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IMPROVEMENT AND GENERALIZATION OF POLYNOMIAL INEQUALITY DUE TO RIVLIN

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Abstract. Let $p(z)$ be a polynomial of degree n having no zero in $|z| < 1$. In this paper, by involving some coefficients of the polynomial, we prove an inequality that not only improves as well as generalizes the well-known result proved by Rivlin but also has some interesting consequences.

1. Introduction

Let $p(z)$ be a polynomial of degree n and let $M(p,r) = \max_{|z|=r} |p(z)|$. Then the following inequalities concerning the maximum modulus of a polynomial on a circle in terms of the maximum modulus of the polynomial on the unit

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circle, are known

$$M(p,R) \leq R^n M(p,1),\ R \geq 1 \tag{1.1}$$

and

$$M(p,r) \geq r^n M(p,1),\ r \leq 1. \tag{1.2}$$

Inequalities (1.1) and (1.2) are sharp and equality holds for $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number. Inequality (1.1) is a simple deduction from maximum modulus principle [12, 16]. Inequality (1.2) is due to Zarantonello and Varga [19].

If we restrict ourselves to the class of polynomials not vanishing in $|z| < 1$, inequality analogous to (1.2) was obtained by Rivlin [17].

Theorem 1.1. *Let $p(z)$ be a polynomial of degree n having no zero in $|z| < 1$. Then for $r \leq 1$,*

$$M(p,r) \geq \left(\frac{1+r}{2}\right)^n M(p,1). \tag{1.3}$$

Inequality (1.3) is sharp and equality holds for the polynomial $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, where $|\alpha| = |\beta|$.

Next, Govil [7, Theorem 1] generalized Theorem 1.1 by proving

Theorem 1.2. *Let $p(z)$ be a polynomial of degree n having no zero in $|z| < 1$. Then for $0 < r \leq R \leq 1$,*

$$M(p,r) \geq \left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{1.4}$$

The result is best possible and equality holds for the polynomial $p(z) = \left(\frac{z+1}{R+1}\right)^n$.

Another generalization of Theorem 1.1 for polynomial not vanishing inside the domain $|z| < k$, $k \geq 1$, was proved by Aziz [1].

Theorem 1.3. *Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$. Then for $0 \leq r < 1$,*

$$M(p,r) \geq \left(\frac{r+k}{1+k}\right)^n M(p,R). \tag{1.5}$$

The result is best possible and equality holds for the polynomial $p(z) = (z+k)^n$.

For the case $k \leq 1$ Aziz [1] further proved.

Theorem 1.4. *Let $p(z)$ be a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$. Then for $0 \leq r \leq k^2$,*

$$M(p,r) \geq \left(\frac{r+k}{1+k}\right)^n M(p,R). \tag{1.6}$$

The result is best possible and equality holds for the polynomial $p(z) = (z+k)^n$.

Qazi [14], obtained a generalization of Theorem 1.2 by taking a more general class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$.

Theorem 1.5. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < 1$. *Then for* $0 \leq r \leq R \leq 1$,

$$M(p,r) \geq \left(\frac{1+r^\mu}{1+R^\mu}\right)^{\frac{n}{\mu}} M(p,R). \tag{1.7}$$

Inequality (1.7) *is best possible and equality holds for* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

Further, Jain [9] obtained a result which provides a generalization of Theorem 1.5 as well as a generalization of Theorem 1.3 and Theorem 1.4 proved by Aziz [1].

Theorem 1.6. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$. *Then for* $0 \leq r \leq R \leq k$,

$$M(p,r) \geq \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n}{\mu}} M(p,R). \tag{1.8}$$

Equality holds in (1.8) *for* $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, *where* n *is a multiple of* μ.

As a generalization and refinement of Theorem 1.2, Govil and Nwaeze [8] obtained the following result.

