal{H}(\xi_1, \eta_1)\Big) \\ &\leq \Big[\max\{s(g\eta_0, g\xi_1), s(g\xi_0, g\eta_1)\}\Big]^{\lambda}. \end{aligned}$$

From above two inequalities, we have

$$\max\Big\{s(g\xi_1,\ g\eta_2),\ s(g\eta_1,\ g\xi_2)\Big\} \leq \Big[\max\{s(g\xi_0,\ g\eta_1),\ s(g\eta_0,\ g\xi_1)\}\Big]^{\lambda}.$$

Again from (2.1) and (2.2), we get

$$\begin{aligned} s\Big(g\xi_2, g\eta_3\Big) &= s\Big(\mathcal{H}(\xi_1,\eta_1),\ \mathcal{H}(\eta_2,\xi_2)\Big) \\ &\leq \Big[\max\{s(g\xi_1, g\eta_2), s(g\eta_1, g\xi_2)\}\Big]^{\lambda} \\ &\leq \Big[\max\{s(g\eta_0, g\xi_1), s(g\xi_0, g\eta_1)\}\Big]^{\lambda^2} \end{aligned}$$

and

$$\begin{aligned} s\Big(g\eta_2, g\xi_3\Big) &= s\Big(\mathcal{H}(\eta_1,\xi_1), \mathcal{H}(\xi_2,\eta_2)\Big) \\ &\leq \Big[\max\{s(g\eta_1, g\xi_2), s(g\xi_1, g\eta_2)\}\Big]^{\lambda} \\ &\leq \Big[\max\{s(g\eta_0, g\xi_1), s(g\xi_0, g\eta_1)\}\Big]^{\lambda^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \max\Big\{s(g\xi_2, g\eta_3), s(g\eta_2, g\xi_3)\Big\} &\leq \Big[\max\{s(g\xi_1, g\eta_2), s(g\eta_1, g\xi_2)\}\Big]^{\lambda} \\ &\leq \Big[\max\{s(g\eta_0, g\xi_1), s(g\xi_0, g\eta_1)\}\Big]^{\lambda^2}. \end{aligned}$$

Inductively, we obtain

$$s\Big(g\xi_n, g\eta_{n+1}\Big) \leq \Big[\max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}\Big]^{\lambda^n} \tag{2.3}$$

and

$$s\Big(g\eta_n, g\xi_{n+1}\Big) \leq \Big[\max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}\Big]^{\lambda^n}. \tag{2.4}$$

From (2.3) and (2.4), as $\lambda \in (0,1)$, we get

$$\lim_{n\to\infty} s(g\xi_n, g\eta_{n+1}) = 1 \ \text{ and } \ \lim_{n\to\infty} s(g\eta_n, g\xi_{n+1}) = 1. \tag{2.5}$$

Now, we define the sequence $\{S_n\}$ by $S_n = s(g\xi_n, g\eta_n)$. By using (2.1) and (2.2), we get

$$\begin{aligned} s\Big(g\xi_1, g\eta_1\Big) &= s\Big(\mathcal{H}(\xi_0,\eta_0), \mathcal{H}(\eta_0,\xi_0)\Big) \\ &\leq \Big[\max\{s(g\xi_0, g\eta_0), s(g\eta_0, g\xi_0)\}\Big]^{\lambda} \\ &= \Big[s(g\xi_0, g\eta_0)\Big]^{\lambda}. \end{aligned} \tag{2.6}$$

Again, by using (2.1), (2.2) and (2.6), we have

$$\begin{aligned} s\Big(g\xi_2, g\eta_2\Big) &= s\Big(\mathcal{H}(\xi_1,\eta_1), \mathcal{H}(\eta_1,\xi_1)\Big) \\ &\le \Big[\max\{s(g\xi_1, g\eta_1), s(g\eta_1, g\xi_1)\}\Big]^{\lambda} \\ &= \Big[s(g\xi_1, g\eta_1)\Big]^{\lambda} \\ &\le \Big[s(g\xi_0, g\eta_0)\Big]^{\lambda^2}. \end{aligned} \tag{2.7}$$

Continuing in this way using mathematical induction, we get

$$s\Big(g\xi_n, g\eta_n\Big) \le \Big[s(g\xi_0, g\eta_0)\Big]^{\lambda^n}. \tag{2.8}$$

