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ON THE SUPERSTABILITY FOR THE p-POWER-RADICAL SINE FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the superstability for the p-power-radical sine functional equation

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y)$$

from an approximation of the p-power-radical functional equation:

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)h(y),$$

where p is an odd positive integer and f, g, h are complex valued functions. Furthermore, the obtained results are extended to Banach algebras.

1. Introduction

In 1940, the stability problem of the functional equation was conjectured by Ulam [27]. In next year, Hyers [13] obtained a partial answer for the case of the additive mapping in this problem: If f satisfies $|f(x+y)-f(x)-f(y)| \le \varepsilon$ for some fixed $\varepsilon > 0$, then f satisfies the additive mapping $f(x+y) = f(x)+f(y)$, which is called the Hyers-Ulam stability.

In 1979, Baker *et al.* [6] announced the superstability as the new concept as follows: If f satisfies $|f(x+y) - f(x)f(y)| \le \varepsilon$ for some fixed $\varepsilon > 0$, then either f is bounded or f satisfies the exponential functional equation $f(x+y) = f(x)f(y)$.

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D'Alembert [1] in 1769 (see Kannappen's book [14]) introduced the cosine (d'Alembert) functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{C}$$

and whose superstability was proved on Abelian group by Baker [5] in 1980.

Badora [3] in 1998 generalized Baker's result to a noncommutative group, and it again was improved by Badora and Ger [4] in 2002 under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \le \varphi(x)$ or $\varphi(y)$ (Găvruta sense).

The cosine (d'Alembert) functional equation (C) was generalized to the following:

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{W}$$
$$f(x+y)+f(x-y)=2g(x)f(y), \tag{K}$$

in which (W) is called the Wilson equation, and (K) was arisen by Kim [15]. The superstability of the cosine (C), Wilson (W) and Kim (K) functional equations was founded in Badora [3], Ger [4], Kannappan and Kim [15], and Kim [21, 22, 25] (see [7, 11, 26]).

Due to the trigonometric formula '$\sin(x+y)-\sin(x-y)=2\cos(x)\sin(y)$' and '$\cos(x+y)-\cos(x-y)=2\sin(x)(-\sin(y))=-2\sin(x)\sin(y)$' arise the functional equations $f(x+y)-f(x-y)=2g(x)f(y) := ({}_{-}gf)$ and $f(x+y)-f(x-y)=-2g(x)g(y)=2g(x)h(y) := ({}_{-}gh)$.

For other applications: hyperbolic trigonometric functions, several exponential functions, and alternative Jensen equation are as follows:

$$\begin{aligned}
\cosh(x+y)-\cosh(x-y) &= 2\sinh(x)\sinh(y),\\
\sinh(x+y)-\sinh(x-y) &= 2\cosh(x)\sinh(y),\\
e^{x+y}-e^{x-y} &= 2\tfrac{e^x}{2}(e^y-e^{-y}) = 2e^x\cosh(y),\\
n(x+y)-n(x-y) &= 2ny : \text{for } f(x)=nx.
\end{aligned}$$

In 1983, Cholewa [9] investigated the superstability of the sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = f(x)f(y) \tag{S}$$

under the condition bounded by constant. His result was improved by Kim ([17, 20]), namely, the generalized sine functional equation

$$f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 = g(x)h(y) \tag{S_{gh}}$$

of (S) is superstable under the condition bounded by a constant or a function.

In 2009, Eshaghi Gordji and Parviz [12] introduced the radical functional equation

$$f\left(\sqrt{x^2+y^2}\right) = f(x) + f(y) \tag{R}$$

related to the additive mapping, proved its stability.

Recently, Almahalebi *et al.* [2] and Kim [24] obtained the superstability in Hyers' sense for the p-power-radical functional equations as follows:

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = 2f(x)f(y), \tag{C^r}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)f(y), \tag{C^r_λ}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda f(x)g(y), \tag{W^r_λ}$$

$$f\left(\sqrt[p]{x^p+y^p}\right) + f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)f(y), \tag{K^r_λ}$$

which are related to cosine (d'Alembert) equation (C), Wilson equation (W) and Kim's equation (K).

