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GENERALIZED LINDLEY DISTRIBUTION USING PROPORTIONAL HAZARD FAMILY AND INFERENCE OF FAILURE TIME DATA

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Abstract. In this paper, we propose a generalization of Lindley distribution (GLD) via a special structure that is concern with progressively Type-II right censoring and time failure data. We study the modern properties that we have built by such combination, for example, survival function, hazard function, moments, and estimation by non-Bayesian methods. Application on some selected data related to Lindley distribution (LD) and (ED) have been employed to find out the best distribution that can fit data comparing with the GLD.

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1. Introduction

In this study, we demonstrated a general scheme of progressively Type-II right censoring that was presented with more details in [1,6,7] on Lindley distribution. Studying new lifetime models become necessary and extensive as many applications appeared in natural sciences.

Recently, many authors focused their studies on generating new lifetime distributions that will fit the experimental data, for example, medical, engineering, social sciences, reliability analysis, and others. According to Balakrishnan and Aggarwala [7], and Balakrishnan [6], they produced n terms of life-testing experiment. They indicate to a failure time by R_i such that $(i = 1, 2, 3, ..., m)$ with m stages and $m(< n)$, R_1 which is the first failure stage is $(n - 1)$ of surviving unites of the experiment, while the second stage R_2 is $(n - 2 - R_1)$ of the surviving units, etc. At the end stages of failure which is m^{th} term $R_m = n - m - R_1 - ... - R_{m-1}$ of the surviving units. Specifically, in this paper we set these new notions that explained in [1] on a continuous distribution called Lindley distribution [6] which is involved in category of lifetime distributions. One of the main reasons to consider the Lindley distribution is to carry out those ideas is its time dependent/increasing hazard rate, so this distribution work on lifetime data like applied sciences [2]. In this work we mainly restrict solution on two objectives, the first one is discover fresh concept from composition of proportional hazards family, reliability and hazard rate functions [4,8] which deals with failure rates with continuous distribution (Lindley distribution) that also deals treats with age of something. The second reason of this paper is to observe the functions form of first objective with judgment process like estimation like maximum likelihood estimation (MLE) for finding the best approximation with least errors, and the rest of these sections are numerical example of estimation that are mentioned in second reasons.

The main concept that is used for these new notions is:

$$\bar{F}_0(.) = 1 - F_0(.), \tag{1.1}$$

where $F_0(.)$ is an arbitrary continuous cdf(cumulative distribution function). While the function of a proportional hazard family is

$$F(x; \theta) = 1 - [\bar{F}_0(x)]^{\theta}, \ \ -\infty \leq c < x < d \leq \infty, \ \ \theta > 0, \tag{1.2}$$

ehere $F_0(c) = 0$ and $F_0(d) = 1$ (see Marshal and Olkin [3]). From (1.2), the probability density function (pdf), the reliability function and hazard rate function are demonstrated as follows:

$$f(x; \theta) = \theta f_0(x)[\bar{F}_0(x)]^{\theta-1}, \ \ -\infty \leq c < x < d \leq \infty \tag{1.3}$$

and

$$H(x) = \theta \frac{f_0(x)}{\bar{F}_0(x)}. \tag{1.4}$$

Now, if we applied above notions on distribution life rate called Lindley distribution, we will get the following: Since the pdf, and the cdf of Lindley distribution are:

$$f_0(x;\lambda) = \frac{\lambda^2}{\lambda+1}(x+1)e^{-\lambda x}, \quad \lambda > 0, \quad x > 0 \tag{1.5}$$

and

$$F_0(x;\lambda) = 1 - \frac{e^{-\lambda x}(1+\lambda+\lambda x)}{1+\lambda}, \tag{1.6}$$

we have

$$\bar{F}_0(x;\lambda) = \frac{e^{-\lambda x}(1+\lambda+\lambda x)}{1+\lambda}. \tag{1.7}$$

By substituting the survival from (1.6) in (1.2)

$$F(x;\theta) = 1 - [\frac{e^{-\lambda x}(1+\lambda+\lambda x)}{1+\lambda}]^{\theta}, \tag{1.8}$$

$$f(x;\lambda,\theta) = \theta[\frac{e^{-\lambda x}(1+\lambda+\lambda x)}{1+\lambda}]^{\theta-1}[\frac{\lambda^2}{\lambda+1}(x+1)e^{-\lambda x}] \tag{1.9}$$

for $0 < x < \infty$, $\lambda > 0$. Furthermore, the structure of reliability function is

$$R(x) = [\frac{e^{-\lambda x}(1+\lambda+\lambda x)}{1+\lambda}]^{\theta}. \tag{1.10}$$

While, we consider the hazard rate function as

$$H(x) = \theta \frac{\lambda^2(1+x)}{1+\lambda+\lambda x}. \tag{1.11}$$

2. Moments

In this section, we find moments on the new pdf of GLD within some calculations as the following.