As $\lambda \in (0,1)$, from (2.8), we have

$$\lim_{n\to\infty} s(g\xi_n, g\eta_n) = 1. \tag{2.9}$$

Now, by triangular inequality, (2.4) and (2.8), we get

$$\begin{aligned} s(g\xi_n, g\xi_{n+1}) &\le s(g\xi_n, g\eta_n)\cdot s(g\eta_n, g\xi_{n+1}) \\ &\le \Big[s(g\xi_0, g\eta_0)\Big]^{\lambda^n} \times \Big[\max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}\Big]^{\lambda^n} \\ &= K^{\lambda^n}, \end{aligned} \tag{2.10}$$

where $K = s(g\xi_0, g\eta_0)\cdot \max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}$. Similarly, by (2.3) and (2.8), we have

$$\begin{aligned} s(g\eta_n, g\eta_{n+1}) &\le s(g\eta_n, g\xi_n)\cdot s(g\xi_n, g\eta_{n+1}) \\ &\le \Big[s(g\xi_0, g\eta_0)\Big]^{\lambda^n} \times \Big[\max\{s(g\xi_0, g\eta_1), s(g\eta_0, g\xi_1)\}\Big]^{\lambda^n} \\ &= K^{\lambda^n}. \end{aligned} \tag{2.11}$$

Instantly, we prove that the sequences $\{g\xi_n\}$ and $\{g\eta_n\}$ are the multiplicative Cauchy in $g(O)$ and $g(P)$.

For $m, n \in \mathbb{N}$ with $n < m$, by using triangular inequality and (2.10), we have

$$\begin{aligned} s(g\xi_n, g\xi_m) &\le s(g\xi_n, g\xi_{n+1}) \times s(g\xi_{n+1}, g\xi_{n+2}) \times \cdots \times s(g\xi_{m-1}, g\xi_m) \\ &\le K^{\lambda^n} \times K^{\lambda^{n+1}} \times \cdots \times K^{\lambda^{m-1}} \\ &< K^{\frac{\lambda^n}{1-\lambda}}. \end{aligned}$$

As $\lambda \in (0,1)$, $s(g\xi_m, g\xi_n) \to 1$ as $m, n \to \infty$. This shows that $\{g\xi_n\}$ is a multiplicative Cauchy sequence in $g(O)$. Similarly, by (2.11), we have

$$\begin{aligned} s(g\eta_n, g\eta_m) &\leq s(g\eta_n, g\eta_{n+1}) \times s(g\eta_{n+1}, g\eta_{n+2}) \times \cdots \times s(g\eta_{m-1}, g\eta_m) \\ &\leq K^{\lambda^n} \times K^{\lambda^{n+1}} \times \cdots \times K^{\lambda^{m-1}} \\ &< K^{\frac{\lambda^n}{1-\lambda}}. \end{aligned}$$

Thus, we get $\{g\eta_n\}$ is a multiplicative Cauchy sequence in $g(P)$.

Since $g(O)$ and $g(P)$ are multiplicative closed in the complete MMS $(\mathcal{G}, s)$, so $g(O)$ and $g(P)$ are multiplicative complete in $(\mathcal{G}, s)$. Therefore, $\{g\xi_n\}$ and $\{g\eta_n\}$ are multiplicative convergent in $g(O)$ and $g(P)$, respectively. Thus, there exists $\varpi \in g(O)$ and $\varrho \in g(P)$ such that

$$g\xi_n \to \varpi \text{ and } g\eta_n \to \varrho \quad \text{as} \quad n \to \infty. \tag{2.12}$$

From (2.9) and (2.12), we get $1 = \lim_{n\to\infty} s(g\xi_n, g\eta_n) = s(\varpi, \varrho)$, thus

$$\varpi = \varrho. \tag{2.13}$$

As $\varpi \in g(O)$ and $\varrho \in g(P)$, there exist $l \in O$ and $p \in P$ such that $gl = \varpi$ and $gp = \varrho$. By using (2.12) and (2.13),

$$g\xi_n \to gl \text{ and } g\eta_n \to gp \tag{2.14}$$

and also

$$gl = gp. \tag{2.15}$$

Thus $gl = gp \in O \cap P$, part (i) is completed.

(ii) Now, by (2.1), (2.2), (2.14), (2.15) and triangle inequality, we get

$$\begin{aligned} s\Big(g(l), \mathcal{H}(l,p)\Big) &\leq s\Big(g(l), g\eta_{n+1}\Big) \times s\Big(g\eta_{n+1}, \mathcal{H}(l,p)\Big) \\ &= s\Big(g(l), g\eta_{n+1}\Big) \times s\Big(\mathcal{H}(\eta_n, \xi_n), \mathcal{H}(l,p)\Big) \\ &\leq s\Big(g(l), g\eta_{n+1}\Big) \times \Big[\max\{s(g\eta_n, gl), s(g\xi_n, gp)\}\Big]^{\lambda} \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

We deduce that

$$\mathcal{H}(l,p) = g(l). \tag{2.16}$$

Now, by using (2.1), (2.2), (2.14), (2.15) and triangle inequality, we have

$$\begin{aligned} s\Big(g(p),\mathcal{H}(p,l)\Big) &\leq s\Big(g(p),g\xi_{n+1}\Big)\times s\Big(g\xi_{n+1},\mathcal{H}(p,l)\Big) \\ &= s\Big(g(p),g\xi_{n+1}\Big)\times s\Big(\mathcal{H}(\xi_n,\eta_n),\mathcal{H}(p,l)\Big) \\ &\leq s\Big(g(p),g\xi_{n+1}\Big)\times\Big[\max\{s(g\xi_n,gp),s(g\eta_n,gl)\}\Big]^{\lambda} \\ &\to 1 \quad \text{as} \quad n\to\infty. \end{aligned}$$

Thus, we have

$$\mathcal{H}(p,l)=g(p). \tag{2.17}$$

Hence, we can say that from (2.16) and (2.17), $\mathcal{H}(l,p)=g(l)$ and $\mathcal{H}(p,l)=g(p)$, and so $(l,p)\in O\times P$ is a coupled coincidence point of $\mathcal{H}$ and g, this completes the part (ii).