The p-radical form of the sine functional equation (S) is represented by

$$f\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - f\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = f(x)f(y). \tag{S^r}$$

Since the function $f(x) = \sin x^p$ is the solution of the equation (S^r), in this paper, this equation is reasonably called the p-power-radical sine functional equation.

The aim of this paper is to investigate the superstability for the p-power-radical sine functional equation (S^r) from an approximation of the p-power-radical functional equation:

$$f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) = \lambda g(x)h(y) \tag{${}_-gh^r_\lambda$}$$

related to $({}_-gh)$. Furthermore, the obtained results are extended to Banach algebras.

In this paper, let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{C}$ be the field of complex numbers. We may assume that f, g, h are nonzero functions, ε is a nonnegative real number, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a given nonnegative function and p is an odd positive integer.

2. Superstability for the p-power-radical sine functional equation

In this section, we investigate the superstability in Găvruta's sense for p-power-radical sine functional equation (S^r).

Theorem 2.1. *Assume that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)h(y)| \leq \varphi(x). \tag{2.1}$$

Then, either h (with $g(0) = 0$ or $f(x) = f(-x)$) is bounded or g satisfies (S^r). In particular, if h satisfies $({}_{-}ff_{\lambda}^{r})$, then g and h satisfy (W_{λ}^{r}).

Proof. Assume that h with $g(0) = 0$ is unbounded. Then we can choose $\{y_n\}$ such that $0 \neq |h(y_n)| \to \infty$ as $n \to \infty$.

Putting $y = y_n$ in (2.1) and dividing both sides by $\lambda h(y_n)$, we have

$$\left| \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) - f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda h(y_n)} - g(x) \right| \leq \frac{\varphi(x)}{\lambda h(y_n)}. \tag{2.2}$$

As $n \to \infty$ in (2.2), we get

$$g(x) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x^p + y_n^p}\right) - f\left(\sqrt[p]{x^p - y_n^p}\right)}{\lambda h(y_n)} \tag{2.3}$$

for all $x \in \mathbb{R}$. Replacing y by $\sqrt[p]{y^p + y_n^p}$ and $\sqrt[p]{y^p - y_n^p}$ in (2.1), we obtain

$$\left|f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) - f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right) - \lambda g(x)h(\sqrt[p]{y^p + y_n^p})\right| \leq \varphi(x),$$

$$\left|f\left(\sqrt[p]{x^p + (y^p - y_n^p)}\right) - f\left(\sqrt[p]{x^p - (y^p - y_n^p)}\right) - \lambda g(x)h(\sqrt[p]{y^p - y_n^p})\right| \leq \varphi(x),$$

for all $x, y, y_n \in \mathbb{R}$.

Then, it implies that

$$\begin{aligned} &\Bigg| \frac{f\left(\sqrt[p]{x^p + (y^p + y_n^p)}\right) - f\left(\sqrt[p]{x^p + (y^p - y_n^p)}\right)}{\lambda h(y_n)} \\ &\quad + \frac{f\left(\sqrt[p]{x^p - (y^p - y_n^p)}\right) - f\left(\sqrt[p]{x^p - (y^p + y_n^p)}\right)}{\lambda h(y_n)} \\ &\quad - \lambda g(x) \frac{h(\sqrt[p]{y^p + y_n^p}) - h(\sqrt[p]{y^p - y_n^p})}{\lambda h(y_n)} \Bigg| \leq \frac{2\varphi(x)}{\lambda h(y_n)} \end{aligned} \tag{2.4}$$

for all $x, y, y_n \in \mathbb{R}$. In (2.4), taking the limit as $n \longrightarrow \infty$ and by (2.3), we conclude that, for every $x \in \mathbb{R}$, there exists the limit

$$L_1(y) := \lim_{n\to\infty} \frac{h(\sqrt[p]{y^p + y_n^p}) - h(\sqrt[p]{y^p - y_n^p})}{\lambda h(y_n)}, \tag{2.5}$$

where the obtained function $L_1 : \mathbb{R} \to \mathbb{C}$ satisfies the equation

$$g(\sqrt[p]{x^p + y^p}) + g(\sqrt[p]{x^p - y^p}) = \lambda g(x)L_1(y), \quad \forall x, y \in \mathbb{R}. \tag{2.6}$$

