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EXTENSIONS OF ORDERED FIXED POINT THEOREMS

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Abstract. Our long-standing Metatheorem in Ordered Fixed Point Theory is applied to some well-known order theoretic fixed point theorems. In the first half of this article, we introduce extended versions of the Zermelo fixed point theorem, Zorn's lemma, and the Caristi fixed point theorem based on the Brøndsted-Jachymski principle and our 2023 Metatheorem. We show some of their applications to other fixed point theorems or theorems on the existence of maximal elements in partially ordered sets. In the second half, we collect and improve order theoretic fixed point theorems in the collection of Howard-Rubin in 1991 and others. In fact, we improve or extend several ordering principles or fixed point theorems due to Brézis-Browder, Brøndsted, Knaster-Tarski, Tarski-Kantorovitch, Turinici, Granas-Horvath, Jachymski, and others.

1. Introduction

Since the appearances of the Ekeland variational principle [16,17] in 1972-74 and the Caristi fixed point theorem [7] in 1976, almost one thousand works followed on their equivalents, modifications, generalizations, applications, and related topics. Many of them are related to new spaces extending complete metric spaces, new metrics or topologies on them, and new order relations extending the so-called Caristi order.

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While the author was working on the same subject in 1985-2000, we found a Metatheorem on ordered fixed point theory. It claims that certain order theoretic maximal element statements are equivalently formulated to theorems on fixed points, stationary points, common fixed points, common stationary points of families of maps or multimaps. In 2022 [40-46], we applied certain extended versions of Metatheorem to various statements in ordered fixed point theory. Later we found a new version of the ones in 1985-2000 [38,39] and 2022 [40-45], and it is called the 2023 Metatheorem in [46,47].

Recall that, for progressive selfmaps on ordered sets, Brøndsted [4] in 1976 observed when maximal elements are fixed points, and Jachymski [27] in 2003 studied situations when periodic points are same to fixed points. Combining their methods in 2022, we introduced the Brøndsted-Jachymski principle on partially ordered sets stating that maximal elements are fixed and periodic points of a progressive selfmap; see [46].

Recently, in 2022, we obtained extended versions of our Metatheorem in [40-42, 44] and the 2023 Metatheorem in [46,47]. Motivated by this, in the first half of the present article, we obtain some improved versions of the Zermelo fixed point theorem, Zorn's lemma, and the Caristi fixed point theorem with their equivalent formulations and applications to known results.

As a continuation of [46,47], in the second half of the present article, we obtain improved versions of another known order theoretic fixed point theorems. Moreover, we comment their relations to our previous works or improve them based on our new results. In fact, we collect and improve order theoretic fixed point theorems in the collection of Howard-Rubin [21] in 1991 and others. Moreover, we improve or extend several ordering principles or fixed point theorems due to Brézis-Browder, Brøndsted, Knaster-Tarski, Tarski-Kantorovitch, Turinici, Granas-Horvath, Jachymski, and others; see the references in the end of this article.

This article is organized as follows: In Section 2 for preliminaries, we introduce some known order theoretic results from the works of Li [33], Jachymski [23], and Granas-Dugundji [12, 18]. In Section 3, we improve the Caristi fixed point theorem in several forms, and introduce the Brøndsted principle and the Brøndsted-Jachymski principle. Section 4 devotes to obtain improved versions of Zermelo's fixed point theorem and others with the aid of the Axiom of Choice. In Section 5, we obtain equivalent formulations of the Caristi theorem based on the 2023 Metatheorem. Section 6 is to introduce the main theorem of Jachymski [27, Theorem 2] and its new applications to improve theorems of Zermelo and Caristi.

In Section 7, based on our results in the previous sections, we try to improve various order theoretic results appeared in the collection of Howard-Rubin [21] and others. Consequently a large number of them becomes more practical one. In addition to them, Section 8 devotes to improve several ordering principles or fixed point theorems due to Brézis-Browder, Brøndsted, Knaster-Tarski, Tarski-Kantorovitch, Turinici, Granas-Horvath, and Jachymski. Finally, Section 9 is for conclusion.

This article is mainly based on our previous works with many new results. Our arguments are based on ZF, but on ZFC sometimes. All spaces are non-empty and all multimaps are non-empty valued.

2. Preliminaries

In this section, we follow mainly Dugundji and Granas [12,18] and Li [33]:

A binary relation $\preccurlyeq$ in a nonempty set X is called a *preorder* if (a) $x \preccurlyeq x$ for every $x \in X$, (b) $x \preccurlyeq y$ and $y \preccurlyeq z$ implies $x \preccurlyeq z$; it is called a *partial order* if also (c) $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$. A set X together with a preorder (resp. partial order) $\preccurlyeq$ is called a *preordered* (resp. *partially ordered*) set. An element $x_0 \in X$ is said to be *maximal* in X if there is no $x \in X$ with $x \neq x_0 \preccurlyeq x$; a minimal element is defined similarly.

Let $(P, \preccurlyeq)$ be a preordered set and $M \subset P$ a non-empty subset. Recall that an upper (lower) bound for M is an element $p \in P$ with $m \preccurlyeq p$ ($p \preccurlyeq m$) for each $m \in M$; the supremum of M, if it exists, is an upper bound for M which is a lower bound for the set of all upper bounds of M, that is, the least upper bound of M. A map $f : P \to P$ is *isotone* if $f(x) \preccurlyeq f(y)$ whenever $x \preccurlyeq y$.