Theorem 2.1. *The r^{th} moment of the random variable X that has the pdf in* (1.9) *is given by*

$$E(X^r) = \frac{\theta\lambda^2}{(1+\lambda)^{\theta}} \sum_{i=1}^{\infty} \binom{\theta-1}{i} \lambda^i (1+\lambda)^{(\theta-1)-i} \times [\Gamma(r+i+1)(\lambda\theta)^{-(r+i+1)} + \Gamma(r+i+2)(\lambda\theta)^{-(r+i+2)}]. \tag{2.1}$$

Proof. It is given the pdf in (1.9). Then

$$\begin{aligned}E(X^r) &= \int_0^\infty x^r f(x;\lambda;\theta)dx \\ &= \frac{\theta\lambda^2}{(1+\lambda)^\theta}\int_0^\infty (1+\lambda+\lambda x)^{(\theta-1)} e^{-\lambda\theta x} x^r (1+x)dx \\ &= \frac{\theta\lambda^2}{(1+\lambda)^\theta}\Big[\int_0^\infty (1+\lambda+\lambda x)^{(\theta-1)} e^{-\lambda\theta x} x^r dx \\ &\quad + \int_0^\infty (1+\lambda+\lambda x)^{(\theta-1)} e^{-\lambda\theta x} x^{r+1} dx\Big].\end{aligned}$$

By using the binomial expansion series, we can see that

$$(1+\lambda+\lambda x)^{(\theta-1)} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(\lambda x)^i (1+\lambda)^{(\theta-1)-i}.$$

Thus,

$$\begin{aligned}E(X^r) &= \frac{\theta\lambda^2}{(1+\lambda)^\theta}\sum_{i=0}^{\infty}\binom{\theta-1}{i}(\lambda)^i(1+\lambda)^{(\theta-1)-i} \\ &\quad \times \Big[\int_0^\infty e^{-\lambda\theta x} x^{r+i} dx + \int_0^\infty e^{-\lambda\theta x} x^{r+i+1} dx\Big]. \qquad (2.2)\end{aligned}$$

By comparing the integrals in (2.2) with the well-known gamma distribution, it is clear that

$$\begin{aligned}\int_0^\infty e^{-\lambda\theta x} x^{r+i} dx &= \int_0^\infty e^{-\lambda\theta x} x^{r+i+1-1} dx \\ &= \Gamma(r+i+1)(\lambda\theta)^{-(r+i+1)} \\ &\quad \times \int_0^\infty \frac{1}{\Gamma(r+i+1)(\lambda\theta)^{-(r+i+1)}} e^{-\lambda\theta x} x^{r+i+1-1} dx \\ &= \Gamma(r+i+1)(\lambda\theta)^{-(r+i+1)},\end{aligned}$$

where,

$$\int_0^\infty \frac{1}{\Gamma(r+i+1)(\lambda\theta)^{-(r+i+1)}} e^{-\lambda\theta x} x^{r+i+1-1} dx = 1.$$

Similarly, we obtain that

$$\int_0^\infty e^{-\lambda\theta x} x^{r+i+1} dx = \Gamma(r+i+2)(\lambda\theta)^{-(r+i+2)}.$$

Therefore, this yields the results of r^{th} moment in (2.1), and this completes the proof. □

In particular, let $r = 1$, and a summation for $i = 0, 1$. Then

$$E(x) = \frac{\theta\lambda^2}{(1+\lambda)^\theta} \sum_{i=0}^{1} \lambda^i (1+\lambda)^{(\theta-1)-i} [\Gamma(2+i)(\theta\lambda)^{-(2+i)}] + \Gamma(3+i)(\theta\lambda)^{-(3+i)} \quad (2.3)$$

and

$$E(x) = \frac{\theta\lambda^2}{(1+\lambda)^\theta} [(1+\lambda)^{(\theta-1)} ((\theta\lambda)^{-2} + 2(-\theta\lambda)^{-3}) + (\theta-1)\lambda(1+\lambda)^{(\theta-2)} (2(\theta\lambda)^{-3} + 6(\theta\lambda)^{-4})]. \quad (2.4)$$