Now (2.15) and since g is injective map imply that $l=p$, so $\mathcal{H}$ and g have a strong coupled coincidence point, that is, $\mathcal{H}(l,l)=g(l)$ and for the uniqueness, we suppose that there exist two strong coupled coincidence points $w,z\in O\cap P$ of $\mathcal{H}$ and g, then

$$\mathcal{H}(w,w)=g(w) \quad \text{and} \quad \mathcal{H}(z,z)=g(z). \tag{2.18}$$

From (2.1), we have

$$\begin{aligned} s\Big(g(w),g(z)\Big) &= s\Big(\mathcal{H}(w,w),\mathcal{H}(z,z)\Big) \\ &\leq \Big[\max\{s(gw,gz),s(gw,gz)\}\Big]^{\lambda} \\ &= \Big[s(g(w),g(z))\Big]^{\lambda}. \end{aligned}$$

Since $\lambda\in(0,1)$, we deduce that $s(g(w),g(z))=1$, that is, $g(w)=g(z)$, so $w=z$. Hence, g is injective map. Also $\mathcal{H}$ and g have the unique strong coupled coincidence point in $O\cap P$. □

The above result can be well understood by the following illustrated example.

Example 2.4. Let $\mathcal{G}=\mathbb{R}$ and metric $s(\xi,\eta)=e^{|\xi-\eta|}$. Take $O=[0,2]$ and $P=[0,3]$. Let $\mathcal{H}$ be defined as $\mathcal{H}(\xi,\eta)=\frac{\xi+\eta}{10}$, where $\xi,\eta\in\mathcal{G}$. We define $g:\mathcal{G}\to\mathcal{G}$ by $g(\xi)=\frac{\xi}{2}$. Then clearly $g(O)=[0,1]$ and $g(P)=[0,\frac{3}{2}]$, so $g(O)$ and $g(P)$ have multiplicative closed subsets of $\mathcal{G}$. Also, O and P are invariant by g as $g(O)\subseteq O$ and $g(P)\subseteq P$. Now it remains to prove that $\mathcal{H}$ is a g-coupling. As $g(O)\cap P=[0,1]$ and $g(P)\cap O=[0,\frac{3}{2}]$, so for all $\xi\in O$ and $\eta\in P$, we have $0\leq\mathcal{H}(\xi,\eta)\leq\frac{1}{2}$ and $0\leq\mathcal{H}(\eta,\xi)\leq\frac{1}{2}$, that is,

$\mathcal{H}(\xi,\eta)\in g(O)\cap P$ and $\mathcal{H}(\eta,\xi)\in g(P)\cap O$. So, $\mathcal{H}$ is a g-coupling with respect to O and P. Again, for $\xi,\varrho\in O$ and $\eta,\varpi\in P$, we have

$$\begin{aligned} s(\mathcal{H}(\xi,\eta),\mathcal{H}(\varpi,\varrho)) &= e^{|\frac{\xi+\eta}{10}-\frac{\varpi+\varrho}{10}|} \\ &= e^{\frac{1}{5}|(\frac{\xi}{2}-\frac{\varpi}{2})-(\frac{\varrho}{2}-\frac{\eta}{2})|} \\ &\le \Big\{\max\{e^{\frac{1}{5}|(\frac{\xi}{2}-\frac{\varpi}{2})|}, e^{\frac{1}{5}|(\frac{\varrho}{2}-\frac{\eta}{2})|}\}\Big\}^2 \\ &= \Big[\max\{s(g\xi,g\varpi),s(g\eta,g\varrho)\}\Big]^\lambda, \end{aligned}$$

where $\lambda=\frac{2}{5}\in(0,1)$. Thus all the assumptions of Theorem 2.3 hold. So, there exists $(l,m)\in O\times P$ such that $\mathcal{H}(l,m)=g(l)$ and $\mathcal{H}(m,l)=g(m)$, that is

$$\frac{l+m}{10}=\frac{l}{2} \quad\text{and}\quad \frac{m+l}{10}=\frac{m}{2}.$$

Then $l=m=0$. Since g is injective map, so $(0,0)$ is the unique strong g-coupled coincidence point of $\mathcal{H}$ and g.

The following example show that the uniqueness does not necessarily to be satisfied if g is not injective map.