The partially ordered set is *totally ordered* (or a *chain*) if for each $x, y \in X$, either $x \preccurlyeq y$ or $y \preccurlyeq x$; it is *well-ordered* if each non-empty subset has a minimal element.

For the reader's convenience, in this section, based on Jinlu Li [33], we recall some properties of partially ordered sets (posets), lattices and some fixed point theorems. The fixed point theorems recalled in this section will be used in the later sections. Here we closely follow the notations in [22,23].

We say that a poset $(P, \preccurlyeq)$ is chain-complete iff every chain in P has an upper bound in P. The poset $(P, \preccurlyeq)$ is called a complete lattice iff both $\vee S$ and $\wedge S$ exist in P, for any nonempty subset S of P.

Let $(P, \preccurlyeq)$ be a chain-complete lattice. Then $\vee S$ exists in P for every non-empty subset S of $(P, \preccurlyeq)$, see [23].

Given posets $(X, \preccurlyeq^X)$ and $(U, \preccurlyeq^U)$, we say that a multimap $F : X \multimap U$ is order-increasing upward if $x \preccurlyeq^X y$ in X and $z \in F(x)$, then there is $w \in F(y)$ such that $z \preccurlyeq^U w$. F is order-increasing downward if $x \preccurlyeq^X y$ in X and

$w \in F(y)$, then there is $z \in F(x)$ such that $z \preccurlyeq^U w$. F is said to be order-increasing whenever F is both (order) increasing upward and downward.

As a special case, a single-valued map f from a poset $(X, \preccurlyeq^X)$ to another poset $(U, \preccurlyeq^U)$ is said to be order-increasing (or order-preserving) whenever $x \preccurlyeq^X y$ implies $f(x) \preccurlyeq^U f(y)$. An order-increasing map $f : X \to U$ is said to be strictly order-increasing whenever $x \prec^X y$ implies $f(x) \prec^U f(y)$. A map f from $(X, \preccurlyeq^X)$ to $(U, \preccurlyeq^U)$ is said to be order-decreasing (or order-reversing) whenever $x \preccurlyeq^X y$ implies $f(y) \preccurlyeq^U f(x)$.

We recall some fixed point theorem on posets below. The first one is about multimaps and the last two are about single-valued maps. We need the following notation: Let $(P, \preccurlyeq)$ be a lattice and $F : P \multimap P$ a multimap. For every $x \in P$, we denote

$$SF(x) := \{z \in P : z \preccurlyeq u \text{ for some } u \in F(x)\}.$$

Extension of Tarski's Fixed Point Theorem in [21]. Let $(P, \preccurlyeq)$ be a complete lattice and $F : P \multimap P$ a multimap. If F satisfies the following two conditions:

A1. F is order-increasing upward (isotone).

A2. $(SF(x), \succcurlyeq)$ is an inductively ordered set for each $x \in P$.

Then F has a fixed point, that is, there exists $x^* \in P$ such that $x^* \in F(x^*)$.

The Knaster-Tarski-Davis Fixed Point Theorem. Let $(P, \preccurlyeq)$ be a lattice. Then every order-increasing self-map on P has a fixed point if, and only if, $(P, \preccurlyeq)$ is a complete lattice.

The Abian-Brown Fixed Point Theorem. Let $(P, \preccurlyeq)$ be a chain-complete poset and let $f : P \to P$ be an order-increasing map. If there is an $x^* \in P$ with $x^* \preccurlyeq f(x^*)$, then f has a fixed point ([2]).

Jachymski [25] introduced the following after giving enough preparation:

Tarski-Kantorovitch Theorem. Let $(P, \preccurlyeq)$ be a $\preccurlyeq$-complete partially ordered set and a map $f : P \to P$ be $\preccurlyeq$-continuous. If there exists $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$, then f has a fixed point; moreover, $p_* = \sup\{f^n(p_0) : n \in \mathbb{N}\}$ is fixed under f.

Axiom of Choice introduced by Zermelo in 1904 has the following equivalent forms; see Dugundji and Granas [12,18]:

Zorn's Lemma. A partially ordered set in which every totally ordered subset has an upper bound contains at least one maximal element.

Zermelo's Well-Ordering Principle. Every set can be well-ordered.

Note that Zorn's lemma has several different forms.

Recall that a real-valued function $f : X \to \mathbb{R}$ on a topological space X is said to be *lower (resp. upper) semi-continuous* (l.s.c.) (resp. u.s.c.) whenever

$$\{x \in X : f(x) > r\} \quad (\text{resp. } \{x \in X : f(x) < r\}$$

is open for each $r \in \mathbb{R}$.

Later, l.s.c. is extended to l.s.c. *from above* by Chen-Cho-Yang [8] in 2002 and others:

Definition 2.1. Let X be a metric space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* at a point $x \in X$ if $x_n \to x$ as $n \to \infty$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that $\lim_{n\to\infty} f(x_n) \geq f(x)$.

This concept was applied in [8] to improve the Ekeland variational priciple, the Caristi fixed point theorem, and some others.

3. Caristi, Brøndsted, and Jachymski

In this section, we improve the Caristi fixed point theorem [7] in several forms, and introduce the Brøndsted principle and the Brøndsted-Jachymski principle, which will be very important in our study.