First, let us consider the case $g(0) = 0$. Then it forces by (2.6) that g is odd. Putting $y = x$ in (2.6), we get

$$g(\sqrt[p]{2}x) = g(x)L_1(x), \quad \forall x \in \mathbb{R}. \tag{2.7}$$

From (2.6), the oddness of g and (2.7), we obtain the equation

$$\begin{aligned} g(\sqrt[p]{x^p + y^p})^2 - g(\sqrt[p]{x^p - y^p})^2 &= g(x)L_1(y)[g(\sqrt[p]{x^p + y^p}) - g(\sqrt[p]{x^p - y^p})] \\ &= g(x)[g(\sqrt[p]{x^p + 2y^p}) - g(\sqrt[p]{x^p - 2y^p})] \\ &= g(x)[g(\sqrt[p]{2y^p + x^p}) + g(\sqrt[p]{2y^p - x^p})] \\ &= g(x)g(\sqrt[p]{2}y)L_1(x) \\ &= g(\sqrt[p]{2}x)g(\sqrt[p]{2}y), \end{aligned} \tag{2.8}$$

that holds true for all $x, y \in \mathbb{R}$. Putting $x = \frac{x}{\sqrt[p]{2}}$ and $y = \frac{y}{\sqrt[p]{2}}$ in (2.8), we get nothing else but (S^r).

For next case $f(-x) = f(x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case. Then in what follows, without loss of generality, we may assume that $g(0) = c$: constant. Putting $x = 0$ in (2.1), from the above assumption, we obtain the inequality

$$|h(y)| \le \frac{\varphi(0)}{|\lambda c|}, \quad \forall\, y \in G.$$

This inequality means that h is globally bounded which is a contradiction by unboundedness assumption. Thus the claimed $g(0) = 0$ holds.

In particular, if h satisfies $({}_{-}ff^r_\lambda)$, from (2.5) and (2.6), g and h satisfy (W^r_λ). Hence, the proof of the theorem is completed. □

Theorem 2.2. *Assume that $f, g, h : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)h(y)\right| \le \varphi(y). \tag{2.9}$$

Then, either g(with $h(0) = 0$) is bounded or h satisfies (S^r). In particular, if g satisfies (C^r_λ), then h and g satisfy (W^r_λ).

Proof. Assume that g is unbounded. Then we can choose $\{x_n\}$ such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$. In (2.9), we deduce

$$h(y) = \lim_{n\to\infty} \frac{f\left(\sqrt[p]{x_n^p + y^p}\right) - f\left(\sqrt[p]{x_n^p - y^p}\right)}{\lambda g(x_n)} \tag{2.10}$$

for all $y \in \mathbb{R}$.

Replace (x,y) by $(\sqrt[p]{x_n^p+y^p},x)$ and replace (x,y) by $(\sqrt[p]{x_n^p-y^p},x)$ in (2.9). Let's go through the same procedure as Theorem 2.1. Then we obtain

$$\begin{aligned}
\Bigg|&\frac{f\left(\sqrt[p]{(x_n^p+y^p)+x^p}\right)-f\left(\sqrt[p]{(x_n^p-y^p)-x^p}\right)}{\lambda g(x_n)}\\
&+\frac{f\left(\sqrt[p]{(x_n^p-y^p)+x^p}\right)-f\left(\sqrt[p]{(x_n^p+y^p)-x^p}\right)}{\lambda g(x_n)}\\
&-\lambda\frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{\lambda g(x_n)}h(x)\Bigg|\le\frac{2\varphi(x)}{\lambda g(x_n)}.
\end{aligned}\tag{2.11}$$

Taking the limit as $n\to\infty$ with the use of $|g(x_n)|\to\infty$ in (2.11), we conclude that, for every $x\in\mathbb{R}$, there exists the limit

$$L_2(y):=\frac{g(\sqrt[p]{x_n^p+y^p})+g(\sqrt[p]{x_n^p-y^p})}{\lambda g(x_n)},\tag{2.12}$$

where the obtained function $L_2:\mathbb{R}\to\mathbb{C}$ satisfies the equation

$$h(\sqrt[p]{x^p+y^p})+h(\sqrt[p]{x^p-y^p})=\lambda h(x)L_2(y),\quad\forall x,y\in\mathbb{R}.\tag{2.13}$$

In which, (2.13) is none other than (2.6). So, the remainder of the proof goes through the same procedure as in (2.7) and (2.8) of Theorem 2.1. □

The following corollaries follow from Theorems 2.1 and 2.2.