Example 2.5. Let($\mathcal{G}=[-2\pi,2\pi]$,s) be the MMS when $s(\xi,\eta)=e^{|\xi-\eta|}$ for all $\xi,\eta\in\mathcal{G}$. Take $O=(-\pi,\pi)$ and $P=(-\frac{3\pi}{4},\frac{3\pi}{4})$. We define $\mathcal{H}:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ by $\mathcal{H}(\xi,\eta)=\frac{\sin\xi+\sin\eta}{5}$ and $g:\mathcal{G}\to\mathcal{G}$ by $g(\xi)=\sin\xi$. Clearly $g(O)=g(P)=[-1,1]$. Also, O and P are invariant by g as $g(O)\subseteq O$ and $g(P)\subseteq P$.

Now, it needs to prove that $\mathcal{H}$ is a g-coupling. As $g(O)\cap P=g(P)\cap O=[-1,1]$, so for all $\xi\in O$ and $\eta\in P$, we have

$$-\frac{2}{5}\le\mathcal{H}(\xi,\eta)\le\frac{2}{5} \;\;\Rightarrow\;\; \mathcal{H}(\xi,\eta)\in g(O)\cap P$$

and

$$-\frac{2}{5}\le\mathcal{H}(\eta,\xi)\le\frac{2}{5} \;\;\Rightarrow\;\; \mathcal{H}(\eta,\xi)\in g(P)\cap O.$$

Thus, $\mathcal{H}$ is a g-coupling with respect to O and P. Again, for $\xi,\varrho\in O$ and $\eta,\varpi\in P$, we have

$$\begin{aligned} s(\mathcal{H}(\xi,\eta),\mathcal{H}(\varpi,\varrho)) &= e^{|\frac{\sin\xi+\sin\eta}{5}-\frac{\sin\varpi+\sin\varrho}{5}|} \\ &= e^{\frac{1}{5}|(\sin\xi-\sin\varpi)-(\sin\varrho-\sin\eta)|} \\ &\le \Big\{\max\{e^{\frac{1}{5}|(\sin\xi-\sin\varpi)|}, e^{\frac{1}{5}|(\sin\varrho-\sin\eta)|}\}\Big\}^2 \\ &= \Big[\max\{s(g\xi,g\varpi),s(g\eta,g\varrho)\}\Big]^\lambda, \end{aligned}$$

where $\lambda = \frac{2}{5} \in (0,1)$. Hence we can say that all the conditions of Theorem 2.3 are satisfied, and there are several strong g-coupled coincidence points of $\mathcal{H}$ and g.

3. Fixed point results in ordered multiplicative metric spaces having t-property

First, we give definition before proceeding the result.

Definition 3.1. ([24]) Let $(\mathcal{G}, s, \preceq)$ be an ordered metric space. $\mathcal{G}$ is said to be have t-property, if every strictly increasing cauchy sequence $\{\xi_n\}$ in $\mathcal{G}$ has a strict upper bound in $\mathcal{G}$, that is, there exists $\varpi \in \mathcal{G}$ such that $\xi_n \prec \varpi$, for all $n \in \mathbb{N}$.

Now, we give some examples on Definition 3.1.

Example 3.2. ([24]) Let $\mathcal{G} = \mathbb{R}, \ \mathbb{Q}, \ (l,m], \ l,m \in \mathbb{R}$ be equipped with the natural ordering $\leq$ and the usual metric. Then $\mathcal{G}$ has the t-property.

Example 3.3. ([24]) Let $\mathcal{G} = \{(\xi,\eta) : \xi,\eta \in \mathbb{Q}\}$. We define $\preceq$ in $\mathcal{G}$ by $(\xi_1,\xi_2) \preceq (\eta_1,\eta_2)$ if and only if $\xi_1 \leq \eta_1$ and $\xi_2 \leq \eta_2$. Let s be the Euclidean metric on $\mathcal{G}$. Then $(\mathcal{G}, s, \preceq)$ has the t-property.

Example 3.4. ([24]) Let $\mathcal{G} = C[l,m]$ be equipped with the metric s defined as $s(f,g) = \int_l^m \mid f-g \mid d\xi$. Then $(\mathcal{G}, s)$ is not a complete metric space. We define $\preceq$ in $\mathcal{G}$ as $f \preceq g$ if and only if $f(\xi) \leq g(\xi)$ for each $\xi \in [l,m]$. Obviously, $(C[l,m], \ s, \preceq)$ has t-property.

Now, we present some fixed point results in ordered MMS having t-property.

Theorem 3.5. *Let $(\mathcal{G}, s, \preceq)$ be an ordered MMS having t-property. Let $f : \mathcal{G} \to \mathcal{G}$ be a self-mapping such that for every $\xi, \eta \in \mathcal{G}$ with $\xi \prec \eta$ and for $\lambda \in (0,1)$, we have*

$$s(\eta, f\eta) \leq [s(\xi, f\xi)]^{\lambda}. \tag{3.1}$$

Further, we consider that f is non-decreasing and there exists $\xi_0 \in \mathcal{G}$ such that $\xi_0 \preceq f(\xi_0)$. Then f has at least one fixed point in $\mathcal{G}$. Moreover, any strict upper bound of a fixed point of f is a fixed point.