Recall the new Caristi fixed point theorem appeared in [8]:

Theorem 3.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a map such that for all $x \in X$,*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), \tag{3.1}$$

where a function $\varphi : X \to \mathbb{R}^+ = [0, \infty)$ is lower semicontinuous from above. Then f has a fixed point.

A map f satisfying (3.1) is called a Caristi map. Now, Theorem 3.1 simply tells that: Any Caristi map on a complete metric space has a fixed point.

For a preordered set $(X, \preccurlyeq)$, a selfmap $f : X \to X$ is said to be *progressive* if $x \preccurlyeq f(x)$ for all $x \in X$.

In Theorem 3.1, (X, d) is a partially ordered set by defining

$$x \preccurlyeq y \quad \text{iff} \quad d(x, y) \leq \varphi(x) - \varphi(y) \quad \text{for} \quad x, y \in X.$$

A Caristi map f is progressive by defining $x \preccurlyeq f(x)$ if and only if (3.1) holds for any $x \in X$.

Recently, motivated by Brøndsted [4], we adopted the following in [43]:

Brøndsted Principle. Let $(X, \preccurlyeq)$ be a preordered set and $f : X \to X$ be a progressive map. Then a maximal element $v \in X$ is a fixed point of f.

From now on $\mathrm{Max}(\preccurlyeq)$ denotes the set of maximal elements of the order $\preccurlyeq$, and $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$ denote the sets of all fixed points and periodic points of a map $f : X \to X$, respectively; that is, $\mathrm{Fix}(f) = \{x \in X : x = f(x)\}$ and $\mathrm{Per}(f) = \bigcup\{\mathrm{Fix}(f^n) : n \geq 1\}$.

Proposition 3.2. (Jachymski [27]) *Let* $(X, \preccurlyeq)$ *be a partially ordered set and* $f : X \to X$ *be progressive. Then* $\mathrm{Per}(f) = \mathrm{Fix}(f)$.

Proof. Let $x_0 = f^k(x_0)$ for some $k \in \mathbb{N}$. Since f is progressive, we may infer that $f^{j-1}(x_0) \preccurlyeq f^j(x_0)$ for $j = 1, \ldots, k$ (under the convention that $f^0 = \mathrm{id}_X$). Hence, by the transitivity,

$$x_0 \preccurlyeq f(x_0) \quad \text{and} \quad f(x_0) \preccurlyeq f^k(x_0) = x_0.$$

which implies that $x_0 = f(x_0)$. □

Combining this with the Brøndsted principle, we obtain the following:

Brøndsted-Jachymski Principle. Let $(X, \preccurlyeq)$ be a partially ordered set and $f : X \to X$ be a progressive map. Then a maximal element $v \in X$ is a fixed and periodic point of f, that is,

$$\mathrm{Max}(\preccurlyeq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

This is not claiming the non-emptiness of the three sets. However, Zorn's lemma and the Brøndsted-Jachymski principle imply the following:

Theorem 3.3. *Let* $(X, \preccurlyeq)$ *be a partially ordered set in which every totally ordered subset has an upper bound. If* $f : X \to X$ *is progressive, then*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

This can be improved later.

4. Improvements of Zermelo fixed point theorem

In this section, we obtain improved versions of Zermelo's fixed point theorem and others with the aid of the Axiom of Choice.

A fundamental fixed point theorem of Zermelo (see, e.g., [13], p.5) says that if $(X, \preccurlyeq)$ is a partially ordered set in which every chain has a supremum (least upper bound) and a selfmap $f : X \to X$ is progressive, then f has a fixed point. This was given implicitly in 1908 [53] and formulated by Bourbaki [3] in 1949-50.

As an application of the Brøndsted-Jachymski principle, we improve Zermelo's fixed point theorem as follows:

Theorem 4.1. *Let $(X, \preccurlyeq)$ be a partially ordered set in which each nonempty well-ordered subset has a supremum. Then every progressive map $f : X \to X$ has*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

This is an equivalent form of Theorem 3.3.

Jachymski [27] noted: "Under the Axiom of Choice, the assumption of Theorem 4.1 can be weakened to "each nonempty well-ordered subset has an upper bound." This improves Kneser's fixed point theorem [32], which turns out to be equivalent to the Axiom of Choice as shown by Abian [1]."

However we have the following particular form of our 2023 Metatheorem in [46,47]:

Theorem 4.2. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$ has an upper bound $v \in A$. Then the following equivalent statements hold:*

- (α) *$v \in A$ is a maximal element of $(X, \preccurlyeq)$, that is, $v \not\preccurlyeq w$ for any $w \in X \backslash \{v\}$.*
- (β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $x \preccurlyeq y$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*
- (γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preccurlyeq f(x)$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*
- (δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash F(x)$, there exists $y \in X\backslash\{x\}$ satisfying $x \preccurlyeq y$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*
- (ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ satisfying $x \preccurlyeq y$ for any $x \in A$ and any $y \in F(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $T \in \mathfrak{F}$.*
- (η) *If Y is a subset of X such that, for each $x \in A\backslash Y$,there exists a $z \in X\backslash\{x\}$ satisfying $x \preccurlyeq z$, then there exists an element $v \in A \cap Y$.*

When $\mathfrak{F}$ is a singleton, each of $(\beta)-(\epsilon)$ is denoted by $(\beta 1)-(\epsilon 1)$, respectively.