Theorem 1.7. *If* $p(z) = \sum_{\nu=\mu}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no in* $|z| < k$, $k \geq 1$. *Then for* $0 < r < R \leq 1$,

$$M(p,r) \geq \frac{(1+r)^n}{(1+r)^n + (R+k)^n - (r+k)^n}\left[M(p,R) + nm \ln\left(\frac{R+k}{r+k}\right)\right]. \tag{1.9}$$

Recently, Mir et al. [11] have considered a more general class of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, not vanishing in $|z| < k$, $k \geq 1$ and proved an extension as well as sharpening of Rivlin's inequality (1.3).

Theorem 1.8. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$. *Then for* $0 < r < R \le 1$,

$$M(p,r) \ \ge \ \frac{(1+r^\mu)^{\frac{n}{\mu}}}{(1+r^\mu)^{\frac{n}{\mu}} + \{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}\}} \times \left\{ M(p,R) + m \ln \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} \right\}, \tag{1.10}$$

where $m = \min_{|z|=k} |p(z)|$.

The improvement and generalization of the inequalities concerning the maximum modulus of a polynomial on a circle is a widely studied topic, and for more informations in this direction, we refer to the recently published papers [2], [3], [4], [5], [10], [18], etc.

2. Main result

In this paper, by involving certain coefficients of the polynomial, we prove the following inequality which improves as well as generalizes the bound given by Theorem 1.8. More precisely, we obtain:

Theorem 2.1. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$. *Then for* $0 < r \le R \le \rho$, $\rho \le k$ *and for every non-negative real number* λ *with* $0 \le \lambda < 1$,

$$\begin{aligned} M(p,r) \ \ge \ & \Bigg[M(p,R) + n\lambda m \int_r^R \frac{t^{\mu-1}}{k^\mu + t^\mu} \\ & \times \exp\left\{ n \int_r^R \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m} k^{\mu+1} x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)} dx \right\} dt \\ & -(\lambda-1)\frac{n}{\mu} m \ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right) \Bigg] \\ & \times \frac{1}{1+n\left[\int_r^R \frac{t^{\mu-1}}{k^\mu+t^\mu} \exp\left\{ n\int_r^R \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)} dx \right\} dt\right]}, \end{aligned} \tag{2.1}$$

where $m = \min_{|z|=k} |p(z)|$.

Putting $\lambda = 0$ in inequality (2.1) of Theorem 2.1, we get a result recently proved by Devi et al. [4].

Corollary 2.2. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$. *Then for* $0 < r \le R \le \rho$, $\rho \le k$,

$$M(p,r) \ge \left\{ M(p,R) + m\frac{n}{\mu} ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)\right\} \tag{2.2}$$
$$\times \frac{1}{1+n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} exp\left\{ n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt\right]},$$

where $m = \min_{|z|=k} |p(z)|$.

Putting $\rho = 1$ in Corollary 2.2, we have the following interesting result recently proved by Chanam [2].

Corollary 2.3. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *be a polynomial of degree* n *having no zero in* $|z| < k$, $k \ge 1$. *Then for* $0 < r \le R \le 1$,

$$M(p,r) \ge \left\{ M(p,R) + m\frac{n}{\mu} ln\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)\right\} \tag{2.3}$$
$$\times \frac{1}{1+n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} exp\left\{ n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt\right]},$$

where $m = \min_{|z|=k} |p(z)|$.

Remark 2.4. By inequality (3.27) for $\rho = 1$ of Lemma 3.9, it is evident that the right hand side of (2.3) dominates over that of (1.10) and thus Corollary 2.3 gives better bound than that of Theorem 1.8.

Remark 2.5. Putting $\mu = 1$ and $R = k = 1$ in Theorem 1.8, we have under the same hypotheses, the following improvement of the famous result due to Rivlin [17].

Corollary 2.6. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z| < 1$. *Then for* $0 < r \le 1$,

$$M(p,r) \ge \left(\frac{1+r}{2}\right)^n \left\{ M(p,1) + nm \ln\left(\frac{2}{1+r}\right)\right\}, \tag{2.4}$$

where $m = \min\limits_{|z|=1}|p(z)|$.

Inequality (2.4) *is sharp and equality holds for the polynomial* $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, *where* $|\alpha| = |\beta|$.