Corollary 2.3. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)g(y)\right|\le\varphi(x).$$

Then, either g(with $g(0)=0$ or $f(-x)=-f(x)$) is bounded or g satisfies (S^r).

Corollary 2.4. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)g(y)\right|\le\varphi(y).$$

Then, either g (with $g(0)=0$) is bounded or g satisfies (S^r).

Corollary 2.5. *Assume that $f,g:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$\left|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)g(y)\right|\le\varepsilon.$$

Then, either g(with $g(0)=0$ or $f(-x)=-f(x)$) is bounded or g satisfies (S^r).

Corollary 2.6. ([23, Theorem 1]) *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)f(y)| \leq \begin{cases} (i) \;\; \varphi(x), \\ (ii) \;\; \varphi(y) \;\; and \;\; \varphi(x). \end{cases} \tag{2.14}$$

Then

(i) *either f is bounded or g satisfies (C_λ^r),*

(ii) *either g is bounded or g satisfies (C_λ^r), and f and g satisfy (K_λ^r) and (W_λ^r).*

Proof. (i) By replacing h to f in (2.1) of Theorem 2.1, and replacing y by $\sqrt[p]{y^p+y_n^p}$ and $\sqrt[p]{y^p-y_n^p}$ in (2.1), then, by considering equations and inequalities (2.2) $\sim$ (2.6), it forces that the limit L_1 of (2.5) is none other than g. Hence it follows from (2.6) that g satisfies (C_λ^r).

(ii) First, we can show that f (or g) is bounded if and only if g (or f) is bounded (see [23, Theorem 1]).

Hence, in the case $\varphi(y)$, by (i), that g also satisfies (C_λ^r).

Then we can choose $\{x_n\}$ and $\{y_n\}$ such that $0 \neq |g(x_n)| \to \infty$ and $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$, simultaneously. Replace h to f in (2.9) of Theorem 2.2.

(a) For $\varphi(x)$ in (ii) of (2.14), replace y by $\sqrt[p]{y^p+y_n^p}$ and $\sqrt[p]{y^p-y_n^p}$.

(b) For $\varphi(y)$ in (ii) of (2.14), replace (x, y) by $(\sqrt[p]{x_n^p+y^p}, x)$, and replace (x, y) by $(\sqrt[p]{x_n^p-y^p}, x)$.

By considering equations and inequalities (2.10)$\sim$(2.13), and by applying of the result (i), it forces that the limit L_2 of (2.12) is none other than g. g satisfies (C_λ^r).

Finally, the replaced $f = h$ in (2.6) with limit $L_1 = g$, and the replaced $f = h$ in (2.13) with limit $L_2 = g$ force that g and f satisfy the required (K_λ^r) and (W_λ^r). □

The following corollary also holds by the same logic as in Corollary 2.6.

Corollary 2.7. ([23, Theorem 2]) *Assume that $f, g : \mathbb{R} \to \mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)g(y)| \leq \begin{cases} (i) \;\; \varphi(y), \\ (ii) \;\; \varphi(x) \;\; and \;\; \varphi(y). \end{cases}$$

Then

(i) *either f(: odd) is bounded or g satisfies $({}_-ff_\lambda^r)$,*

(ii) *either g(with f:odd) is bounded or g satisfies $({}_-ff_\lambda^r)$, and f and g satisfy $({}_-fg_\lambda^r)$.*

Corollary 2.8. *Assume that $f:\mathbb{R}\to\mathbb{C}$ satisfy the inequality*

$$|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda f(x)f(y)|\le\begin{cases}(i)\ \varphi(x),\\(ii)\ \varphi(y),\\(iii)\ \varepsilon.\end{cases}$$

Then either f(:odd) is bounded or f satisfies $({}_{-}ff_\lambda^r)$.

