Nonlinear Functional Analysis and Applications
Vol. 28, No. 3 (2023), pp. 851-863
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2023.28.03.17
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON IMPULSIVE SYMMETRIC Ψ-CAPUTO FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We study the appropriate conditions for the findings of uniqueness and existence for a group of boundary value problems for impulsive Ψ-Caputo fractional nonlinear Volterra-Fredholm integro-differential equations (V-FIDEs) with symmetric boundary non-instantaneous conditions in this paper. The findings are based on the fixed point theorem of Krasnoselskii and the Banach contraction principle. Finally, the application is provided to validate our primary findings.

1. Introduction

Fractional calculus (FC) theory, which covers differential equations of fractional order, has produced substantial advancements in research and development in recent decades, chiefly by offering appropriate answers for models, particularly for real-world issues. Furthermore, fractional differential equations are thought to be generalized differential equations. It is possible to consider a field of mathematical physics that deals with integro-differential equations (IDEs) in which the integrals are of the convolution form and also have predominantly power law or logarithm type single kernels, the IDE being an operator that contains both integer-order integrals and integer-order derivatives as special cases, which is the reason why FC is becoming increasingly popular and many applications arise from the term in the present [2, 8, 9, 10, 11, 12, 21].

[0]Received December 30, 2022. Revised February 20, 2023. Accepted February 24, 2023.
[0]2020 Mathematics Subject Classification: 34A08, 34B37, 47H10.
[0]Keywords: ψ-Caputo fractional derivative, V-FIDE, boundary value problem, fixed point approach.

The boundary value problems (BVPs) posing for parabolic nonlinear differential and IDEs can explain many natural phenomena in mathematical modeling (see, for example [1, 4, 5, 13, 14, 15, 16, 23, 25, 26, 27, 28, 29, 30]). This justifies the use of the fractional Caputo operator and its extension to IDEs.

Many academics have examined the uniqueness, existence, and stability of various BVPs using Caputo operators and their generalization in recent years.

In [6], Asawasamrit et al. investigated the ψ-Caputo (or, more precisely, ψ-Caputo-Liouville) FD and non-instantaneous BVPs. In [17], Ivaz et al. studied the fractional impulsive ψ-Hilfer derivative with boundary constraints. In [3], Ali et al. used the HOBW approach to solve fractional V-FIDEs with mixed boundary conditions. In [18], Kailasavalli et al. proved the existence of solutions to fractional BVPs using IDEs in Banach spaces. Karthikeyan et al. studied existence results for FIDEs with Katugampola type integral conditions in [19].

For the purpose of resolving the non-instantaneous new BVP, Long et al. [21] addressed the fractional differential equations listed below:

$$\begin{cases} {}^{c}\mathcal{D}^{p}_{0,\zeta}\Omega(\zeta) = E(\zeta,\Omega(\zeta)), \;\; \zeta \in (\Phi_{\varrho},\zeta_{\varrho+1}] \subset [0,\Lambda], \;\; p \in (0,1), \\ \Omega(\zeta) = \Upsilon_{\varrho}(\zeta,\Omega(\zeta)), \;\; \zeta \in (\zeta_{\varrho},\Phi_{\varrho}], \quad \varrho = 1,\dots,\mathrm{m}, \\ \Omega(\Lambda) = \Omega(0) + \chi \int_0^{\Lambda} \Omega(\Phi) d\Phi, \end{cases}$$

where E, Υ_{ϱ} are continuous and χ is constant.

Nuchpong et al. described the fractional Hilfer derivative with boundary non-local conditions in [22]:

$$\begin{cases} {}^{\Upsilon}\mathcal{D}^{\mathrm{p,q}}\Omega(\zeta) = \Xi\left(\zeta,\Omega(\zeta),\mathcal{I}^{\delta}\Omega(\zeta)\right), \quad \zeta \in [\Lambda_1,\Lambda_2], \\ \Omega(\Lambda_1) = 0, \;\; \varpi + \int_{\Lambda_1}^{\Lambda_2} \Omega(l)\mathrm{d}l = \sum_{\mathrm{k}=1}^{\varrho-2} \varsigma_{\mathrm{k}}\Omega(\vartheta_{\mathrm{k}}), \end{cases}$$

where the $\mathcal{I}^{\delta}$-R-L, and ${}^{\Upsilon}\mathcal{D}^{\mathrm{p,q}}$-fractional Hilfer derivative.