Remark 2.7. If we put $\lambda = 0$, $\rho = \mu = 1$ and $R = k = 1$ in Lemma 3.9, then we have, in particular, for $0 < r \leq 1$,

$$\left(\frac{2}{1+r}\right)^n - 1 \geq \frac{n}{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}r + r^2\right)^{\frac{n}{2}}} \int_r^1 \frac{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}t + t^2\right)^{\frac{n}{2}}}{1+t} dt, \tag{2.5}$$

where $m = \min\limits_{|z|=1}|p(z)|$.

Further, if we put $\mu = \rho = 1$ and $R = k = 1$ in Corollary 2.2, we have:

Corollary 2.8. *If* $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for* $0 < r \leq 1$,

$$\begin{aligned} M(p,r) \geq & \left\{M(p,1) + mn \ln\left(\frac{2}{r+1}\right)\right\} \\ & \times \frac{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}r + r^2\right)^{\frac{n}{2}}}{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}r + r^2\right)^{\frac{n}{2}} + n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}t+t^2\right)^{\frac{n}{2}}}{1+t} dt}, \end{aligned} \tag{2.6}$$

where $m = \min\limits_{|z|=1}|p(z)|$.

Again, by inequality (2.5) of Remark 2.7, the quantity

$$\left\{\frac{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}r + r^2\right)^{\frac{n}{2}}}{\left(1 + \frac{2}{n}\frac{|a_1|}{|a_0|}r + r^2\right)^{\frac{n}{2}} + n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}t+t^2\right)^{\frac{n}{2}}}{1+t} dt}\right\}$$

appearing in the right hand side of (2.6) is greater than or equal to $\left(\frac{1+r}{2}\right)^n$ and hence Corollary 2.8 further improves Corollary 2.6, which in turn, improves Theorem 1.1 due to Rivlin [17].

Remark 2.9. In the same way as the previous two joint conclusions in Remarks 2.5 and 2.7, we are further clear that Corollary 2.2 is an improvement as well as generalization of Theorem 1.2.

Remark 2.10. Theorem 1.8 has a limitation in the sense that for k in $(0,1)$, we do not have analogous bound of inequality (1.10) for $0 < r \le R \le k$. It is easily seen that this has been supplemented by our result. Moreover, for $k > 1$, the limit of r and R extends from $(0,1]$ to $(0,k]$.

Remark 2.11. If we put $\lambda = \rho = \mu = 1$ and $R = k = 1$ in Lemma 3.9, then we have, in particular, for $0 < r \le 1$,

$$\left(\frac{2}{1+r}\right)^n - 1 \ge \frac{n}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}}} \int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}t+t^2\right)^{\frac{n}{2}}}{1+t}dt, \quad (2.7)$$

where $m = \min\limits_{|z|=1}|p(z)|$.

If we put $\lambda = \rho = \mu = 1$ and $R = k = 1$ in Theorem 2.1, we have:

Corollary 2.12. *If* $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for* $0 < r \le 1$,

$$\begin{aligned} M(p,r) \ge & \left[M(p,1)+m\frac{n}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}}}\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}t+t^2\right)^{\frac{n}{2}}}{1+t}dt\right] \\ & \times \left\{\frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}}}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}} + n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}t+t^2\right)^{\frac{n}{2}}}{1+t}dt}\right\}, \end{aligned} \quad (2.8)$$

where $m = \min\limits_{|z|=1} |p(z)|$.

Again, by inequality (2.7) of Remark 2.11, the quantity

$$\left\{\frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}}}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}r+r^2\right)^{\frac{n}{2}} + n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|-m}t+t^2\right)^{\frac{n}{2}}}{1+t}dt}\right\}$$

appearing in the right hand side of (2.8) is greater than or equal to $\left(\frac{1+r}{2}\right)^n$ and hence Corollary 2.12 further improves Corollary 2.6 and Corollary 2.8, which in turn, improves Theorem 1.1 due to Rivlin [17].

3. Lemmas

To prove the theorem, the following lemmas are required.

The following lemma is due to Pukhta [13].

Lemma 3.1. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p^{'}(z)| \leq \frac{n}{1+k^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{3.1}$$

The next result is due to Rahman and Stankiewicz [15, Theorem $2^{'}$, p. 180].