Remark 2.9. In obtained results, by applying $p=1$ or $\lambda=2$, and replacing g or h to g or f, each cases arrive the sine, cosine, Wilson, Kim, variety trigonometric, exponential, hyperbolic sine(cosine), Jensen equations, etc.((S), $({}_{-}ff_\lambda)$, $({}_{-}fg_\lambda)$, $({}_{-}gf_\lambda)$, $({}_{-}gg_\lambda)$, $({}_{-}gh_\lambda)$, $({}_{-}ff_\lambda^r)$, $({}_{-}fg_\lambda^r)$, $({}_{-}gf_\lambda^r)$, $({}_{-}gg_\lambda^r)$, $({}_{-}gh_\lambda^r)$). Hence, we obtain the stability of Hyers-Ulam with $\varphi(x),\varphi(y)=\varepsilon$ and Găvruta sense for the derived functional equations.

Namely, the applied results appear in many papers (Badora [3], Badora and Ger [4], Baker [5], Fassi, et al. [10], Kannappan and Kim [15], Kim [16, 18, 21, 22, 24], and Almahalebi, et al. [2]), etc.

The above noted equations are as follows:

$$f(x+y)-f(x-y)=\lambda f(x)f(y), \tag{${}_{-}ff_\lambda$}$$
$$f(x+y)-f(x-y)=\lambda f(x)g(y), \tag{${}_{-}fg_\lambda$}$$
$$f(x+y)-f(x-y)=\lambda g(x)f(y), \tag{${}_{-}gf_\lambda$}$$
$$f(x+y)-f(x-y)=\lambda g(x)g(y), \tag{${}_{-}gg_\lambda$}$$
$$f(x+y)-f(x-y)=\lambda g(x)h(y), \tag{${}_{-}gh_\lambda$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)f(y), \tag{${}_{-}ff_\lambda^r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda f(x)g(y), \tag{${}_{-}fg_\lambda^r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)f(y), \tag{${}_{-}gf_\lambda^r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)g(y), \tag{${}_{-}gg_\lambda^r$}$$
$$f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)=\lambda g(x)h(y). \tag{${}_{-}gh_\lambda^r$}$$

3. Extension to Banach algebras

All the results in Section 2 can be extended to Banach algebras. Since the same applies to all results, Theorems 2.1 and 2.2 are only grouped together and the rest of the results will be omitted.

Theorem 3.1. *Let $(E,\|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f,g,h:\mathbb{R}\to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p+y^p}\right)-f\left(\sqrt[p]{x^p-y^p}\right)-\lambda g(x)h(y)\|\le\begin{cases}(i)\ \ \varphi(x),\\(ii)\ \ \varphi(y).\end{cases} \tag{3.1}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional. Then

(i) *if $z^* \circ h$ with $g(0) = 0$ is unbounded, then g satisfies (S^r),*

(ii) *if $z^* \circ g$ with $h(0) = 0$ is unbounded, then h satisfies (S^r).*

Proof. (i) Assume that (3.1) holds and let $z^* \in E^*$ be a linear multiplicative functional. Since $\|z^*\| = 1$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\varphi(x) &\geq \left\|f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)h(y)\right\| \\ &= \sup_{\|w^*\|=1} \left|w^*\Big(f\left(\sqrt[p]{x^p+y^p}\right) - f\left(\sqrt[p]{x^p-y^p}\right) - \lambda g(x)h(y)\Big)\right| \\ &\geq \left|z^*\Big(f\left(\sqrt[p]{x^p+y^p}\right)\Big) - z^*\Big(f\left(\sqrt[p]{x^p-y^p}\right)\Big) - \lambda \cdot z^*\big(g(x)\big) \cdot z^*\big(h(y)\big)\right|,\end{aligned}$$

which states that the superpositions $z^* \circ g$ and $z^* \circ h$ yield solutions of the inequalities (2.1) in Theorem 2.1. Hence, we can apply to Theorem 2.1.