The BVP for generalized fractional Hilfer derivatives with impulses non-instantaneous was researched by Salim et al. [24]:

$$\begin{cases} \left({}^{\alpha}\mathcal{D}^{\mathrm{p,q}}_{\tau^+}\Omega\right)(\zeta) = \Xi\left(\zeta,\Omega(\zeta),\left({}^{\alpha}\mathcal{D}^{\mathrm{p,q}}\Omega\right)(\zeta)\right), \quad \zeta \in \zeta_k, \\ \Omega(\zeta) = \mp_{\iota}(\zeta,\Omega(\zeta)), \quad \zeta \in (\psi_{\iota},\vartheta_{\iota}], \quad \iota = 1,\cdots,\eta, \\ \varphi_1\left({}^{\alpha}\mathcal{I}^{1-\epsilon}_{a_1^+}\right)(\mathrm{a}_1) + \varphi_2\left({}^{\alpha}\mathcal{I}^{1-\epsilon}_{\tau^+}\right)(\mathrm{a}_2) = \varphi_3, \end{cases}$$

where ${}^{\alpha}\mathcal{D}^{\mathrm{p,q}}_{\tau^+}$ and ${}^{\alpha}\mathcal{I}^{1-\epsilon}_{a_1^+}$ are the generalized derivative of fractional Hilfer-type and integral of fractional order.

The non-instantaneous BVP for generalized fractional Hilfer derivatives with impulses was researched by Salim et al. [24]:

$$\begin{cases} {}^{c}\mathcal{D}_{\zeta}^{p;\Psi}\Omega(\zeta)=\mu(\zeta,\Omega(\zeta)), \quad 0<\zeta\leq\xi, \\ G(\Omega(0),\Omega(\xi))=0; \end{cases}$$

where ${}^{c}\mathcal{D}_{\zeta}^{p;\Psi}$ – Ψ-Caputo derivative and μ is continuous.

Inspire by the aforementioned studies, we provide some uniqueness and existence results for the following problem:

$$\begin{cases} {}^{c}\mathcal{D}^{p;\Psi}\varpi(\zeta)=\mu(\zeta,\varpi(\zeta),A\varpi(\zeta),B\varpi(\zeta)), \quad \zeta\in(s_i,\zeta_{i+1}], \\ \varpi(\zeta)=\Upsilon_i(\zeta,\varpi(\zeta)), \ \zeta\in(\zeta_i,s_i], \quad i=1,\ldots,m, \\ a\varpi(0)+b\varpi(\xi)=c, \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{p;\Psi}$ is the fractional of Ψ-Caputo derivative of order $0<p<1$, and $c,b,a\in\mathbb{R}$ with $0\neq b+a$ and $\mu:[0,\xi]\times\mathbb{R}^3\longrightarrow\mathbb{R}$, and $\Upsilon_i:[\zeta_i,s_i]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous, and $0=s_0<\zeta_1\leq\zeta_2<\ldots<\zeta_m\leq s_m\leq s_{m+1}=\xi$, pre-fixed. Also, $A\varpi(\zeta)=\int_0^{\zeta}k(\zeta,s)\varpi(s)ds$, $B\varpi(\zeta)=\int_0^{\xi}k_1(\zeta,s)\varpi(s)ds$, $k\in C(\mathrm{D},\mathbb{R}^+)$, and $\mathrm{D}=\left\{(\zeta,s)\in\mathbb{R}^2:0\leq s\leq\zeta\leq\xi\right\}$.

2. An auxiliary result

Let the space $PC([0,\xi],\mathbb{R})=\{\varpi:[0,\xi]\to\mathbb{R}:\varpi\in C(\zeta_\zeta,\zeta_{\mathfrak{k}+1}]\}$ be continuous and there exist $\varpi\left(\zeta_{\mathfrak{k}}^-\right)$, $\varpi\left(\zeta_{\mathfrak{k}}^+\right)$ such that $\varpi\left(\zeta_{\mathfrak{k}}^-\right)=\varpi\left(\zeta_{\mathfrak{k}}^+\right)$ with a norm $\|\varpi\|_{PC}=\sup\{|\varpi(\zeta)|:0\leq\zeta\leq\xi\}$ [5].

Definition 2.1. ([25]) For a continuous function μ, the R-L fractional derivative of order $\mathfrak{p}>0$, is

$$\mathcal{D}_{0^+}^{\mathfrak{p}}\mu(\zeta)=\frac{1}{\Gamma(\varepsilon-\mathfrak{p})}\left(\frac{\mathfrak{d}}{\mathfrak{d}\zeta}\right)^{\varepsilon}\int_0^{\zeta}(\zeta-\Phi)^{\varepsilon-\mathfrak{p}-1}\mu(\Phi)\mathfrak{d}\Phi, \quad \varepsilon-1<\mathfrak{p}<\varepsilon.$$

Definition 2.2. ([25]) The R-L fractional integral for a continuous function μ, is

$$\mathcal{J}^{\mathfrak{p}}\mu(\zeta)=\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}(\zeta-\Phi)^{\mathfrak{p}-1}\mu(\Phi)\mathfrak{d}\Phi,$$

where, $\Gamma(\mathfrak{p})=\int_0^{\infty}\mathfrak{e}^{-\Phi}\Phi^{\mathfrak{p}-1}\mathfrak{d}\Phi$, $\mathfrak{p}>0$.