Proof. Note that each of $(\beta), (\delta), (\epsilon)$ implies (γ), and that $(\beta) - (\epsilon)$ imply $(\beta 1) - (\epsilon 1)$, respectively. We prove (α) and $(\alpha) \Longrightarrow (\gamma 1)$ as follows:

(α) Since A has an upper bound $v \in A$, for each $x \in A$, we have $x_0 \preccurlyeq x \preccurlyeq v$. If $v \preccurlyeq w$ for some $w \in X$, then $w \in S(x_0) = A$ and $w \preccurlyeq v$. Since $(X, \preccurlyeq)$ is partially ordered, we have $w = v$. Hence v is maximal.

The equivalency of $(\alpha)-(\eta)$ can be seen as follows:

$(\alpha) \Longrightarrow (\delta 1)$: Suppose $v \notin F(v)$ for the $v \in A$ in (α). Then there exists a $y \in X\backslash\{v\}$ satisfying $v \not\preccurlyeq y$. This is a contradiction.

$(\delta 1) \Longrightarrow (\beta 1)$: Clear.

$(\beta 1) \Longrightarrow (\gamma 1)$: Clear.

$(\gamma 1) \Longrightarrow (\epsilon 1)$: Suppose F has no stationary element, that is, $F(x)\backslash\{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{F(x)\backslash\{x\} : x \in A\}$. Then f has no fixed element by its definition. However, $x \preccurlyeq f(x)$ for any $x \in A$, Therefore, by $(\gamma 1)$, f has a fixed element, a contradiction.

$(\epsilon 1) \Longrightarrow (\gamma)$: Define a multimap $F : A \multimap X$ by $F(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $x \preccurlyeq f(x)$ for any $x \in A$ and any $f \in \mathfrak{F}$, by $(\epsilon 1)$, F has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

$(\gamma) \Longrightarrow (\alpha)$: Suppose that for any $x \in A$, there exists a $y \in X\backslash\{x\}$ satisfying $x \preccurlyeq y$. Choose $f(x)$ to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $x \preccurlyeq f(x)$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By (γ), f has a fixed element, a contradiction.

$(\alpha) + (\epsilon 1) \Longrightarrow (\epsilon)$: By (α), there exists a $v \in A$ such that $v \not\preccurlyeq w$ for all $w \in A\backslash\{v\}$. For each $F \in \mathfrak{F}$, by $(\epsilon 1)$, we have a $v_F \in A$ such that $\{v_F\} = F(v_F)$. Suppose $v \neq v_F$. Then $v \not\preccurlyeq v_F$ holds by (α) and $v \preccurlyeq v_F$ holds by assumption on (ϵ). This is a contradiction. Therefore $v = v_F$ for all $F \in \mathfrak{F}$.

$(\epsilon) \Longrightarrow (\epsilon 1)$: Clear.

$(\alpha) \Longrightarrow (\eta)$: By (α), there exists a $v \in A$ such that $v \not\preccurlyeq w$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

$(\eta) \Longrightarrow (\alpha)$: For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ x \preccurlyeq y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (η) is satisfied. Therefore, by (η), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $v \not\preccurlyeq w$ for all $w \neq v$. Hence (α) holds. This completes our proof. □

Note that all v's in Theorem 4.2 are the same maximal element in (α).

Remark 4.3. It should be emphasized that (α) improves Zorn's lemma, and $(\beta 1)$ or $(\gamma 1)$ improves Theorem 4.1 and the Zermelo fixed point theorem. In fact, Zorn's Lemma requires every totally ordered subset has an upper bound.

However, (α) in Theorem 4.2 requires only one A has an upper bound. Moreover, Zermelo's theorem requires that every chain has a supremum. However, ($\gamma 1$) requires only one chain A has a supremum.

Example 4.4. We give an example of Theorem 4.2.($\gamma 1$) and the Brøndsted-Jachymski principle. Let $X = A = [0,3] \subset (\mathbb{R}, \leq)$, x_0 any point in [0,3], and $f : [0,3] \to [0,3]$ be as follows:

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{if } x \in [0,1], \\ x & \text{if } x \in [1,2], \\ 3 & \text{if } x \in (2,3]. \end{cases} \tag{4.1}$$

Then f is progressive, $S(x_0) \subset S(0) = [0,3]$, and

$$\mathrm{Fix}(f) = \mathrm{Per}(f) = [1,2] \cup \{3\} \supset \{3\} = \mathrm{Max}(\preccurlyeq).$$

Note that $v = 3$ is a maximal fixed point of f.

Example 4.5. Let $C = [0,1] \times \{1\}$ and $D = \mathbb{R} \times \{0\}$ in $\mathbb{R}^2$ with their natural orders. Let $X = C \cup D$ be the partially ordered set and $f : X \to X$ a progressive map defined by

$$f(x,y) = \begin{cases} \left(\frac{1}{2}(x+1), 1\right) & \text{if } (x,y) = (x,1) \in C, \\ (x+1, 0) & \text{if } (x,y) = (x,0) \in D. \end{cases} \tag{4.2}$$

Then the chain or totally ordered subset $A = S(0,1)$ has an upper bound or a supremum (1,1) $\in A$, which is a maximal element and a fixed point of a progressive map f. Note that any progressive map on C has a fixed point and that the chain D does not have any upper bound.