Proof. By given condition, we have $\xi_0 \preceq f(\xi_0)$. If $\xi_0 = f(\xi_0)$, the proof is completed. Otherwise, choose $\xi_1 = f(\xi_0)$ such that $\xi_0 \prec \xi_1$. By monotonicity of f, we have $f(\xi_0) \preceq f(\xi_1)$, that is, $\xi_1 \preceq f(\xi_1)$. If $\xi_1 = f(\xi_1)$, the proof is completed. Otherwise, there is $\xi_2 = f(\xi_1)$ such that $\xi_1 \prec \xi_2$. Again, by monotonicity of f, we have $f(\xi_1) \preceq f(\xi_2)$. Continuing in this process, we have a strictly increasing sequence $\{\xi_n\}$ in $\mathcal{G}$ such that

$$\xi_{n+1} = f(\xi_n). \tag{3.2}$$

As $\xi_0 \prec \xi_1$, by (3.1), we get

$$s(\xi_1, f(\xi_1)) \leq [s(\xi_0, f(\xi_0))]^{\lambda}. \tag{3.3}$$

Again as $\xi_1 \prec \xi_2$, by (3.1), we have

$$s(\xi_2, f(\xi_2)) \leq [s(\xi_1, f(\xi_1))]^{\lambda}. \tag{3.4}$$

Using (3.3) in (3.4), we have

$$s(\xi_2, f(\xi_2)) \leq [s(\xi_0, f(\xi_0))]^{\lambda^2}.$$

Continuing in this way, we have

$$s(\xi_n, f(\xi_n)) \leq [s(\xi_0, f(\xi_0))]^{\lambda^n}. \tag{3.5}$$

Now, we prove that $\{\xi_n\}$ is a multiplicative Cauchy sequence in $\mathcal{G}$. For $n < m$, by using triangular inequality, (3.2) and (3.5), we get

$$\begin{aligned} s(\xi_n, \xi_m) &\leq s(\xi_n, \xi_{n+1}) \times s(\xi_{n+1}, \xi_{n+2}) \times \cdots \times s(\xi_{m-1}, \xi_m) \\ &= s(\xi_n, f(\xi_n)) \times s(\xi_{n+1}, f(\xi_{n+1})) \times \cdots \times s(\xi_{m-1}, f(\xi_{m-1})) \\ &\leq [s(\xi_0, f(\xi_0))]^{\lambda^n} \times [s(\xi_0, f(\xi_0))]^{\lambda^{n+1}} \times \cdots \times [s(\xi_0, f(\xi_0))]^{\lambda^{m-1}} \\ &= K^{\lambda^n} \times K^{\lambda^{n+1}} \times \cdots \times K^{\lambda^{m-1}} \\ &\leq K^{\frac{\lambda^n}{1-\lambda}}, \end{aligned}$$

where $K = s(\xi_0, f(\xi_0))$.

As $\lambda \in (0, 1)$, $s(\xi_n, \xi_m) \to 1$ as $n, m \to \infty$. This shows that $\{\xi_n\}$ is an increasing multiplicative Cauchy sequence in $\mathcal{G}$, which has the t-property, so there exists $\varpi \in \mathcal{G}$ such that $\xi_n \prec \varpi$ for all n. Thus, from (3.1) and (3.5), we have

$$\begin{aligned} s(\varpi, f\varpi) &\leq [s(\xi_n, f(\xi_n))]^{\lambda} \\ &\leq [s(\xi_0, f(\xi_0))]^{\lambda^n} \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\varpi = f(\varpi)$. Now, let ϱ be any other strict upper bound of a fixed point of f, say ϖ, that is, $\varpi \prec \varrho$. By (3.1), we have

$$s(\varrho, f\varrho) \leq [s(\varpi, f\varpi)]^{\lambda} = 1.$$

This shows that ϱ is again a fixed point of f in $\mathcal{G}$. □

Example 3.6. Let $\mathcal{G} = \{a_n : \ a_{n+1} = 3a_n + 1 \text{ for } n \geq 0 \text{ and } a_0 = -1\} \cup (-1, 0]$. Then $\mathcal{G} = \{..., -41, -14, -5, -2, -1\} \cup (-1, 0]$. Endow $\mathcal{G}$ with the usual multiplicative metric on $\mathbb{R}$, that is, $s(\xi, \eta) = e^{|\xi - \eta|}$ and the natural ordering $\leq$. Clearly, $(\mathcal{G}, s, \preceq)$ has the t-property. Define $f : \mathcal{G} \to \mathcal{G}$ by

$$f(\xi) = \begin{cases} 3\xi + 1, & if\ \xi < -1, \\ \xi, & if\ \xi \geq -1. \end{cases}$$