Definition 2.3. ([25]) The Caputo derivative for a function $\mu:[0,\infty)\to\mathbb{R}$ is defined by

$${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p}}\mu(\zeta)=\frac{1}{\Gamma(\varepsilon-\mathfrak{p})}\int_0^{\zeta}\frac{\mu^{(\varepsilon)}(s)}{(\zeta-s)^{\mathfrak{p}+1-\varepsilon}}ds=\mathfrak{I}^{\varepsilon-\mathfrak{p}}\mu^{(\varepsilon)}(\zeta), \quad \zeta>0, \ \varepsilon-1<\mathfrak{p}<\varepsilon.$$

Definition 2.4. ([23]) The fractional integral and FD for a function μ are defined as:

$$\mathfrak{I}^{\mathfrak{p};\Psi}\mu(\zeta)=\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi)\mathfrak{d}\Phi$$

and

$$\mathfrak{D}^{\mathfrak{p};\Psi}\mu(\zeta)=\frac{1}{\Gamma(\varepsilon-\mathfrak{p})}\left(\frac{1}{\Theta'(\zeta)}\frac{\mathfrak{d}}{\mathfrak{d}\zeta}\right)^{\varepsilon}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\varepsilon-\mathfrak{p}-1}\mu(\Phi)\mathfrak{d}\Phi,$$

respectively.

Lemma 2.5. ([2]) *Let $\mu:\mathfrak{J}\longrightarrow\mathbb{R}$ be continuous and $0<\alpha<1$. The ϖ is a solution of the Ψ-fractional integral equation(FIE):*

$$\varpi(\zeta)=\begin{cases}\Upsilon_{\mathfrak{m}}\left(\Phi_{\mathfrak{m}}\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\zeta)\mathfrak{d}\Phi+\varpi_0, & \zeta\in[0,\zeta_1],\\ \Upsilon_\rho(\zeta), \quad \zeta\in(\zeta_\rho,\Phi_\rho], \quad \rho=1,2,\ldots,\mathfrak{m}, \\ \Upsilon_\rho\left(\Phi_\rho\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\zeta)\mathfrak{d}\Phi \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho}\Theta'(\Phi)\left(\Theta_{\Phi_\rho}-\Theta_{\Phi}\right)^{\mathfrak{p}-1}\omega(\zeta)\mathfrak{d}\Phi, & \zeta\in(\Phi_\rho,\zeta_{\rho+1}],\ \rho=1,2,\ldots,\mathfrak{m}\end{cases}\tag{2.1}$$

if and only if ϖ is a solution of initial value problem (IVP) of the DEs:

$$\begin{cases}{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\varpi(\zeta)=\omega(\zeta),\ \zeta\in(\Phi_\rho,\zeta_{\rho+1}]\subset[0,\xi],\quad 0<\mathfrak{p}<1,\\ \varpi(\zeta)=\Upsilon_\rho(\zeta),\ \zeta\in(\zeta_\rho,\Phi_\rho], \quad \rho=1,\ldots,\mathfrak{m},\\ \varpi(0)=\varpi_0.\end{cases}\tag{2.2}$$

Lemma 2.6. *A function $\varpi\in PC([0,\xi],\mathbb{R})$ is given by*

$$\varpi(\zeta)=\begin{cases}\Upsilon_{\mathfrak{m}}\left(\Phi_{\mathfrak{m}}\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi \\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^{\xi}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi-\mathfrak{c}\right], \quad \zeta\in[0,\zeta_1],\\ \Upsilon_\rho(\zeta), \quad \zeta\in(\zeta_\rho,\Phi_\rho],\rho=1,2,\ldots,\mathfrak{m},\\ \Upsilon_\rho\left(\Phi_\rho\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho}\Theta'(\Phi)\left(\Theta_{\Phi_\rho}-\Theta\Phi\right)^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi,\ \zeta\in(\Phi_\rho,\zeta_{\rho+1}],\ \rho=1,2,\ldots,\mathfrak{m}\end{cases}\tag{2.3}$$

is a solution of

$$\begin{cases}{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\varpi(\zeta)=\omega(\zeta),\ \zeta\in(\Phi_\rho,\zeta_{\rho+1}],\quad 0<\mathfrak{p}<1,\\ \varpi(\zeta)=\Upsilon_\rho(\zeta),\ \zeta\in(\zeta_\rho,\Phi_\rho], \quad \rho=1,\ldots,\mathfrak{m},\\ \mathfrak{a}\varpi(0)+\mathfrak{b}\varpi(\xi)=\mathfrak{c}.\end{cases}\tag{2.4}$$