Lemma 3.2. *Let* $p_n(z) = \prod_{\nu=1}^{n} (1 - z_\nu z)$ *be a polynomial of degree* n *not vanishing in* $|z| < 1$ *and let* $p_n^{'}(0) = p_n^{''}(0) = ... = p_n^{l}(0) = 0$. *If* $\Phi(z) = \{p_n(z)\}^\epsilon = \sum_{k=0}^{n} b_{k,\epsilon} z^k$, *where* $\epsilon = 1$ *or* -1, *then*

$$|b_{k,\epsilon}| \leq \frac{n}{k}, \ (l+1 \leq k \leq 2l+1)$$

and

$$|b_{2l+2,1}| \leq \frac{n}{2(l+1)^2}(n+l-1), \ |b_{2l+2,-1}| \leq \frac{n}{2(l+1)^2}(n+l+1).$$

The following lemma is due to Gardner et al. [6].

Lemma 3.3. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$, *then*

$$|p(z)| \geq m \text{ for } |z| \leq k, \tag{3.2}$$

where $m = \min_{|z|=k} |p(z)|$.

Lemma 3.4. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *and if* $m = \min_{|z|=k} |p(z)|$, *then for every real or complex number* λ *with* $|\lambda| < 1$,

$$\frac{\mu}{n} \frac{|a_\mu| k^\mu}{|a_0| - \lambda m} \leq 1. \tag{3.3}$$

Proof. Without loss of generality, we can assume $a_0 > 0$ for otherwise, we can consider the polynomial $P(z) = e^{-iarga_0}p(z)$, which clearly also has no zero in $|z| < k$ and $M(P, R) = M(p, R)$. Since $p(z) \neq 0$ for $|z| < k$, hence, by Lemma 3.3, $|p(z)| \geq m$ for $|z| \leq k$, $m = \min_{|z|=k} |p(z)|$. Now, for every real or complex number λ such that $|\lambda| < 1$, $|p(z)| > m \geq |\lambda|m$, $\forall$ z in $|z| < k$. Therefore, by Rouche's Theorem, the polynomial $p(z) - \lambda m \neq 0$ in $|z| < k$, and hence the polynomial $Q(z) = p(kz) - \lambda m \neq 0$ for $|z| < 1$. Applying Lemma 3.2 to the polynomial $\frac{Q(z)}{a_0 - \lambda m}$, which clearly satisfies the hypothesis of Lemma 3.2, we get

$$\frac{|a_\mu|k^\mu}{a_0 - \lambda m} \leq \frac{n}{\mu},$$

which is equivalent to

$$\frac{\mu}{n}\frac{|a_\mu|k^\mu}{a_0 - \lambda m} \leq 1.$$

This completes the proof of Lemma 3.4. □

Lemma 3.5. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then the function*

$$f(x) = \frac{1 + \frac{\mu}{n}\frac{|a_\mu|}{x}k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{x}(k^{\mu+1} + k^{2\mu})} \tag{3.4}$$

is a non-increasing function of $x > 0$.

Proof. The proof follows simply by using first derivative test. □

The next lemma is due to Qazi [14].

Lemma 3.6. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq n\frac{1 + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}}{1 + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(k^{\mu+1} + k^{2\mu})}\max_{|z|=1} |p(z)| \tag{3.5}$$

and

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \leq 1. \tag{3.6}$$

Lemma 3.7. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \le R \le \rho$, $\rho \le k$ *and for every real number* λ *with* $0 \le \lambda < 1$,

$$M(p,R) \le \{M(p,r) - \lambda m\} r \times \exp\left\{n \int_r^R \frac{t^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m} k^{\mu+1} t^{\mu-1}}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu + k^{2\mu}t)} dt\right\} + \lambda m, \tag{3.7}$$

where $m = \min_{|z|=k} |p(z)|$.

Proof. Since the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, has no zero in $|z| < k$, $k > 0$, for every real or complex number α with $|\alpha| < 1$, by Rouche's Theorem, for $0 < t \le k$, the polynomial $P(z) = p(tz) + \alpha m$ has no zero in $|z| < \frac{k}{t}$, $\frac{k}{t} \ge 1$, where $m = \min_{|z|=k} |p(z)|$.