Example 4.6. Let $X = [0,1] \subset \mathbb{R}$ and $F : [0,1] \multimap [0,1]$ a multimap defined by

$$F(x) = [(1/2)(x+1), (1/4)(x+3)] \subset X \quad \text{if } x \in X. \tag{4.3}$$

Then $F(0) = [\frac{1}{2}, \frac{3}{4}]$ and $F(1) = \{1\}$.

In Theorem 4.2, x_0 can be any point in $X = [0,1]$ and F satisfies ($\delta 1$) and ($\epsilon 1$).

Remark 4.7. Note that ($\beta 1$) $\Longrightarrow$ ($\gamma 1$) and (γ) $\Longrightarrow$ (α) adopt the Axiom of Choice. Consequently, we showed that improved versions of Zorn's lemma, Zermelo's theorem, and other theorems are equivalent by the aid of the Axiom of Choice.

5. Equivalents of the Caristi theorem

The following extends a previous one in [42]:

Theorem 5.1. *Let* (X,d) *be a complete metric space, and* $\varphi : X \to \mathbb{R}^+$ *lower semicontinuous from above. Define a partial order* $\preccurlyeq$ *on* X *by*

$$x \preccurlyeq y \ \text{ if and only if } \ d(x,y) \leq \varphi(x) - \varphi(y).$$

Then the following statements hold:

- (α) *There exists a maximal element* $v \in X$*; that is,* $d(v,w) > \varphi(x) - \varphi(y)$ *for any* $w \in X\backslash\{v\}$.
- (β) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *such that, for any* $x \in X\backslash\{f(x)\}$, *there exists a* $y \in X\backslash\{x\}$ *satisfying* $d(x,y) \leq \varphi(x) - \varphi(y)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in X$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.
- (γ) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $d(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ *for all* $x \in X$ *with* $x \neq f(x)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in X$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.
- (δ) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \multimap X$ *such that, for any* $x \in X\backslash T(x)$, *there exists a* $y \in X\backslash\{x\}$ *satisfying* $d(x,y) \leq \varphi(x) - \varphi(y)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v \in T(v)$ *for all* $T \in \mathfrak{F}$.
- (ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \multimap X$ *such that* $d(x,y) \leq \varphi(x) - \varphi(y)$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in X$, *that is,* $\{v\} = T(v)$ *for all* $T \in \mathfrak{F}$.
- (η) *If* Y *is a subset of* X *such that, for each* $x \in X\backslash Y$, *there exists a* $z \in X\backslash\{x\}$ *satisfying* $d(x,y) \leq \varphi(x) - \varphi(y)$, *then there exists an element* $v \in Y$.

Recall that (α) is originated from Brunner [6] in 1987. Note that (γ1) is originated from the celebrated Caristi fixed point theorem which implies (α), and others except (α) and (η) are its generalizations. Therefore (α) $-$ (η) hold by the 2023 Metatheorem [46,47].

Now we strengthen the conclusion of the Caristi theorem as follows by the Brøndsted-Jachymski principle:

Theorem 5.2. *Let* (X,d) *be a complete metric space, and* $\varphi : X \to \mathbb{R}^+$ *lower semicontinuous from above. Define a partial order* $\preccurlyeq$ *on* X *by*

$$x \preccurlyeq y \ \text{ if and only if } \ d(x,y) \leq \varphi(x) - \varphi(y).$$

Then for any progressive map (or Caristi map) $f : X \to X$, *we have the following:*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

Actually, any progressive map f on X has a maximal fixed point. Recall that Turinici [52] in 2009 noticed that $\mathrm{Fix}(f) = \mathrm{Per}(f) \neq \emptyset$ for a particular case of Theorem 5.2.

From Theorem 5.1(β1), Caristi's theorem has the following extension:

Theorem 5.3. *Let (X, d) be a complete metric space, and $\varphi : X \to \mathbb{R}^+$ lower semicontinuous from above. If $f : X \to X$ is a map such that, for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y),$$

then f has a maximal fixed element $v \in X$, that is, $v = f(v)$, and we have

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \supset \{v\}.$$

Here we can define a partial order (or a Caristi order) on X by using φ. Note that Caristi theorem takes $y = f(x)$ for all $x \in X$.

From Theorem 5.1(δ1), we have a multimap version of the Caristi theorem:

Theorem 5.4. *Let (X, d) be a complete metric space, and $\varphi : X \to \mathbb{R}^+$ lower semicontinuous from above. If $T : X \multimap X$ is a multimap such that $d(x, y) \leq \varphi(x) - \varphi(y)$ holds for any $x \in X$ and any $y \in T(x) \backslash \{x\}$, then T has a maximal stationary element $v \in X$, that is, $\{v\} = T(v)$ and*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \supset \{v\}.$$

Note that Theorem 5.1(γ), (ϵ) are extensions of the Caristi theorem for families of maps or multimaps.

6. Jachymski's 2003 theorem

In this article, we introduce many examples of maps $f : X \to X$ satisfying $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$. Such sets X can have more rich properties by applying the main theorem of Jachymski ([27], Theorem 2) as follows:

Theorem 6.1. *Let X be a nonempty abstract set and $T : X \to X$. The following statements are equivalent:*

(a) $\mathrm{Per}(T) = \mathrm{Fix}(T) \neq \emptyset$.