Proof. Assume that $\varpi(\zeta)$ is satisfied (2.4). Integrating in (2.4) for $\zeta \in [0,\zeta_1]$, we get

$$\varpi(\zeta)=\varpi(\xi)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi \tag{2.5}$$

and if $\zeta \in (\Phi_\rho,\zeta_{\rho+1}]\,,\rho=1,2,\dots,\mathfrak{m}$ and again integrating (2.4), we get

$$\varpi(\zeta)=\varpi\left(\Phi_\rho\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_{\Phi_\rho}^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi. \tag{2.6}$$

Now, by applying impulsive condition, $\varpi(\zeta)=\Upsilon_\rho(\zeta)$, $\zeta \in (\zeta_\rho,\Phi_\rho]$, we get,

$$\varpi\left(\Phi_\rho\right)=\Upsilon_\rho\left(\Phi_\rho\right). \tag{2.7}$$

Consequently, from (2.6) and (2.7), we obtain

$$\varpi(\zeta)=\Upsilon_\rho\left(\Phi_\rho\right)+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi \tag{2.8}$$

and

$$\begin{aligned}\varpi(\zeta)=\Upsilon_\rho\left(\Phi_\rho\right)&+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi\\ &-\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho}\left(\Theta'(\Phi)\Psi_{\Phi_\rho}-\Psi_{\Phi}\right)^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi,\ \ \zeta\in\left(\Phi_\rho,\zeta_{\rho+1}\right].\end{aligned} \tag{2.9}$$

Using the BCs $\mathfrak{a}\varpi(0)+\mathfrak{b}\varpi(\xi)=\mathfrak{c}$, we get

$$\varpi(\xi)=\Upsilon_{\mathfrak{m}}\left(\Phi_{\mathfrak{m}}\right)-\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^{\xi}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\omega(\Phi)\mathfrak{d}\Phi-\mathfrak{c}\right]. \tag{2.10}$$

As a result, the FDs, lemmas and definitions have immediate applicability, It is obvious that (2.5),(2.9) and (2.10) imply (2.3). Hence the proof is completed. □

Theorem 2.7. *(Banach FPT)* ([23]) *Let Q be a nonempty closed subset of a Banach space $\mathbb{B}$ and $N : Q \to Q$, be a contraction mapping. Then N has a unique fixed point in Q.*

Theorem 2.8. *(Krasnoselkii's FPT)* ([20]) *Let ϑ be a nonempty, convex, bounded and closed subset of a Banach space $\mathbb{B}$. Let A_1 and A_2 be functions from ϑ into itself with the following conditions:*

(1) *$A_1x + A_2y \in \vartheta$ with $x, y \in \vartheta$;*
(2) *A_1 be continuous and compact;*
(3) *A_2 be a contraction.*

Then, there exist $z \in \vartheta$ such that $z = A_1z + A_2z$.

3. Main results

Theorem 3.1. *Aassume the following condition is satisfied:*
(Al1): *There exist $L, G, h, M, M_1, L_{\mathfrak{h}_\rho} > 0$ such that*

$$\begin{aligned} |\mu(\zeta, \varpi_1, \omega_1, u_1) - \mu(\zeta, \varpi_2, \omega_2, u_2)| &\leq L|\varpi_1 - \varpi_2| + G|\omega_1 - \omega_2| + h|u_1 - u_2|, \\ |k(\zeta, \Phi, \vartheta) - k(\zeta, \Phi, v)| &\leq M|\vartheta - v|, \text{ for } \zeta \in [\zeta_\rho, \Phi_\rho], \vartheta, v \in \mathbb{R}, \\ |k_1(\zeta, \Phi, \vartheta) - k_1(\zeta, \Phi, v)| &\leq M_1|\vartheta - v|, \text{ for } \zeta \in [\zeta_\rho, \Phi_\rho], \vartheta, v \in \mathbb{R}, \\ |\Upsilon_\rho(\zeta, \mathfrak{v}_1) - \Upsilon_\rho(\zeta, \mathfrak{v}_2)| &\leq L_{\mathfrak{h}_\rho}|\mathfrak{v}_1 - \mathfrak{v}_2|, \text{ for } \mathfrak{v}_1, \mathfrak{v}_2 \in \mathbb{R}. \end{aligned}$$

If

$$\begin{aligned} &Z : \max\left\{\max_{\rho=1,2,\ldots,\mathfrak{m}} L_{\mathfrak{h}_\rho} + \frac{(L + GM + hM_1)}{\Gamma(\mathfrak{p}+1)}\left(\zeta_{\rho+1}^{\mathfrak{p}} + \Phi_\rho^{\mathfrak{p}}\right),\right. \\ &\left. L_{\mathfrak{h}_\rho} + \frac{(L + GM + hM_1)(\Theta(\xi))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1 + \frac{|\mathfrak{b}|(L + GM + hM_1)}{|\mathfrak{a} + \mathfrak{b}|}\right]\right\} < 1, \quad (3.1) \end{aligned}$$

then problem (1.1) has a unique solution on $[0, \xi]$.