Since, by assumption, the superposition $z^* \circ h$ with $g(0) = 0$ is unbounded, an appeal to Theorem 2.1 shows that the superposition $z^* \circ g$ is a solution of (S^r), that is,

$$(z^* \circ g)\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - (z^* \circ g)\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = \lambda(z^* \circ g)(x)(z^* \circ g)(y).$$

This means by a linear multiplicativity of z^* that the differences

$$\mathcal{D}S^r(x,y) := g\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - g\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 - g(x)g(y)$$

falls into the kernel of z^*. That is, $z^*\left(\mathcal{D}S^r(x,y)\right) = 0$. Hence an unrestricted choice of z^* implies that

$$\mathcal{D}S^r(x,y) \in \bigcap\{\ker z^* : z^* \in E^*\}.$$

Since the space E is a semisimple, $\bigcap\{\ker z^* : z^* \in E^*\} = 0$, which means that g satisfies the claimed equation (S^r).

(ii) By assumption, the superposition $z^* \circ g$ with $h(0) = 0$ is unbounded, an appeal to Theorem 2.2 shows that the results hold.

$z^* \circ h$ is solution of the equation (S^r), that is,

$$(z^* \circ h)\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - (z^* \circ h)\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 = \lambda(z^* \circ h)(x)(z^* \circ h)(y).$$

As in (i), a linear multiplicativity of z^* and semisimplicity imply

$$h\left(\sqrt[p]{\frac{x^p+y^p}{2}}\right)^2 - h\left(\sqrt[p]{\frac{x^p-y^p}{2}}\right)^2 - h(x)h(y) \in \bigcap\{\ker z^* : z^* \in E^*\} = 0,$$

which means that h satisfies (S^r). $\square$

As shown in Section 2, the following results are naturally derived from the above theorem.

Corollary 3.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)g(y)\| \leq \begin{cases} (i) \ \ \varphi(x), \\ (ii) \ \ \varphi(y). \end{cases}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional. Then

(i) *if $z^* \circ g$ with $g(0) = 0$ or $f(-x) = -f(x)$ is unbounded, then h satisfies (S^r),*

(ii) *if $z^* \circ g$ with $g(0) = 0$ is unbounded, then g satisfies (S^r).*

Corollary 3.3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda g(x)f(y)\| \leq \begin{cases} (i) \ \ \varphi(x), \\ (ii) \ \ \varphi(y) \ \ and \ \ \varphi(x). \end{cases}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional. Then

(i) *if $z^* \circ f$ is unbounded, then g satisfies (C^r_λ),*

(ii) *if $z^* \circ g$ is unbounded, then g satisfies (C^r_λ), and f and g satisfy (K^r_λ) and (W^r_λ).*

Corollary 3.4. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)g(y)\| \leq \begin{cases} (i) \ \ \varphi(y), \\ (ii) \ \ \varphi(x) \ \ and \ \ \varphi(y). \end{cases}$$

Let $z^* \in E^*$ be an arbitrary linear multiplicative functional. Then

(i) *if $z^* \circ f$ (f:odd) is unbounded, then g satisfies $({}_{-}ff^r_\lambda)$,*

(ii) *if $z^* \circ g$ (or $z^* \circ f$:odd) is unbounded, then g satisfies $({}_{-}ff^r_\lambda)$, and f and g satisfy $({}_{-}gf^r_\lambda)$ and $({}_{-}fg^r_\lambda)$.*

Corollary 3.5. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f : \mathbb{R} \to E$ satisfy the inequality*

$$\|f\left(\sqrt[p]{x^p + y^p}\right) - f\left(\sqrt[p]{x^p - y^p}\right) - \lambda f(x)f(y)\| \leq \begin{cases} (i) \ \ \varphi(x), \\ (ii) \ \ \varphi(y), \\ (iii) \ \ \varepsilon. \end{cases}$$

Then either the superposition $z^* \circ f$ (f:odd) is bounded for each linear multiplicative functional $z^* \in E^*$ or f satisfies $({}_{-}ff^r_\lambda)$.

Remark 3.6. As in remark 2.9 of Section 2, stability results to Banach algebras for the functional equations applied $p = 1$ or $\lambda = 2$, $\varphi(x), \varphi(y) = \varepsilon$ are found in Badora [3], Badora and Ger [4], Baker [5], Fassi, et al. [10], Kannappan and Kim [15], Kim [16, 18, 21, 22, 24], and Almahalebi, et al. [2].

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