(b) (Zermelo) *There exists a partial ordering $\preccurlyeq$ such that every chain in $(X, \preccurlyeq)$ has a supremum and T is progressive with respect to $\preccurlyeq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that T satisfies the condition* (3.1)*.*

(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \to \mathbb{R}^+$ such that T satisfies condition* (3.1) *and T is nonexpansive with respect to d; that is,*

$$d(Tx, Ty) \le d(x, y) \quad for\ all \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0, 1)$, there exists a complete metric d such that T is nonexpansive with respect to d and*

$$d(Tx, T^2x) \le \alpha d(x, Tx) \quad for\ all \quad x \in X.$$

(f) *There exists a complete metric d such that T is continuous with respect to d and for each $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

(g) *There exists a partition of X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$, such that all the sets X_γ are nonempty, T-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $T|_{X_\gamma}$ has a unique periodic point.*

(h) *For each $\alpha \in (0, 1)$, there exists a partition of X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; T-invariant and pairwise disjoint; and*

$$d_\gamma(Tx, Ty) \le \alpha d_\gamma(x, y) \quad for\ all\ x, y \in X.$$

Remark 6.2. (Jachymski [27]) Implication (a) $\Longrightarrow$ (b) is a converse to Zermelo's theorem. Implication (a) $\Longrightarrow$ (c) is a reciprocal to (the new form of) Caristi's theorem; in fact, a stronger result, (a) $\Longrightarrow$ (d) can be obtained here. Implication (a) $\Longrightarrow$ (e) is a converse to a fixed point theorem of Hicks-Rhoades. Finally (a) $\Longrightarrow$ (f) answers a question posed by Matkowski.

Remark 6.3. Each of (a)–(h) seems to be order theoretic fixed point theorems. For them, we state our own comments.

(a) This could be $\mathrm{Fix}(T) = \mathrm{Per}(T) \supset \mathrm{Max}(\preccurlyeq) \ne \emptyset$ by defining $\preccurlyeq$ on X.

(b) Zermelo's theorem is improved by Theorem 4.1, Theorem 4.2(γ) and its equivalents there. Note that its conclusion should be as above (a).

(c) Caristi's theorem is improved by Theorem 5.1 and its conclusion should be as above (a).

(d) This is a variant of Caristi's theorem and its conclusion should be as (a).

(e) Here nonexpansiveness is redundant; see Theorem H in [46].

From Theorems 4.1, 4.2, and 6.1, we have the following improved form of a part of Theorem 6.1:

Theorem 6.4. *Let $(X, \preccurlyeq)$ be a nonempty partially ordered set and $f : X \to X$. The following statements are equivalent:*

(a) $\mathrm{Per}(f) = \mathrm{Fix}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$.

(b) (Zermelo) *There exists an $x_0 \in X$ such that $S(x_0) = \{x \in X : x_0 \preccurlyeq x\}$ has an upper bound and f is progressive with respect to $\preccurlyeq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ from above such that f satisfies the Caristi condition or* (3.1).

Note that this theorem implies Theorem 3.1 of the Caristi type, Theorems 3.3 and 4.1 of the Zermelo type, Theorems 5.2 and 5.3 of the Caristi type and some others in Theorems 4.2 and 5.1.

7. Applications to ordered fixed point theorems

Recall that Howard-Rubin [21] collected a large number of consequences of the Axiom of Choice. In this section, we introduce examples of ordered fixed point theorems in [21] and other sources, which can be applied or improved by our results in Section 4. Some of them may not be consequences the Axiom and are consequences of our Theorem 6.2.

(I) (Zermelo) *For every poset $(X, \preccurlyeq)$ if every well-ordered subset has a least upper bound then every progressive map $f : X \to X$ has a fixed point.* Zermelo [53](1908), Abian [1](1980), Manka [34](1988).

For a poset, as we have seen in Theorems 5.1 and 5.2, existence of a chain with an upper bound is enough for the existence of a fixed point of progressive maps. Moreover, we have $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$ by the Brøndsted-Jachymski principle and Theorem 6.2.

Such better formulations of the Zermelo theorem appear our previous works [46] and others.

The Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem [3] or the Bourbaki-Kneser fixed point theorem [32]. It implies the (old form of) Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma. See Toyoda [50].

(II) (Caristi) *If (X, ρ) is a complete metric space and $\phi : X \to \mathbb{R}^+$ lower semi-continuous (from above), then in the Brøndsted order ($x \preccurlyeq y$ iff $\rho(x, y) \leq \phi(x) - \phi(y)$) every progressive map $f : X \to X$ has a fixed point.* Caristi [7](1976), Jachymski [21](1998), Chen-Cho-Yang [8](2002).

Then the Caristi theorem has equivalent formulation $(\alpha)-(\eta)$ as in Theorem 4.2 and to $\mathrm{Fix}(\phi) = \mathrm{Per}(\phi) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$ as in Theorem 6.2. See also Park [44,46].

(III) (Manka) *For a poset $(X, \preccurlyeq)$, if there is a sup function σ on the well-ordered subsets of X then every progressive map $f : X \to X$ has a fixed point.* Manka [34](1988).

Manka's theorem also can be improved as for the Caristi theorem.