Proof. Let $N : PC([0, \xi], \mathbb{R}) \longrightarrow PC([0, \xi], \mathbb{R})$ be define by

$$(N\varpi)(\zeta) = \begin{cases} \Upsilon_{\mathfrak{m}}(\Phi_{\mathfrak{m}}, \varpi(\Phi_{\mathfrak{m}})) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta \Theta'(\Phi)(\Theta(\zeta) - \Theta(\Phi))^{\mathfrak{p}-1} \\ \times\mu(\Phi, \varpi(\Phi), A\varpi(\Phi), B\varpi(\Phi))\mathfrak{d}\Phi \\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^\xi \Theta'(\Phi)(\Theta(\zeta) - \Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi, \varpi(\Phi), A\varpi(\Phi), B\varpi(\Phi))\mathfrak{d}\Phi\right], \\ \\ \zeta \in [0, \zeta_1], \Upsilon_\rho(\zeta), \quad \zeta \in (\zeta_\rho, \Phi_\rho], \quad \rho = 1, 2, \ldots, \mathfrak{m}, \\ \\ \Upsilon_\rho(\Phi_\rho) + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta \Theta'(\Phi)(\Theta(\zeta) - \Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi, \varpi(\Phi), A\varpi(\Phi), B\varpi(\Phi))\mathfrak{d}\Phi \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_i} \Theta'(\Phi)(\Theta\Phi_\rho - \Theta\Phi)^{\mathfrak{p}-1}\mu(\Phi, \varpi(\Phi), A\varpi(\Phi), B\varpi(\Phi))\mathfrak{d}\Phi, \\ \zeta \in (\Phi_\rho, \zeta_{\rho+1}]. \end{cases}$$

Then it is obvious that $N\varpi \in PC([0, \xi], \mathbb{R})$ and N is well defined. Now we show that N is a contraction mapping.

Case 1: For $\zeta \in [0, \zeta_1]$ and $\varpi, \bar{\varpi} \in PC([0, \xi], \mathbb{R})$, we get

$$
\begin{aligned}
&|(N\varpi)(\zeta)-(N\bar{\varpi})(\zeta)|\\
&\quad\le L_{\mathfrak{h}_\rho}\left|\varpi\left(\Phi_{\mathfrak{m}}\right)-\bar{\varpi}\left(\Phi_{\mathfrak{m}}\right)\right|\mathfrak{d}\Phi\\
&\qquad+\frac{(L+GM+hM_1)}{\Gamma(\mathfrak{p}+1)}\int_0^{\zeta}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}|\varpi-\bar{\varpi}|\mathfrak{d}\Phi\\
&\qquad+\frac{|\mathfrak{b}|(L+GM+hM_1)}{|\mathfrak{a}+\mathfrak{b}|\Gamma(\mathfrak{p})}\int_0^{\xi}\Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}|\varpi-\bar{\varpi}|\mathfrak{d}\Phi\\
&\quad\le L_{\mathfrak{h}_\rho}+\frac{(L+GM+hM_1)(\Theta(\xi))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right]\|\varpi-\bar{\varpi}\|_{PC}.
\end{aligned}
$$

Case 2: For $\zeta\in(\zeta_\rho,\Phi_\rho]$, we get

$$
\begin{aligned}
|(N\varpi)(\zeta)-(N\bar{\varpi})(\zeta)| &\le |\Upsilon_\rho(\zeta,\varpi(\zeta))-\Upsilon_{\mathfrak{i}}(\zeta,\bar{\varpi}(\zeta))|\\
&\le L_{\mathfrak{h}_\rho}\|\varpi-\bar{\varpi}\|_{PC}.
\end{aligned}
$$

Case 3: For $\zeta\in(\Phi_\rho,\zeta_{\rho+1}]$, we get

$$
\begin{aligned}
&|(N\varpi)(\zeta)-(N\bar{\varpi})(\zeta)|\\
&\quad\le|\Upsilon_\rho(\Phi_\rho,\varpi(\Phi_\rho)-\Upsilon_\rho(\Phi_\rho,\bar{\varpi}(\Phi_\rho)|\\
&\qquad+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\zeta}(\zeta-\Phi)^{\mathfrak{p}-1}|\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))-\mu(\Phi,\bar{\varpi}(\Phi),\\
&\qquad\qquad A\bar{\varpi}(\Phi),B\bar{\varpi}(\Phi))|\mathfrak{d}\Phi\\
&\qquad+\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho}(\Phi_\rho-\Phi)^{\mathfrak{p}-1}|\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))-\mu(\Phi,\bar{\varpi}(\Phi),\\
&\qquad\qquad A\bar{\varpi}(\Phi),B\bar{\varpi}(\Phi))|\mathfrak{d}\Phi,\\
&\quad\le\left[L_{\mathfrak{h}_\rho}+\frac{(L+GM+hM_1)}{\Gamma(\mathfrak{p}+1)}\left(\zeta_{\rho+1}^{\mathfrak{p}}+\Phi_\rho^{\mathfrak{p}}\right)\right]\|\varpi-\bar{\varpi}\|_{Pb}.
\end{aligned}
$$