Obviously, f is non-decreasing. Now, it remains to prove that f satisfies (3.1). Let $\xi, \eta \in \mathcal{G}$ with $\xi < \eta$. If $\eta \geq -1$, then $f(\eta) = \eta$, so $s(\eta, f(\eta)) = 1$ and the proof is completed. Assume now that $\xi < \eta \leq -2$. Then $s(\eta, f(\eta)) = e^{|-(2\eta+1)|}$ and $s(\xi, f(\xi)) = e^{|-(2\xi+1)|}$. It should be noted that for $\xi, \eta \in X$ with $\xi < \eta \leq -2$, we have $\eta \geq \frac{5}{12}\xi$. Then

$$\begin{aligned} s(\eta, f(\eta)) &= e^{|-(2\eta+1|)} \\ &\leq e^{|-\frac{5}{6}\xi - 1|} \\ &= \left[e^{|-[\frac{5\xi+6}{3}]|}\right]^{\frac{1}{2}} \\ &\leq \left[e^{|-(2\xi+1)|}\right]^{\frac{1}{2}} \\ &= [s(\xi, f(\xi))]^{\lambda}, \end{aligned}$$

where $\lambda = \frac{1}{2}$. Hence we say that all the conditions of Theorem 3.5 are satisfied. Therefore f has at least one fixed point in $\mathcal{G}$. In fact, any element in the set $[-1, 0]$ is a fixed point of f.

4. Application to integral equations

In this section, we present an application in support of our main results.

Let $\mathcal{G} = C([0, 1], \mathbb{R})$ be the set of all continuous functions defined on $[0, 1]$ and equipped with the metric $s(\xi, \eta) = e^{\sup\{|\xi(t) - \eta(t)|\}}$ for all $t \in [0, 1]$. Clearly, $(\mathcal{G}, s)$ is a complete MMS. Now consider the following integral equation:

$$\xi(\varsigma) = h(\varsigma) + \int_0^1 k(\varsigma, \vartheta)[f_1(\vartheta, \xi(\vartheta)) + f_2(\vartheta, \xi(\vartheta))]d\vartheta, \ \ \varsigma \in [0, 1], \tag{4.1}$$

where $h \in C([0, 1], \mathbb{R})$, $k : [0, 1] \times [0, 1] \to [0, \infty)$ and $f_1, f_2 : [0, 1] \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$. Consider two non-empty subsets $O, P \subseteq \mathcal{G}$. Let $\xi(t) \in O$ and $\eta(t) \in P$, where $t \in [0, 1]$. Suppose there exist $0 < \alpha, \beta \leq \frac{1}{4}$ such that for each $t, \varsigma, \vartheta \in [0, 1]$, we have

$$| f_1(\varsigma, \xi(t)) - f_1(\varsigma, \eta(t)) | \leq \alpha \mid \xi(t) - \eta(t) |, \tag{4.2}$$

$$| f_2(\varsigma, \xi(t)) - f_2(\varsigma, \eta(t)) | \leq \beta \mid \xi(t) - \eta(t) |, \tag{4.3}$$

$$k(\varsigma,\vartheta)\leq\frac{1}{2}. \tag{4.4}$$

Theorem 4.1. *By using conditions* (4.2)-(4.4), *the equation* (4.1) *has the unique solution in* $C([0,1],\mathbb{R})$.

Proof. Let $\mathcal{H}:\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ and $g:\mathcal{G}\to\mathcal{G}$ be defined as

$$\begin{aligned}\mathcal{H}(\xi,\eta)(\varsigma)=h(\varsigma)&+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\xi(\vartheta))+f_2(\vartheta,\xi(\vartheta))\big]d\vartheta\\&+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\eta(\vartheta))+f_2(\vartheta,\eta(\vartheta))\big]d\vartheta\end{aligned}$$

and

$$g(\xi)(\varsigma)=\begin{cases}\xi(\varsigma), & \xi(\varsigma)\in O,\\ 2\xi(\varsigma)+3, & \text{elsewhere},\end{cases}$$

where $\varsigma\in[0,1]$ and $O=\{\xi(\varsigma)\in\mathcal{G}:\xi(\varsigma)\leq h(\varsigma)+s, s\in\mathbb{R}\}$ is a multiplicative closed subset in $\mathcal{G}$. Taking $P=O$, we have $g(O)\subset O$ and $g(P)\subseteq P$. Also, $g(O)$ and $g(P)$ are multiplicative closed subsets in $\mathcal{G}$. Now, we show that $\mathcal{H}$ is a g-coupling. For $\xi(\varsigma)\in O$ and $\eta(\varsigma)\in P$, we have

$$\begin{aligned}\mathcal{H}(\xi,\eta)(\varsigma)=h(\varsigma)&+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\xi(\vartheta))+f_2(\vartheta,\xi(\vartheta))\big]d\vartheta\\&+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\eta(\vartheta))+f_2(\vartheta,\eta(\vartheta))\big]d\vartheta\\&\leq h(\varsigma)+r_1+r_2\\&\in g(O)\cap P=O,\quad r_1,r_2\in\mathbb{R}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(\eta,\xi)(\varsigma)\;&=\;h(\varsigma)+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\eta(\vartheta))+f_2(\vartheta,\eta(\vartheta))\big]d\vartheta\\&\quad+\int_0^1\frac{k(\varsigma,\vartheta)}{2}\big[f_1(\vartheta,\xi(\vartheta))+f_2(\vartheta,\xi(\vartheta))\big]d\vartheta\\&\leq\;h(\varsigma)+s_1+s_2\\&\in\;g(P)\cap O=P,\quad s_1,s_2\in\mathbb{R}.\end{aligned}$$