(IV) (Turinici) *If $(X, \preccurlyeq)$ is a directed poset and τ is a Hausdorff topology on X such that*

(a) *$(X, \preccurlyeq)$ is upper semi-continuous with respect to τ,*

(b) *every well-ordered subset of $(X, \preccurlyeq)$ has a unique limit as a net,*

then every progressive map $f : X \to X$ has a fixed point. Manka [34](1988).

This can have equivalent formulations, and be improved by applying our method. Here (a) means that $\{y \in X : x_0 \preccurlyeq y\}$ is closed for every $x_0 \in X$. See Note 38 of Howard-Rubin [21](1998).

(V) *For every poset $(X, \preccurlyeq)$, if every well ordered subset is bounded above then every progressive map $f : X \to X$ has a fixed point.* Manka [34](1988) and Abian [1](1980).

Then $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$. Moreover, (V) is a simple consequence of Theorems 4.1 and 6.2, and has equivalent formulations by following Theorem 4.2.

(VI) *Let $\mathcal{F}$ be a family of selfmaps of a poset $(X, \preccurlyeq)$ such that $\forall f \in \mathcal{F}, \forall x \in X, f(x) \preccurlyeq x$. If for some element $e \in X$ each chain in X containing e has a lower bound, then the family $\mathcal{F}$ has a common fixed point.* Kasahara [29](1976).

This is the dual of Theorem 4.2(γ) and has equivalent dual forms to $(\alpha)-(\eta)$. Moreover, we have $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset$ for all $f \in \mathcal{F}$. See [46,47].

(VII) *Assume P is a poset in which every non-empty chain has an upper bound and $f : P \to P$ satisfies $\forall x \in f(P) \cup \{a \in P : a$ is an upper bound for some chain in $f(P)\}$, $x \leq f(f(x))$. Then f has a fixed apex u (that is, there is some $v \in P$ such that $f(u) = v$ and $f(v) = u$).* Tasković [48,49](1988, 1992).

Note that f^2 is progressive and hence has a fixed point $v = f^2(v)$ by (I). Let $u = f(v)$. Then $f(u) = f^2(v) = v$. Moreover, we have $\mathrm{Max}(\preccurlyeq) \subset \mathrm{Fix}(f^2) = \mathrm{Per}(f^2) \neq \emptyset$.

(VIII) *If a linearly ordered set $(A, \preccurlyeq)$ has the fixed point property then it is complete.* [*$(A, \preccurlyeq)$ has the fixed point property if every map $f : A \to A$*

satisfying ($x \preccurlyeq y \Longrightarrow f(x) \preccurlyeq f(y)$) has a fixed point, and it is complete if every subset of A has a least upper bound.] Höft/Howard [20](1994).

(IX) *If (X,ρ) is a complete metric space and $\phi : X \to \mathbb{R}$ is bounded above and upper semi-continuous then in the Brøndsted order ($x \preccurlyeq y$ iff $\rho(x,y) \leq \phi(y) - \phi(x)$), there is a maximal element.* Brunner [6](1987).

This is equivalent to Caristi's fixed point theorem, see [47]. The upper semi-continuity can be weakened to upper semi-continuity from below.

8. Applications to other ordering principles

Theorems in Section 4 have shown to be applied in our previous works on several ordering principles. We introduce some of them.

(I) (Brézis-Browder) *Let $(X,\preceq)$ be a preordered set, $x_0 \in X$, and $A = S(x_0) = \{x \in X : x_0 \preceq x\}$. Let $\phi : X \to \mathbb{R}$ be a function satisfying*

(1) *ϕ is bounded above;*

(2) *$x \preceq y$ and $x \neq y$ imply $\phi(x) < \phi(y)$;*

(3) *For any $\preceq$-increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ (i.e. $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$), there exists some $y \in X$ such that $x_n \preceq y$ for all $n \in \mathbb{N}$.*

Then there exists a maximal element $v \in A$ such that

$$v \preceq w \ \text{ implies } \ \phi(v) = \phi(w)$$

for any $w \in X\backslash\{v\}$, that is, $S(v) = \{v\}$. [5]

In our previous work [41], this is equivalently reformulated as in Theorem 4.2.

(II) (Brøndsted) *Let $(E,\preccurlyeq)$ be a partially ordered set which admits at least one maximal element. Let $f : E \to E$ be a map such that $x \preccurlyeq f(x)$ for all $x \in E$. Then f admits at least one fixed point.*

This was equivalently formulated in [43] and is the origin of our Brøndsted-Jachymski Principle.

(III) (Knaster-Tarski) *Let $(P,\preccurlyeq)$ be a partially ordered set in which every chain has a supremum. Assume that $f : P \to P$ is isotone and there is an element $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$. Then f has a fixed point.* Knaster [31](1928), Jachymski [23](1998).

This is given as [25, Theorem 2.1] where its history is also given. In [46], its equivalent formulations are given.

(IV) (Tarski-Kantorovitch) *Let $(P,\preccurlyeq)$ be a $\preccurlyeq$-complete partially ordered set and a map $f : P \to P$ be $\preccurlyeq$-continuous. If there exists $p_0 \in P$ such that*

$p_0 \preccurlyeq f(p_0)$*, then f has a fixed point; moreover,* $p_* := \sup\{f^n(p_0) : n \in \mathbb{N}\}$ *is fixed under f.* Jachymski et al. [22,24](1998, 2000).

