Nonlinear Functional Analysis and Applications
Vol. 18, No. 2 (2013), pp. 279-289

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COMMON FIXED POINT FOR A PAIR OF MAPS IN COMPLEX VALUED METRIC SPACES

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Abstract. Common fixed point theorems for a pair of maps under contraction involving rational expressions in the setting of complex valued metric spaces are proved. Some consequence have also been given of the main results. To illustrate our results and to distinguish them from the existing ones, we equip the paper with example.

1. Introduction

The Banach contraction principle [2] is a very popular tool in solving existence problems in many branches of mathematical analysis. Due to simplicity and usefulness of this celebrated theorem, it has become a very popular source of existence and uniqueness theorems in different branches of mathematical analysis. This theorem provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplines. This famous theorem can be stated as follows.

There are in the literature a great number of generalizations of the Banach contraction principle exist. Some generalizations of the notion of a metric space have been proposed by some authors, such as, rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, Quasi metric spaces, Quasi semi metric spaces, D-metric spaces, cone metric spaces, partially ordered metric spaces (see [3]- [19]).

[0]Received January 6, 2013. Revised March 26, 2013.
[0]2000 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Common fixed point, contractive type mapping, complex valued metric space.

Azam et al. [1] introduce the notion of complex valued metric spaces and established some fixed point results for mappings satisfying a rational inequality. Though complex valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, one can study improvements of a host of results of analysis involving divisions.

In this paper, an attempt has been made to establish common fixed point results for a pair of mappings satisfying contractive condition involving a rational expression, more general than in [1], in the frame of complex valued metric spaces. An examples is given to support the usability of our results.

2. Preliminaries

Recall the following definitions.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

$$0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$$

and

$$z_1 \preceq z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

The following definition is recently introduced by Azam et al. [1].

Definition 2.1. Let $\mathcal{X}$ be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, satisfies the following conditions:

(d_1) $0 \precsim d(x, y)$, for all $x, y \in \mathcal{X}$ and $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$, for all $x, y, z \in \mathcal{X}$.

Then d is called a complex valued metric on $\mathcal{X}$, and $(\mathcal{X}, d)$ is called a complex valued metric space.

Definition 2.2. Let $(\mathcal{X}, d)$ be a complex valued metric space and $\mathcal{B} \subseteq \mathcal{X}$.

(i) $b \in \mathcal{B}$ is called an interior point of a set $\mathcal{B}$ whenever there is $0 \prec r \in \mathbb{C}$ such that

$$\mathcal{N}(b,r) \subseteq \mathcal{B},$$

where $\mathcal{N}(b,r) = \{y \in \mathcal{X} : d(b,y) \prec r\}$.

(ii) A point $x \in \mathcal{X}$ is called a limit point of $\mathcal{B}$ whenever for every $0 \prec r \in \mathbb{C}$,

$$\mathcal{N}(x,r) \cap (\mathcal{B} \backslash \mathcal{X}) \neq \emptyset.$$

(iii) A subset $\mathcal{A} \subseteq \mathcal{X}$ is called open whenever each element of $\mathcal{A}$ is an interior point of $\mathcal{A}$.

A subset $\mathcal{B} \subseteq \mathcal{X}$ is called closed whenever each limit point of $\mathcal{B}$ belongs to $\mathcal{B}$. The family

$$\mathcal{F} = \{\mathcal{N}(x,r) : x \in \mathcal{X}, 0 \prec r\}$$

is a sub-basis for a topology on $\mathcal{X}$. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 2.3. Let $(\mathcal{X}, d)$ be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in $\mathcal{X}$ and $x \in \mathcal{X}$. We say that

(i) the sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) the sequence$\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,

(iii) the metric space $(\mathcal{X}, d)$ is a complete complex valued metric space if every Cauchy sequence is convergent.

Definition 2.4. (cf. [10]) Two families of self-mappings $\{\mathcal{T}_i\}_{i=1}^m$ and $\{\mathcal{S}_i\}_{i=1}^n$ are said to be pairwise commuting if:

(I) $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$, $i, j \in \{1, 2, ...m\}$;

(II) $\mathcal{S}_k\mathcal{S}_l = \mathcal{S}_l\mathcal{S}_k$, $k, l \in \{1, 2, ...n\}$;

(III) $\mathcal{T}_i\mathcal{S}_k = \mathcal{S}_k\mathcal{T}_i$, $i \in \{1, 2, ...m\}$, $k \in \{1, 2, ...n\}$.

In [1], Azam et al. established the following two lemmas.

Lemma 2.5. (cf. [1]) *Let $(\mathcal{X}, d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.6. (cf. [1]) *Let $(\mathcal{X}, d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3. Main Result

We will prove now common fixed point results which generalize the results of Azam et al. [1].