Hence, $\mathcal{H}$ is a g-coupling with respect to O and P. We will prove that $\mathcal{H}$ is g-contraction. For this, let $\xi,\varrho\in O$ and $\eta,\varpi\in P$. Using (4.2)–(4.4) and the fact that $e^{|l-m|} \leq \{\max\{e^{|l|}, e^{|m|}\}\}^2$, we have

$$\begin{aligned}
\Big|\mathcal{H}(\xi,\eta)(\varsigma) - \mathcal{H}(\varpi,\varrho)(\varsigma)\Big| &= \Big|\int_0^1 \frac{k(\varsigma,\vartheta)}{2}\Big[f_1(\vartheta,\xi(\vartheta)) + f_2(\vartheta,\xi(\vartheta)) + f_1(\vartheta,\eta(\vartheta)) \\
&\quad + f_2(\vartheta,\eta(\vartheta))\Big]d\vartheta - \Big[\int_0^1 \frac{k(\varsigma,\vartheta)}{2}[f_1(\vartheta,\varpi(\vartheta)) \\
&\quad + f_2(\vartheta,\varpi(\vartheta)) + f_1(\vartheta,\varrho(\vartheta)) + f_2(\vartheta,\varrho(\vartheta))]\Big]d\vartheta\Big| \\
&= \Big|\int_0^1 \frac{k(\varsigma,\vartheta))}{2}\Big\{\big([f_1(\vartheta,\xi(\vartheta)) - f_1(\vartheta,\varpi(\vartheta))] \\
&\quad + [f_2(\vartheta,\xi(\vartheta)) - f_2(\vartheta,\varpi(\vartheta))]\big) \\
&\quad - \big([f_1(\vartheta,\varrho(\vartheta)) - f_1(\vartheta,\eta(\vartheta))] \\
&\quad + [f_2(\vartheta,\varrho(\vartheta)) - f_2(\vartheta,\eta(\vartheta))]\big)\Big\}d\vartheta\Big| \\
&\leq \frac{1}{4}\Big|\int_0^1 \Big[\big\{[f_1(\vartheta,\xi(\vartheta) - f_1(\vartheta,\varpi(\vartheta))] \\
&\quad + [f_2(\vartheta,\xi(\vartheta)) - f_2(\vartheta,\varpi(\vartheta))]\big\} \\
&\quad - \big\{[f_1(\vartheta,\varrho(\vartheta)) - f_1(\vartheta,\eta(\vartheta))] \\
&\quad + [f_2(\vartheta,\varrho(\vartheta)) - f_2(\vartheta,\eta(\vartheta))]\big\}\Big]d\vartheta\Big|.
\end{aligned}$$

Now, using the fact that $e^{|l-m|} \leq \{\max\{e^{|l|}, e^{|m|}\}\}^2$, we have

$$\begin{aligned}
e^{|\mathcal{H}(\xi,\eta)\varsigma - \mathcal{H}(\varpi,\varrho)\varsigma|} &\leq \Big\{\max\{e^{\frac{1}{4}|\int_0^1[([f_1(\vartheta,\xi(\vartheta) - f_1(\vartheta,\varpi(\vartheta))] + [f_2(\vartheta,\xi(\vartheta)) - f_2(\vartheta,\varpi(\vartheta))]d\vartheta|}, \\
&\quad e^{\frac{1}{4}|\int_0^1[([f_1(\vartheta,\varrho(\vartheta) - f_1(\vartheta,\eta(\vartheta))] + [f_2(\vartheta,\varrho(\vartheta)) - f_2(\vartheta,\eta(\vartheta))]d\vartheta|}\}\Big\}^2 \\
&\leq \Big\{\max\{e^{\frac{1}{4}[\alpha|\xi-\varpi| + \beta|\xi-\varpi|]}, e^{\frac{1}{4}[\alpha|\eta-\varrho| + \beta|\eta-\varrho|]}\}\Big\}^2 \\
&= \Big[\max\{e^{|\xi-\varpi|}, e^{|\eta-\varrho|}\}\Big]^{\frac{\alpha+\beta}{2}} \\
&\leq \Big[\max\{s(g\xi, g\varpi), s(g\eta, g\varrho)\Big]^{\lambda},
\end{aligned}$$

where $\lambda = \frac{1}{4} \in (0,1)$. Now we say that all the conditions of Theorem 2.3 are satisfied. Also, since g is one-one, there exists a unique $j(t) \in O$ such that $F(j(t), j(t)) = g(j(t)) = j(t)$. Thus (4.1) has a unique solution in O. □

Acknowledgments: We would like to appreciate all referees fruitful comments and suggestions for the improvement in our manuscript.