As can be seen from the inequality above, N is a contraction.

$$
Z=\left[L_{\mathfrak{h}_\rho}+\frac{(L+GM+hM_1)}{\Gamma(\mathfrak{p}+1)}\left(\zeta_{\rho+1}^{\mathfrak{p}}+\Phi_\rho^{\mathfrak{p}}\right)\right]<1.
$$

Then, problem (1.1) has a unique solution for all $\varpi\in PC([0,\xi],\mathbb{R})$. □

Theorem 3.2. *Let's assume the* (Al_1) *is satisfied and the following premise below is also satisfied:*

Al2 : *There exist* $L_{g_i}>0$ *such that*

$$
|\mu\left(t,w_1,\omega_1,u_1\right)|\le L_{g_i}\left(1+|w_1|+|\omega_1|+|u_1|\right),\ t\in\left[s_i,t_{i+1}\right],\ \forall w_1,\omega_1,u_1\in\mathbb{R}.
$$

Al3 : *There exist* $\kappa_i(t),i=1,2,\ldots,m$ *such that*

$$
|\Upsilon_i\left(t,w_1\right)|\le\kappa_i(t),\quad t\in\left[t_i,s_i\right],\ \ \forall w_1\in\mathbb{R}.
$$

Let $M_i := \sup_{t\in[t_i,s_i]} \kappa_i(t) < \infty$, *and* $K := \max L_{h_i} < 1$ *for all* $i = 1, 2, .., m$. *Then, problem* (1.1) *has a solution on* $[0, \xi]$.

Proof. Assume that $B_{\mathfrak{p},\mathfrak{r}} = \{\varpi \in PC([0,\xi],\mathbb{R}) : \|\varpi\|_{PC} \leq \mathfrak{r}\}$. Let Q and R be two operators on $B_{\mathfrak{p},\mathfrak{r}}$ defined as follows:

$$Q_\varpi(\zeta) = \begin{cases} \Upsilon_{\mathfrak{m}}\left(\Phi_{\mathfrak{m}}, \varpi\left(\Phi_{\mathfrak{m}}\right)\right), & \zeta \in [0,\zeta_1], \\ \Upsilon_\rho(\zeta, \varpi(\zeta)), & \zeta \in (\zeta_\rho, \Phi_\rho], \quad \rho = 1, 2, \ldots, \mathfrak{m}, \\ \Upsilon_\rho\left(\Phi_\rho, \varpi\left(\Phi_\rho\right)\right), & \zeta \in (\Phi_\rho, \zeta_{\rho+1}], \quad \rho = 1, 2, \ldots, \mathfrak{m} \end{cases}$$

and

$$R_\varpi(\zeta) = \begin{cases} \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta \Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))\mathfrak{d}\Phi \\ -\frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^\xi \Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))\mathfrak{d}\Phi\right], \\ \zeta \in [0,\zeta_1], \quad 0, \zeta \in (\zeta_{\rho,}, \Phi_\rho], \quad \rho = 1, 2, \ldots, \mathfrak{m}, \\ \\ \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta \Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))\mathfrak{d}\Phi \\ -\frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho} \Theta'(\Phi)\left(\Psi\Phi_{(i)}-\Theta(\Phi)\right)^{\mathfrak{p}-1}\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))\mathfrak{d}\Phi, \\ \zeta \in (\Phi_\rho, \zeta_{\rho+1}]. \end{cases}$$

Step 1: If $\varpi \in B_{\mathfrak{p},\mathfrak{r}}$ then $Q_\varpi + R_\varpi \in B_{\mathfrak{p},\mathfrak{r}}$.
Case 1: For $\zeta \in [0, \zeta_1]$,

$$\begin{aligned} &|Q_\varpi + R_{\bar{\varpi}}| \\ &\leq \left|\Upsilon_{\mathfrak{m}}\left(\Phi_{\mathfrak{m}}, \varpi\left(\Phi_{\mathfrak{m}}\right)\right)\right| + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta (\zeta-\Phi)^{\mathfrak{p}-1}|\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))|\mathfrak{d}\Phi \\ &\quad + \frac{1}{\mathfrak{a}+\mathfrak{b}}\left[\frac{\mathfrak{b}}{\Gamma(\mathfrak{p})}\int_0^\xi \Theta'(\Phi)(\Theta(\zeta)-\Theta(\Phi))^{\mathfrak{p}-1}\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))\mathfrak{d}\Phi\right] \\ &\leq \left[L_{\mathfrak{g}_\rho} + \frac{L_{\mathfrak{g}_\rho(\Theta(\xi))^{\mathfrak{p}}}}{\Gamma(\mathfrak{p}+1)}\left[1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\right]\right](1+\mathfrak{r}) \leq \mathfrak{r}. \end{aligned}$$