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a complete complex valued metric space and the mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy:*

$$\begin{aligned} d(\mathcal{S}x, \mathcal{T}y) \ \precsim\ & \alpha d(x,y) + \frac{\beta d(y, \mathcal{T}y) d(x, \mathcal{S}x)}{1 + d(x,y)} + \gamma[d(x, \mathcal{S}x) + d(y, \mathcal{T}y)] \\ & + \delta[d(x, \mathcal{T}y) + d(y, \mathcal{S}x)] \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{X}$ where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha + \beta + 2\gamma + 2\delta < 1$. Then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in $\mathcal{X}$ and define

$$x_{2k+1} = \mathcal{S}x_{2k}, \ \ x_{2k+2} = \mathcal{T}x_{2k+1}, \ \ k = 0, 1, 2, \cdots .$$

Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(\mathcal{S}x_{2k}, \mathcal{T}x_{2k+1}) \\ &\precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta d(x_{2k+1}, \mathcal{T}x_{2k+1}) d(x_{2k}, \mathcal{S}x_{2k})}{1 + d(x_{2k}, x_{2k+1})} \\ &\quad + \gamma[d(x_{2k}, \mathcal{S}x_{2k}) + d(x_{2k+1}, \mathcal{T}x_{2k+1})] \\ &\quad + \delta[d(x_{2k}, \mathcal{T}x_{2k+1}) + d(x_{2k+1}, \mathcal{S}x_{2k})] \\ &\precsim \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta d(x_{2k+1}, x_{2k+2}) d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \\ &\quad + \gamma[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\ &\quad + \delta[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})] \\ &= \alpha d(x_{2k}, x_{2k+1}) + \frac{\beta d(x_{2k+1}, x_{2k+2}) d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \\ &\quad + \gamma[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] + \delta d(x_{2k}, x_{2k+2}). \end{aligned}$$

This implies that

$$\begin{aligned}|d(x_{2k+1}, x_{2k+2})| &\leq \alpha|d(x_{2k}, x_{2k+1})| + \frac{\beta|d(x_{2k+1}, x_{2k+2})||d(x_{2k}, x_{2k+1})|}{|1 + d(x_{2k}, x_{2k+1})|} \\ &+\gamma[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|] \\ &+\delta[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|].\end{aligned}$$

Since

$$|d(x_{2k}, x_{2k+1})| \leq |1 + d(x_{2k}, x_{2k+1})|$$

and by the triangular inequality

$$|d(x_{2k}, x_{2k+2})| \leq |d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})|.$$

Therefore

$$\begin{aligned}|d(x_{2k+1}, x_{2k+2})| &\leq \alpha|d(x_{2k}, x_{2k+1})| + \beta|d(x_{2k+1}, x_{2k+2})| \\ &+\gamma[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|] \\ &+\delta[|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|] \\ &= \left(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta}\right)|d(x_{2k}, x_{2k+1})|.\end{aligned}$$

Similarly,

$$\begin{aligned}d(x_{2k+2}, x_{2k+3}) &= d(\mathcal{S}x_{2k+1}, \mathcal{T}x_{2k+2}) \\ &\precsim \alpha(x_{2k+2}, x_{2k+1}) + \frac{\beta d(x_{2k+1}, \mathcal{S}x_{2k+1})d(x_{2k+2}, \mathcal{T}x_{2k+2})}{1 + d(x_{2k+1}, x_{2k+2})} \\ &+\gamma[d(x_{2k+1}, \mathcal{S}x_{2k+1}) + d(x_{2k+2}, \mathcal{T}x_{2k+2})] \\ &+\delta[d(x_{2k+1}, \mathcal{T}x_{2k+2}) + d(x_{2k+2}, \mathcal{S}x_{2k+1})] \\ &\precsim \alpha d(x_{2k+1}, x_{2k+2}) + \frac{\beta d(x_{2k}, x_{2k+2})d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k}, x_{2k+2})} \\ &+\gamma[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\ &+\delta[d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})] \\ &\precsim \alpha d(x_{2k+1}, x_{2k+2}) + \frac{\beta d(x_{2k}, x_{2k+2})d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k}, x_{2k+2})} \\ &+\gamma[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\ &+\delta d(x_{2k+1}, x_{2k+3}).\end{aligned}$$

This implies that

$$\begin{aligned}|d(x_{2k+2}, x_{2k+3})| &\leq \alpha|d(x_{2k+1}, x_{2k+2})| + \frac{\beta|d(x_{2k}, x_{2k+2})||d(x_{2k+2}, x_{2k+3})|}{|1 + d(x_{2k}, x_{2k+2})|} \\ &+\gamma[|d(x_{2k+1}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+3})|] \\ &+\delta|d(x_{2k+1}, x_{2k+3})|.\end{aligned}$$

Since

$$|d(x_{2k}, x_{2k+2})| \leq |1 + d(x_{2k}, x_{2k+2})|$$

and by the triangular inequality

$$|d(x_{2k+1}, x_{2k+3})| \leq |d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})|.$$