References

[1] M. Abbas, B. Ali and Y.I. Suleiman, *Generalized coupled common fixed point results in partially ordered A-metric spaces*, Fixed Point Theory Appl., **64** (2015), 24.

[2] T. Abdeljawad, H. Aydi and E. Karapinar, *Coupled fixed points for Meir-Keeler contractions in ordered partial metric spaces*, Math. Prob. Eng., **2012** (2012), 20, Article ID 327273.

[3] A.A.N. Abdou, *Fixed point theorems for generalized contraction mappings in multiplicative metric spaces*, J. Nonlinear Sci. Appl., **9**(5) (2016), 2347-2363.

[4] S. Aggarwal, I. Uddin and S. Mujahid, *Convergence theorems for SP-iteration scheme in a ordered hyperbolic metric space,*, Nonlinear Funct. Anal. Appl., **26**(5) (2021), 961-969.

[5] A.H. Ansari, J. Nantadilok and M.S. Khan, *Best proximity points of generalized cycle weak (F, ψ, φ) contractions in ordered metric spaces*, Nonlinear Funct. Anal. Appl., **25**(1) (2020), 55-67.

[6] H. Aydi, M. Barakat, A. Felhi and S. Radenovic, *On strong coupled fixed points for nonlinear contractive type couplings*, J. Math. Anal., **8**(5) (2017), 86-96.

[7] H. Aydi, E. Karapinar and W. Shatanawi, *Coupled coincidence points in partially ordered cone metric spaces with a c-distance*, J. Appl. Math., **2012** (2012), 15, Article ID 312078.

[8] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux S. équations intégrales*, Fund. Math., **3** (1922), 133–181.

[9] A.E. Bashirov and E.M. Kurpinar, *Özyapici, A. Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337** (2008), 36-48.

[10] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65**(7) (2006), 1379-1393.

[11] B.S. Choudhury, P. Maity and P. Konar, *Fixed point results for couplings on metric spaces*, U. P. B. Sci. Bull., series A, **79**(1) (2017), 1-12.

[12] D. Guo and V. Lakshmikantham, *Coupled fixed points of non-linear operators with applications*, Nonlinear Anal., **11**(5) (1987), 623-632.

[13] Q.H. Khan and M. Imdad, *Coincidence and common fixed point theorems in a multiplicative metric space*, Adv. Fixed Point Theory, **6**(1) (2016), 1-9.

[14] Q.H. Khan and F. Sk, *Some coincidence point theorems for Prešić-Ćirić type contractions* , Nonlinear Funct. Anal. Appl., **26**(5) (2021), 1091-1104.

[15] S. Kumar and T. Rugumisa, *Fixed point for non-self mappings in multiplicative metric spaces,* Malaya J. Math., **6**(4) (2018), 800-806.

[16] S. Kumar and T. Rugumisa, *Fixed point theorem for mappings satisfying implicit relations in multiplicative metric spaces,* Malaya J. Math., **8**(1) (2020), 216-221.

[17] S. Laishram and R. Yumnam, *Tripled fixed point in ordered multiplicative metric spaces*, J. Nonlinear Anal. Appl., **1** (2017), 56-65, Artcli ID jnaa-00312, DOI:10.5899/2017/jnaa-00312.

[18] N.V. Luong and N.X. Thuan, *Coupled fixed point theorems in partially ordered metric spacesd epended on another function*, Bull. Math. Anal., **3**(3) (2011), 129-140.

[19] H.K. Nashine and W. Shatanawi, *Coupled common fixed point theorems for pair of commuting mappings in partially ordered complete metric spaces*, Comput. Math. Appl., **62**(4) (2011), 1984-1993.

[20] J.J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equationS* Order, **22**(3) (2005), 223–239.

[21] M. Özaşvar and A.C. Çevikel, *Fixed points of multiplicative contraction mappings on multiplicative metric spaces*, Math.GN., **ArXiv : 1205.5131v1** (2012), 14.

[22] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partial ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., **132**(5) (2004), 1435-1443.

[23] T. Rashid, Q.H. Khan and H. Aydi, *On strong coupled coincidence points of g-Couplings and an application*, J. Funct. Spaces, **2018** (2018), 10, Article ID 4034535, DOI.org/10.1155/2018/4034535.

[24] T. Rashid, Q.H. Khan and H. Aydi, *t-property of metric spaces and fixed point theorems*, Ital. J. Pure Appl Math., **41**(1) (2019), 422-433.

[25] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl., **117**(1) (1986), 100-127.