By using inequality (3.5) of Lemma 3.6 to $P(z) = p(tz) + \alpha m$, we have

$$\max_{|z|=1} |P^{'}(z)| \le n \frac{1 + \frac{\mu}{n}\frac{|a_\mu| t^\mu}{|a_0+\alpha m|}(\frac{k}{t})^{\mu+1}}{1+(\frac{k}{t})^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu| t^\mu}{|a_0+\alpha m|}\left((\frac{k}{t})^{\mu+1}+(\frac{k}{t})^{2\mu}\right)} \left\{\max_{|z|=1} |p(tz)+\alpha m|\right\},$$

where

$$m = \min_{|z|=\frac{k}{t}} |P(z)| = \min_{|z|=\frac{k}{t}} |p(tz)| = \min_{|z|=k} |p(z)|,$$

which gives

$$t \max_{|z|=t} |p^{'}(z)| \le n \frac{1 + \frac{\mu}{n}\frac{|a_\mu|}{|a_0+\alpha m|}\frac{k^{\mu+1}}{t}}{1 + \frac{k^{\mu+1}}{t^{\mu+1}} + \frac{\mu}{n}\frac{|a_\mu| t^\mu}{|a_0+\alpha m|}\left(\frac{k^{\mu+1}}{t} + \frac{k^{2\mu}}{t^\mu}\right)} \left\{\max_{|z|=1} |p(tz) + \alpha m|\right\}. \tag{3.8}$$

Let z_0 on $|z| = 1$ be such that

$$\max_{|z|=1} |p(tz) + \alpha m| = |p(tz_0) + \alpha m|.$$

Now, we choose the argument of α suitably such that

$$\begin{aligned} |p(tz_0) + \alpha m| &= |p(tz_0)| - |\alpha| m \\ &\le \max_{|z|=t} |p(z)| - |\alpha| m. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\max_{|z|=t}|p^{'}(z)| \;\leq\; n\frac{t^{\mu}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0+\alpha m|}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0+\alpha m|}\left(k^{\mu+1}t^{\mu}+k^{2\mu}t\right)} \times\left\{\max_{|z|=t}|p(z)|-|\alpha|m\right\}. \tag{3.10}$$

By Lemma 3.3, for $|z|\leq k$

$$|p(z)|\geq m.$$

In particular,

$$|p(0)|\geq m,$$

which implies

$$|a_0|\geq m. \tag{3.11}$$

For any real or complex number α with $|\alpha|<1$, inequality (3.11) gives

$$|a_0|\geq|\alpha|m. \tag{3.12}$$

Now, using inequality (3.12)

$$\begin{aligned}|a_0+\alpha m| &\geq ||a_0|-|\alpha|m| \\ &= |a_0|-|\alpha|m.\end{aligned} \tag{3.13}$$

By Lemma 3.5, $f(x)$ is a non-increasing function of x, and hence

$$f(|a_0+\alpha m|)\leq f(|a_0|-|\alpha|m). \tag{3.14}$$

Using inequality (3.14) in (3.10), we have

$$\max_{|z|=t}|p^{'}(z)| \;\leq\; n\frac{t^{\mu}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0|-|\alpha|m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0|-|\alpha|m}\left(k^{\mu+1}t^{\mu}+k^{2\mu}t\right)} \times\left\{\max_{|z|=t}|p(z)|-|\alpha|m\right\}. \tag{3.15}$$

Since the argument of α is fixed in the above inequality, $|\alpha|$ behaves like a non-negative real number λ with $0\leq\lambda<1$ and we set $|\alpha|=\lambda$, then (3.15) becomes

$$\max_{|z|=t}|p^{'}(z)|\leq n\frac{t^{\mu}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_{\mu}|}{|a_0|-\lambda m}\left(k^{\mu+1}t^{\mu}+k^{2\mu}t\right)}\left\{\max_{|z|=t}|p(z)|-\lambda m\right\}. \tag{3.16}$$