Also, we can find the 2^{nd} moment by setting $r = 2$;

$$E(x^2) = \frac{\theta\lambda^2}{(1+\lambda)^\theta} \frac{\theta^2 - \theta + 2}{2} \sum_{i=0}^{2} \lambda^i (1+\lambda)^{(\theta-1)-i} \times [\Gamma(3+i)(\theta\lambda)^{-(3+i)} + \Gamma(4+i)(\theta\lambda)^{-(4+i)}] \quad (2.5)$$

and

$$E(x^2) = \frac{\theta\lambda^2}{(1+\lambda)^\theta} \frac{\theta^2 - \theta + 2}{2} [(1+\lambda)^{(\theta-1)} (2(-\theta\lambda)^{-3} + 6(\theta\lambda)^{-4}) + \lambda(1+\lambda)^\theta (6(\theta\lambda)^{-4}) + 24(\theta\lambda)^{-5}) + \lambda^2 (1+\lambda)^{(\theta+1)} (24(\theta\lambda)^{-5} + 120(\theta\lambda)^{-6})],$$

$$E(x^2) = \frac{(1 - \theta^{-1} + 2\theta^{-2})}{\lambda} [(1+\lambda)^{-1} (1 + 3(\theta\lambda)^{-1} + \lambda(3(\theta\lambda)^{-1} + \lambda(3(\theta\lambda)^{-1} + 12(\theta\lambda)^{-2}) + (1+\lambda)(12\theta^{-2} + 60\theta^{-3}\lambda^{-1}],$$

$$E(x^2) = \frac{\theta^2 - \theta + 2}{\theta^2} \left[\frac{(1+\lambda)^{-1}}{\lambda} (1 + 3(\theta\lambda)^{-1}) + 3(\theta\lambda)^{-1} (1 + 4(\theta\lambda)^{-1}) + \frac{12\theta^{-2}(1+\lambda)}{\lambda} (1 + 5(\theta\lambda)^{-1}) \right]. \quad (2.6)$$

3. Estimation

In this section, we discuss the estimation of the unknown parameter θ, the reliability function $R(t)$ and hazard rate function $H(t)$ by MLE. It is worth mentioning that the estimated parameter of Lindley distribution is

$$\hat{\lambda} = \frac{-(\bar{X} - 1) + \sqrt{(\bar{X} - 1)^2 + 8\bar{X}}}{2\bar{X}}$$

within MLE.

3.1. **Maximal Likelihood Estimation (MLE).** By rearranging the pdf in (1.9), we obtain the following pdf form:

$$f(x;\theta,\lambda) = \theta\lambda e^{-\theta\lambda x}\Big(\frac{\lambda+1+\lambda x}{\lambda+1}\Big)^{\theta}\Big[1-\frac{1}{\lambda+1+\lambda x}\Big]. \tag{3.1}$$

The likelihood function of the pdf in (3.1) has the following:

$$L(\lambda,\theta,\underline{x}) = \theta^n\lambda^n e^{-\theta\lambda\sum_{i=1}^n x_i}\prod_{i=1}^n\Big(\frac{\lambda+1+\lambda x_i}{\lambda+1}\Big)^{\theta}\Big[1-\frac{1}{\lambda+1+\lambda x_i}\Big]. \tag{3.2}$$

Therefore, the log-likelihood is as shown in (3.3):

$$\begin{aligned}\ln L &= n\ln\theta + n\ln\lambda - \theta\lambda\sum_{i=1}^n x_i - \theta n\ln(\lambda+1)\\ &\quad + \theta\sum_{i=1}^n \ln(\lambda+1+\lambda x_i) + \sum_{i=1}^n \ln\Big(1-\frac{1}{\lambda+1+\lambda x_i}\Big).\end{aligned} \tag{3.3}$$

By differentiating the log-likelihood with respect to the λ, θ, respectively. It yields the results in (3.4) and (3.5).