Case 2: For $\zeta \in (\zeta_\rho, \Phi_\rho]$,

$$|Q_\varpi + R_\varpi| \leq |\Upsilon_{\mathfrak{i}}(\zeta, W_1(\zeta))| \leq M_{\mathfrak{i}}.$$

Case 3: For $\zeta \in (\Phi_\rho, \zeta_{\rho+1}]$,

$$\begin{aligned}
&|Q_\varpi + R_\varpi(\zeta)| \\
&\leq |\Upsilon_\rho(\Phi_\rho, \varpi(\Phi_\rho))| + \frac{1}{\Gamma(\mathfrak{p})}\int_0^\zeta (\zeta-\Phi)^{\mathfrak{p}-1}|\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))|\mathfrak{d}\Phi \\
&\quad + \frac{1}{\Gamma(\mathfrak{p})}\int_0^{\Phi_\rho}(\Phi_\rho-\Phi)^{\mathfrak{p}-1}|\mu(\Phi,\varpi(\Phi),A\varpi(\Phi),B\varpi(\Phi))|\mathfrak{d}\Phi \\
&\leq M_\rho + \left[\frac{L_{\mathfrak{g}_\rho}\left(\Phi_\rho^{\mathfrak{p}} + \zeta_{\rho+1}^{\mathfrak{p}}\right)}{\Gamma(\mathfrak{p}+1)}\right](1+\mathfrak{r}) \leq \mathfrak{r}.
\end{aligned}$$

Thus

$$Q_\varpi + R_\varpi \in B_{\mathfrak{p},\mathfrak{r}}.$$

Step 2: ϖ be contraction on $B_{\mathfrak{p},\mathfrak{r}}$.

Case 1: For $\varpi_1, \varpi_2 \in B_{\mathfrak{p},\mathfrak{r}}, \quad \zeta \in [0,\zeta_1]$,

$$|Q_{\varpi_1}(\zeta) - Q_{\varpi_2}(\zeta)| \leq L_{\mathfrak{g}_\mathfrak{m}}|\varpi_1(\Phi_\mathfrak{m}) - \varpi_2(\Phi_\mathfrak{m})| \leq L_{\mathfrak{g}_\mathfrak{m}}\|\varpi_1-\varpi_2\| PC.$$

Case 2: For $\zeta \in (\zeta_\rho, \Phi_\rho], \quad \rho = 1,2,\ldots,\mathfrak{m}$,

$$|Q_{\varpi_1}(\zeta) - Q_{\varpi_2}(\zeta)| \leq L_{\mathfrak{g}_\rho}\|\varpi_1-\varpi_2\|_{PC}.$$

Case 3: For $\zeta \in (\Phi_i, \zeta_{\rho+1}]$,

$$|Q_{\varpi_1}(\zeta) - Q_{\varpi_2}(\zeta)| \leq L_{\mathfrak{g}_\rho}\|\varpi_1-\varpi_2\|_{PC}.$$

The following may be inferred from the above inequalities:

$$|Q_{\varpi_1}(\zeta) - Q_{\varpi_2}(\zeta)| \leq K\|\varpi_1-\varpi_2\| PC\cdot$$

Then, Q is a contraction.

Step 3: R is continuous:

Let $\{\varpi_\mathfrak{n}\}$ be a sequence such that $\varpi_\mathfrak{n} \to \bar{\varpi}$ in $PC([0,\xi],\mathbb{R})$.

Case 1: For all $\zeta \in [0,\zeta_1]$,

$$\begin{aligned}
|Q_\varpi\mathfrak{n}(\zeta) - Q_\varpi(\zeta)| \leq &\Big[\frac{(\Theta(\xi))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\Big[1+\frac{|\mathfrak{b}|}{|\mathfrak{a}+\mathfrak{b}|}\Big]\Big] \\
&\times ||\mu(.,\varpi_\mathfrak{n}(.),.,.) - \mu(.,\varpi(.),.,.)||_{PC}.
\end{aligned}$$

Case 2: For all $\zeta \in (\zeta_\rho, \Phi_\rho]$, we get

$$|Q_{\varpi_\mathfrak{n}}(\zeta) - Q_\varpi(\zeta)| = 0.$$

Case 3: For all $\zeta \in (\Phi_\rho, \zeta_{\rho+1}]$, $\rho = 1, 2, \ldots, \mathfrak{m}$,

$$|Q_{\varpi_{\mathfrak{n}}}(\zeta) - Q_{\varpi}(\zeta)| \leq \frac{(\zeta_{\rho+1} - \Phi_\rho)}{\Gamma(\mathfrak{p}+1)} \|\mu(., \varpi_{\mathfrak{n}}(.), ., .,) - \mu(., \varpi(.), ., .,)\|_{PC}.$$

Thus, we conclude from the above cases that $\|Q_{\varpi_{\mathfrak{n}}}(\zeta) - Q_{\varpi}(\zeta)\|_{PC} \longrightarrow 0$ as $\mathfrak{n} \to \infty$.