Now, for $0<r\leq R\leq\rho$, $\rho\leq k$ and $0\leq\theta<2\pi$, we have

$$|p(Re^{i\theta})-p(re^{i\theta})|\leq\int_r^R|p^{'}(te^{i\theta})|dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p^{'}(te^{i\theta})|dt$$

from which it follows that

$$\max_{|z|=R} |p(z)| \leq \max_{|z|=r} |p(z)| + \int_r^R \max_{|z|=t} |p^{'}(z)|dt. \tag{3.17}$$

Let $\max_{|z|=r} |p(z)|$ be denoted by $M(p, r)$. Then using (3.16) in the above inequality, we get

$$\begin{aligned} M(p,R) \;\leq\; & M(p,r) + n\int_r^R \frac{t^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^{\mu} + k^{2\mu}t)} \\ & \times \{M(p,t) - \lambda m\}\, dt. \end{aligned} \tag{3.18}$$

If we denote the right hand side of (3.18) by $\varphi(R)$. Then

$$\varphi^{'}(R) = n\Big\{\frac{R^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^{\mu} + k^{2\mu}R)}\Big\}\{M(p,R) - \lambda m\}. \tag{3.19}$$

Also by (3.18), we have

$$M(p,R) \leq \varphi(R). \tag{3.20}$$

Combining (3.19) with (3.20), we conclude that

$$\varphi^{'}(R) - n\Big\{\frac{R^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^{\mu} + k^{2\mu}R)}\Big\}\{\varphi(R) - \lambda m\} \leq 0. \tag{3.21}$$

Multiplying both sides of (3.21) by

$$\exp\Big\{-n\int \frac{R^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^{\mu} + k^{2\mu}R)}dR\Big\},$$

we get

$$\begin{aligned} &\frac{d}{dR}\Big[\{\varphi(R) - \lambda m\} \\ &\times \exp\Big\{-n \times\int \frac{R^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^{\mu} + k^{2\mu}R)}dR\Big\}\Big] \leq 0. \end{aligned} \tag{3.22}$$

It is concluded from (3.22) that the function

$$\psi(R) = \{\varphi(R) - \lambda m\} \times \exp\Big\{ - n \int \frac{R^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}R^{\mu-1}}{R^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}R^{\mu} + k^{2\mu}R)} dR\Big\}$$

is a non-increasing function of R in $(0, \rho]$, $\rho \leq k$. Hence for $0 < r \leq R \leq \rho$, $\rho \leq k$,

$$\psi(r) \geq \psi(R). \tag{3.23}$$

Since $\varphi(R) \geq M(p, R)$ and $\varphi(r) = M(p, r)$, it follows from (3.23) that

$$M(p,r) \geq M(p,R)\exp\Big\{- n\int_r^R \frac{t^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^{\mu} + k^{2\mu}t)} dt\Big\} + \lambda\Big[1 - \exp\Big\{- n\int_r^R \frac{t^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^{\mu}+k^{2\mu}t)} dt\Big\}\Big] m,$$

which is equivalent to

$$M(p,R) \leq \{M(p,r) - \lambda m\} \times \exp\Big\{n \int_r^R \frac{t^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^{\mu} + k^{2\mu}t)} dt\Big\} + \lambda m.$$

This completes the proof of Lemma 3.7. □

Lemma 3.8. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq \rho$, $\rho \leq k$ *and for every real number* λ *with* $0 \leq \lambda < 1$,

$$\exp\Big\{ - n \int_r^R \frac{t^{\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^{\mu} + k^{2\mu}t)} dt\Big\} \geq \left(\frac{k^{\mu} + r^{\mu}}{k^{\mu} + R^{\mu}}\right)^{\frac{n}{\mu}}. \tag{3.24}$$

Proof. Since $p(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(tz) \neq 0$ in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$ where $0 < t \leq k$. Hence, applying inequality (3.3) of Lemma 3.4 to $P(z)$, we get

$$\frac{|a_\mu|t^{\mu}}{|a_0| - \lambda m}\left(\frac{k}{t}\right)^{\mu} \leq \frac{n}{\mu}, \tag{3.25}$$

where

$$m=\min_{|z|=\frac{k}{t}}|P(z)|=\min_{|z|=\frac{k}{t}}|p(tz)|=\min_{|z|=k}|p(z)|.$$