$$\frac{\partial L}{\partial\theta} = \frac{n}{\hat{\theta}} - \hat{\lambda}\sum_{i=1}^n x_i - n\ln(\hat{\lambda}+1) + \sum_{i=1}^n \ln(\hat{\lambda}+1+\hat{\lambda}x_i) = 0, \tag{3.4}$$

$$\begin{aligned}\frac{\partial L}{\partial\lambda} &= \frac{n}{\hat{\lambda}} - \hat{\theta}\sum_{i=1}^n x_i - \frac{n\hat{\theta}}{\hat{\lambda}+1} + \hat{\theta}\sum_{i=1}^n \frac{1+x_i}{\hat{\lambda}+1+\hat{\lambda}x_i}\\ &\quad + \sum_{i=1}^n \frac{1}{1-\frac{1}{\hat{\lambda}+1+\hat{\lambda}x_i}} - \frac{-(1+x_i)}{(\hat{\lambda}+1+\hat{\lambda}x_i)^2}\\ &= 0.\end{aligned} \tag{3.5}$$

It is obvious that the solutions of (3.4) and (3.5) can be found by using numerical analysis. Newton-Raphson is one of the best ways that can find the solution to find the estimated parameters $\hat{\lambda}$, $\hat{\theta}$. Furthermore, the estimated $R(x)$, and $H(x)$ from (1.10) and (1.11) are

$$\hat{R}(t) = \Big[\frac{e^{\hat{\lambda}x}(1+\hat{\lambda}+\hat{\lambda}x)}{1+\hat{\lambda}}\Big]^{\hat{\theta}} \tag{3.6}$$

and

$$\hat{H}(t) = \hat{\theta}\frac{\hat{\lambda}^2(1+x)}{(1+\hat{\lambda}+\hat{\lambda}x)}. \tag{3.7}$$

4. Application

4.1. Main results. In this subsection, we demonstrate the hazard rate and reliability function were estimated in (3.6) and (3.7) on data. Further, we present the value of expectation that was calculated in (2.3). By Table 1. With the estimated value of the parameter $\hat{\theta} = 0.987$, we can see that the properties of each desired distribution have the following results:

Table 1. Comparison between LD and GLD to some of their statistical properties

	$\hat{\lambda}$	$E(X)$	$\hat{R}(t)$	$\hat{H}(t)$
LD	0.189195	9.730199	0.4888441	0.116441
	0.19	9.68598	0.487182	0.117111
	0.2	9.166667	0.467122	0.125476
GLD	0.15	33.4495	0.582017	0.083389
	0.169	29.0069	0.53627	0.098554
	0.1703	28.74031	0.533298	0.099609

The results of reliability, and hazard functions of GLD are much better than the results of LD, respectively, with respect to the estimated parameters. Also, we can note that the values of the 1st moment decrease when the estimated value increase.

4.2. Goodness of-fit of data set. In this part, we implement three distributions on data set to compare the results of goodness-of-fit to decide the best distribution that can fit data. The desired distributions are the exponential distribution (ED) with our distributions LD and GLD. We firstly estimate the parameters of GLD for comparing the GLD distribution with ED and LD that have been used in [5] to analyzed the desired data. The calculations of MLEs of the estimated parameters of the GLD are $\hat{\theta} = 0.987$ and $\hat{\lambda} = 0.189195$.

In [5], the MLEs of unknown parameters of ED and LD were $\hat{\lambda} = 0.189195$, respectively. In order to summarize the testing procedure proposed, we use the well-known test statistics that are called Anderson-Darling (A^2), Cramer-von Mises (W^2) and Kolmogorov-Smirnov (D) [9]. The important testing techniques, borrowed from applied statistics to find out the best distribution that fit data set. The calculations of these tests are shown in Table 2. The values of the test statistics above show that GLD provides a good significance, and has smaller values of the three tests in comparison to the values that have been obtained from ED and LD. This verifies that GLD is suitable to fit data better than ED.

Table 2. Test statistics of ED, LD and GLD

	A^2	W^2	D
ED	0.14	0.692267	0.165689
LD	0.503733	0.059017	0.065
GLD	0.48534	0.055453	0.062943

5. Conclusion

GLD is a successful distribution that can be implemented in various applications. Several properties were found and they almost have different forms and characteristics comparison to the classical LD. The measures of goodness show that the GLD is a good distribution that can be employed to fit data set. There are other properties of estimators and other statistical derivations that can be presented in a similar way to what we have achieved in this study.

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