Step 4: Q is compact:

First Q is bounded uniformly on $B_{\mathfrak{p},\mathfrak{r}}$.

Since $\|Q\varpi\| \leq \frac{L_{\mathfrak{g}_\rho}(\xi)}{\Gamma(1+\mathfrak{p})} < \mathfrak{r}$, we prove that Q maps a bounded set to a $B_{\mathfrak{p},\mathfrak{r}}$ equicontinuous set.

Case 1: For interval $\zeta \in [0, \zeta_1], 0 \leq E_1 \leq E_2 \leq \zeta_1, \varpi \in B_{\mathfrak{r}}$, we obtain

$$|Q_{E_2} - Q_{E_1}| \leq \frac{L_{\mathfrak{g}_\rho}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)} (E_2 - E_1).$$

Case 2: For each $\zeta \in (\zeta_\rho, \Phi_\rho], \zeta_\rho < E_1 < E_2 \leq \Phi_\rho, \varpi \in B_{\mathfrak{p},\mathfrak{r}}$, we obtain

$$|Q_E - Q_{E_1}| = 0.$$

Case 3: For each $\zeta \in (\Phi_\rho, \zeta_{\rho+1}]$, $\Phi_\rho < E_1 < E_2 \leq \zeta_{\rho+1}$, $\varpi \in B_{\mathfrak{p},\mathfrak{r}}$, we establish

$$|Q_{E_2} - Q_E| \leq \frac{L_{\mathfrak{g}_\rho}(1+\mathfrak{r})}{\Gamma(\mathfrak{p}+1)} (E_2 - E_1).$$

The aforementioned instances lead us to $|Q_E - Q_{E_1}| \longrightarrow 0$ as $E_2 \longrightarrow E_1$ and Q is equicontinuous. Then $Q(B_{\mathfrak{p},\mathfrak{r}})$ is compact relatively, which completes the proof. □

4. An example

Assume that the fractional BVP:

$$\begin{aligned} D^{\mathfrak{p}}\varpi(\zeta) &= \frac{\mathfrak{e}^{-\zeta}|\mathfrak{w}|}{9+\mathfrak{e}^{\zeta}(1+|\varpi|} + \frac{1}{3}\int_0^\zeta \mathfrak{e}^{-(\Phi-\zeta)}\varpi(\Phi)\mathfrak{d}\Phi \\ &\quad + \frac{1}{6}\int_0^{\frac{1}{2}} \mathfrak{e}^{(\zeta-\Phi)}\varpi(\Phi)\mathfrak{d}\Phi, \ \zeta \in (0, \frac{1}{2}], \quad (4.1) \\ \varpi(\zeta) &= \frac{|\varpi(\zeta)|}{2(1+|\varpi(\zeta)|)}, \ \zeta \in (\frac{1}{2}, 1], \quad (4.2) \\ \varpi(0) + \varpi(1) &= 0, \quad (4.3) \end{aligned}$$

and $L = G = h = \frac{1}{10}, M = \frac{1}{3}, M_1 = \frac{1}{6}, \mathfrak{p} = \frac{5}{7}, L_{\mathrm{h}_1} = \frac{1}{3}$. We can see that (1.1) is satisfied with $\mathfrak{a} = \mathfrak{b} = \xi = 1$ for $\mathfrak{p} \in (0, 1]$. Indeed, we establish that using

Theorem 3.2

$$L_{\mathfrak{h}_\rho} + \frac{(L+GM+hM_1)}{\Gamma(\mathfrak{p}+1)}\left(\zeta^{\mathfrak{p}}_{\rho+1} + \Phi^{\mathfrak{p}}_{\rho}\right) = 0.59 < 1$$

and

$$\left\{L_{\mathfrak{h}_\rho} + \frac{(L+GM+hM_1)(\Theta(\xi))^{\mathfrak{p}}}{\Gamma(\mathfrak{p}+1)}\left[1 + \frac{|\mathfrak{b}|(L+GM+hM_1)}{|\mathfrak{a}+\mathfrak{b}|}\right]\right\} = 0.67 < 1.$$

Hence, all assumptions of Theorem 3.2 are satisfied, so the FIDEs (4.1)-(4.3) has a unique solution in $[0,\xi]$.

5. Conclusions

In this study, we examined a class of $V-FIDEs$ with an impulsive fractional and a closed linear operator boundary value problems. Then, under some appropriate circumstances, we looked at the current result using the Krasnoselskii fixed point theorem. In addition, we introduced and then demonstrated the results of the uniqueness of the $V-FIDEs$ with non-instantaneous symmetric boundary conditions for the Ψ-Caputo fractional nonlinear equations. The appropriate problem has been proven, and the outcomes have been validated. We will in the future extend the periods of our work.

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