Now (3.25) becomes

$$\frac{|a_\mu|k^\mu}{|a_0|-\lambda m}\leq\frac{n}{\mu},$$

which is equivalent to

$$\frac{t^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu+k^{2\mu}t)}\leq\frac{t^{\mu-1}}{t^\mu+k^\mu}. \tag{3.26}$$

Integrating both sides of (3.26) with respect to t from r to R where $0<r\leq R\leq\rho$, $\rho\leq k$, we have

$$\int_r^R\frac{t^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu+k^{2\mu}t)}\leq\int_r^R\frac{t^{\mu-1}}{t^\mu+k^\mu},$$

which is equivalent to

$$-n\int_r^R\frac{t^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu+k^{2\mu}t)}\geq-n\int_r^R\frac{t^{\mu-1}}{t^\mu+k^\mu}.$$

Hence we have

$$\begin{aligned}&\exp\left\{-n\int_r^R\frac{t^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}t^{\mu-1}}{t^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}t^\mu+k^{2\mu}t)}dt\right\}\\&\geq\exp\left\{-\frac{n}{\mu}\int_r^R\frac{\mu t^{\mu-1}}{t^\mu+k^\mu}dt\right\}\\&=\left(\frac{k^\mu+r^\mu}{k^\mu+R^\mu}\right)^{\frac{n}{\mu}},\end{aligned}$$

which completes the proof of Lemma 3.8. □

Lemma 3.9. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n$, *is a polynomial of degree* n *having no zero in* $|z|<k$, $k>0$, *then for* $0<r\leq R\leq\rho$, $\rho\leq k$ *and for every*

real number λ *with* $0 \leq \lambda < 1$,

$$\left(\frac{R^\mu + k^\mu}{\rho^\mu + r^\mu}\right)^{\frac{n}{\mu}} - \left(\frac{r^\mu + k^\mu}{\rho^\mu + r^\mu}\right)^{\frac{n}{\mu}} \tag{3.27}$$
$$\geq n\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} \exp\left\{n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt.$$

Proof. Since $0 < r \leq t \leq R \leq \rho$, $\rho \leq k$, on applying inequality (3.24) of Lemma 3.8, for $r \leq t$ with $R = t$, we have

$$\left(\frac{k^\mu + t^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} \geq \exp\left\{n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt,$$

which is equivalent to

$$\frac{n(k^\mu + t^\mu)^{\frac{n}{\mu}-1}}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} t^{\mu-1} \tag{3.28}$$
$$\geq \frac{nt^{\mu-1}}{k^\mu + t^\mu} \exp\left\{n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt.$$

Integrating both sides of (3.28) with respect to t from r to R, we have

$$\frac{n}{(k^\mu + r^\mu)^{\frac{n}{\mu}}} \int_r^R (k^\mu + t^\mu)^{\frac{n}{\mu}-1} t^{\mu-1} dt \tag{3.29}$$
$$\geq n\int_r^R \frac{t^{\mu-1}}{k^\mu + t^\mu} \exp\left\{n\int_r^t \frac{x^\mu + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu + k^{2\mu}x)}dx\right\} dt.$$

As $0 < r \leq t \leq R \leq \rho$, $\rho \leq k$, $\rho^\mu + r^\mu \leq r^\mu + k^\mu$ for $\mu = 1, 2, 3..., n$, we have

$$\frac{n}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \int_r^R (t^\mu + k^\mu)^{\frac{n}{\mu}-1} t^{\mu-1} dt \leq \frac{n}{(\rho^\mu + r^\mu)^{\frac{n}{\mu}}} \int_r^R (t^\mu + k^\mu)^{\frac{n}{\mu}-1} t^{\mu-1} dt$$
$$= \frac{1}{(\rho^\mu + r^\mu)^{\frac{n}{\mu}}} \left\{(R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}\right\}. \tag{3.30}$$