In [46], this was equivalently reformulated.

(V) (Turinici) *Let* (X, d) *be a metric space,* $\leq$ *a partial order on* X *such that*

(1) *le is a closed order on* X*,*

(2) (X, d) *is* $\leq$*-asymptotic metric space,*

(3) (X, d) *is* $\leq$*-complete metric space.*

Then, for any $x \in X$*, there is a* $z \in X$ *such that* $x \leq z$ *and* $X(z, \leq) = \{z\}$ *(or, in other words, for every* $x \in X$*, there is a maximal element* $z \in X$ *such that* $x \leq z$*).* Turinici [51](1980).

This result is Brøndsted type in a class of order complete metric spaces extending Caristi's theorem, and has equivalent formulations as in Theorem 4.2.

(VI) (Granas-Horvath) *Let* (X, d) *be a complete metric space endowed with a partial order* $\preccurlyeq$*. Assume that for any* $x \in X$*, the set* $\{y \in X : x \preccurlyeq y\}$ *is closed and given* $\varepsilon > 0$*, there is* $y \in X$ *such that* $x \preccurlyeq y$ *and* $\operatorname{diam}\{z \in X : y \preccurlyeq z\} < \varepsilon$*. Then there exists a maximal element* $v \in X$*, that is,* $v \not\preccurlyeq w$ *for any* $w \in X \backslash \{v\}$*.*

This is known as the order theoretic Cantor theorem due to Granas-Horvath; see [19, Theorem 3] and Granas-Dugundji [18, p.32–33]. In [44], this was reformulated eight equivalent forms.

(VII) (Păcurar-Rus): "By various examples we illustrate that the ordered set theory is far away from having unitary notations and terminology. Some suggestions for unifying the terminology are also presented." .

In this paper [37] (2011), it is pointed that three theorems due to Turinici [51] (1980), Dancs-Hegedüs-Medvegyev [11] (1983), and Granas-Horvath [19] (2000) are actually the same, but expressed each in a different language.

(VIII) (Jachymski) *Let* (X, d) *be a complete metric space endowed with a partial order* $\preccurlyeq$*. Assume that for any* $x \in X$*, the set* $Tx = \{y \in X : x \preccurlyeq y\}$ *is closed and given* $\varepsilon > 0$*, there is* $y \in X$ *satisfying* $x \preccurlyeq y$ *and* $\operatorname{diam} Ty < \varepsilon$*. Then* $(X, \preccurlyeq)$ *has a maximal element.*

Jachymski obtained this [28, Theorem 3] as an extension of (VI). This can be equivalently reformulated by our Metatheorem [46].

A map $f : X \to X$ on a metric space (X, d) is said to be *contractive* if

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$.

(IX) (Edelstein) *Let* (X, ρ) *be a metric space and* $f : X \to X$ *be a contractive map. If there exists* $x \in X$ *such that the sequence of iterates* $f^n(x)$ *has a limit point* $\zeta \in X$*, then* ζ *is the unique fixed point of f.* Edelstein [14,15](1962), Kirk [30](2001).

This is a typical example of Theorem 4.2 with the following important Corollary.

(X) (Nemytskiĭ) *If the metric space* (X, ρ) *is compact, then every contractive map* $f : X \to X$ *has a unique fixed point in X. Moreover, for any* $x_1 \in X$ *the sequence defined by* $x_{n+1} = f(x_n),\ n \in \mathbb{N}$*, converges to the fixed point of the map* f*.* Nemytskiĭ [36](1936).

We obtained this from Cobzaş [9](2016).

In addition, there are a huge number of literature on the Banach contraction principle and some of its multi-valued versions due to Nadler [35] and Covitz-Nadler [10]. For them, see [40,46].

9. Conclusion

In 2022, we extended our earlier Metatheorem in [38,39] by adding more equivalent statements in [40-47]. In fact, we showed that the maximal elements in certain ordered sets can be reformulated to maximal fixed points or maximal stationary points of maps or multimaps and to common fixed points or common stationary points of a family of maps or multimaps, and conversely. Actually such points are same as we have seen in proofs of Metatheorem in [46] or Theorem 4.2. Therefore, if we have a theorem on any maximal points, it can be converted to at least ten equivalent theorems on other types of points without any serious argument.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem in [46,47]. Some of such theorems can be seen in our previous works [40-47] and the present article. Therefore, our Metatheorem is a machine to expand our knowledge easily.

In this article, we presented and improved relatively old and well-known examples. Actually we collected improved versions of known ordered fixed point theorems based on Metatheorem or the Brøndsted-Jachymski Principle due to ourselves. Such theorems are due to Zermelo (1980), Zorn (1930), Knaster-Tarski (1927-55), Tarski-Kantorovitch (1939), Nadler (1969), Covitz-Nadler (1970), Caristi (1976), Kasahara (1976), Abian (1980), Brunner (1987), Manka

(1988), Tasković (1988, 1992), Granas-Horvath (2000), Jachymski (2011) and other ordered fixed point theorems.

In fact, in many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our 2023 Metatheorem. Many of such theorems can be seen in our previous works [38-47] and the present article. Therefore, our Metatheorem or Theorem 4.2 are practical machines to expand our knowledge easily. In this article we presented relatively old and well-known such examples.

Some part of this article improves corresponding one appeared in [46].

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