Combining (3.29) and (3.30), we obtain the required result. □

4. Proof of the theorem

Proof of Theorem 2.1. For $0 < t \leq k$, $\frac{k}{t} \geq 1$. Here, $p(z)$ has no zero in $|z| < k$, the polynomial $P(z) = p(tz)$ has no zero in $|z| < \frac{k}{t}$, where $\frac{k}{t} \geq 1$. Hence, applying Lemma 3.1 to the polynomial $P(z)$, we get

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{1 + \left(\frac{k}{t}\right)^{\mu}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=\frac{k}{t}} |P(z)| \right\},$$

which implies

$$\max_{|z|=t} |p^{'}(z)| \leq \frac{nt^{\mu-1}}{(k^{\mu} + t^{\mu})} \left\{ \max_{|z|=1} |p(tz)| - \min_{|z|=\frac{k}{t}} |p(tz)| \right\}.$$

Hence we have

$$\max_{|z|=t} |p^{'}(z)| \leq \frac{nt^{\mu-1}}{(k^{\mu} + t^{\mu})} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\}. \tag{4.1}$$

Now for $0 < r \leq t \leq R \leq \rho$, $\rho \leq k$ and $0 \leq \theta \leq 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| \leq \int_r^R |p^{'}(te^{i\theta})| dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p^{'}(te^{i\theta})| dt.$$

Using (4.1) leads to

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \left\{ \int_r^R \frac{nt^{\mu-1}}{(k^{\mu} + t^{\mu})} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=k} |p(z)| \right\} dt \right\},$$

which implies on considering maximum over θ that

$$M(p,R) \leq M(p,r) + \int_r^R \frac{nt^{\mu-1}}{(k^{\mu} + t^{\mu})} M(p,t) dt - \min_{|z|=k} |p(z)| \int_r^R \frac{nt^{\mu-1}}{(k^{\mu} + t^{\mu})} dt. \tag{4.2}$$

Since $r \leq t$, by applying Lemma 3.7 with $R = t$, we have

$$M(p,t) \leq \{M(p,r) - \lambda m\} \times \exp\left\{ n \int_r^t \frac{x^{\mu} + \frac{\mu}{n} \frac{|a_{\mu}|}{|a_0| - \lambda m} k^{\mu+1} x^{\mu-1}}{x^{\mu+1} + k^{\mu+1} + \frac{\mu}{n} \frac{|a_{\mu}|}{|a_0| - \lambda m} (k^{\mu+1} x^{\mu} + k^{2\mu} x)} dx \right\} + \lambda m. \tag{4.3}$$

Using (4.3) to (4.2), we obtain

$$\begin{aligned}
&M(p,R)\\
&\leq M(p,r)+n\left\{M(p,r)-\lambda m\right\}\int_r^R\frac{t^{\mu-1}}{(k^\mu+t^\mu)}\\
&\quad\times\left[\exp\left\{n\int_r^R\frac{x^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu+k^{2\mu}x)}dx\right\}+\lambda m\right]dt\\
&\quad-m\frac{n}{\mu}\ln\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right).
\end{aligned}\tag{4.4}$$

Inequality (4.4) is equivalent to

$$\begin{aligned}
M(p,r)\geq&\left[M(p,R)+n\lambda m\int_r^R\frac{t^{\mu-1}}{k^\mu+t^\mu}\right.\\
&\times\exp\left\{n\int_r^R\frac{x^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu+k^{2\mu}x)}dx\right\}dt\\
&\left.-(\lambda-1)\frac{n}{\mu}m\ln\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)\right]\\
&\times\frac{1}{1+n\left[\int_r^R\frac{t^{\mu-1}}{k^\mu+t^\mu}exp\left\{n\int_r^R\frac{x^\mu+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}k^{\mu+1}x^{\mu-1}}{x^{\mu+1}+k^{\mu+1}+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-\lambda m}(k^{\mu+1}x^\mu+k^{2\mu}x)}dx\right\}dt\right]},
\end{aligned}\tag{4.5}$$

and hence the proof of Theorem 2.1 is completed. □

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