Therefore

$$\begin{aligned} |d(x_{2k+2}, x_{2k+3})| &\leq \alpha|(x_{2k+1}, x_{2k+2})| + \beta|d(x_{2k+2}, x_{2k+3})| \\ &\quad +\gamma[|d(x_{2k+1}, x_{2k+2})| + |d(x_{2k+2}, x_{2k+3})|] \\ &\quad +\delta[|d(x_{2k+1}, x_{2k+2})| + d(x_{2k+2}, x_{2k+3})|] \\ &= \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)|d(x_{2k+1}, x_{2k+2})|. \end{aligned}$$

Put $\delta = \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$, we have

$$|d(x_{n+1}, x_{n+2})| \leq \delta|d(x_n, x_{n+1})| \leq \cdots \leq \delta^{n+1}|d(x_0, x_1)|.$$

So that for any $m > n$,

$$\begin{aligned} |d(x_n, x_m)| &\leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \\ &\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}]|d(x_0, x_1)| \\ &\leq \frac{\delta^n}{1-\delta}|d(x_0, x_1)| \end{aligned}$$

and so

$$|d(x_m, x_n)| \leq \frac{\delta^n}{1-\delta}|d(x_0, x_1)| \to 0, \;\; as \;\; m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, there exists $u \in \mathcal{X}$ such that $x_n \to u$ as $n \to \infty$. Let on contrary $u \neq \mathcal{S}u$, so that $d(u, \mathcal{S}u) = z > 0$ and we can have

$$\begin{aligned} z &\precsim d(u, x_{2k+2}) + d(x_{2k+2}, \mathcal{S}u) \\ &\precsim d(u, x_{2k+2}) + d(\mathcal{T}x_{2k+1}, \mathcal{S}u) \\ &\precsim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta d(u, \mathcal{S}u)d(x_{2k+1}, \mathcal{T}x_{2k+1})}{1 + d(u, x_{2k+1})} \\ &\quad +\gamma[d(u, \mathcal{S}u) + d(x_{2k+1}, \mathcal{T}x_{2k+1})] + \delta[d(u, \mathcal{T}x_{2k+1}) + d(x_{2k+1}, \mathcal{S}u)] \\ &\precsim d(u, x_{2k+2}) + \alpha d(x_{2k+1}, u) + \frac{\beta d(u, \mathcal{S}u)d(x_{2k+1}, x_{2k+2})}{1 + d(u, x_{2k+1})} \\ &\quad +\gamma[d(u, \mathcal{S}u) + d(x_{2k+1}, x_{2k+2})] + \delta[d(u, x_{2k+2}) + d(x_{2k+1}, \mathcal{S}u)]. \end{aligned}$$

This implies that

$$\begin{aligned} |z| &\leq |d(u, x_{2k+2})| + \alpha|d(x_{2k+1}, u)| + \frac{\beta|z||d(x_{2k+1}, x_{2k+2})|}{|1 + d(u, x_{2k+1})|} \\ &\quad +\gamma[|z| + |d(x_{2k+1}, x_{2k+2})|] + \delta|d(u, x_{2k+2}) + d(x_{2k+1}, \mathcal{S}u)|. \end{aligned}$$

Passing limit to $n \to \infty$, one gets

$$|z| \leq (\gamma + \delta)|z| \leq (\alpha + \beta + 2\gamma + 2\delta)|z| < |z|,$$

which is a contradiction so that $u = \mathcal{S}u$. Similarly, one can show that $u = \mathcal{T}u$.

We now show that $\mathcal{S}$ and $\mathcal{T}$ have unique common fixed point. For this, assume that u^* in $\mathcal{X}$ is a second common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Then

$$\begin{aligned} d(u, u^*) &= d(\mathcal{S}u, \mathcal{T}u^*) \\ &\precsim \alpha d(u, u^*) + \frac{\beta d(u, \mathcal{S}u) d(u^*, \mathcal{T}u^*)}{1 + d(u, u^*)} + \gamma[d(u, \mathcal{S}u) + d(u^*, \mathcal{T}u^*)] \\ &\quad + \delta[d(u, \mathcal{T}u^*) + d(u^*, \mathcal{S}u)] \end{aligned}$$

so that

$$\begin{aligned} |d(u, u^*)| &\leq \alpha|d(u, u^*)| + \frac{\beta|d(u, \mathcal{S}u)||d(u^*, \mathcal{T}u^*)|}{|1 + d(u, u^*)|} \\ &\quad + \gamma[|d(u, \mathcal{S}u)| + |d(u^*, \mathcal{T}u^*)|] + \delta[|d(u, \mathcal{T}u^*)| + |d(u^*, \mathcal{S}u)|] \\ &= (\alpha + 2\delta)|d(u, u^*)|. \end{aligned}$$

This implies that $u^* = u$, which proves the uniqueness of common fixed point. This completes the proof of the Theorem 3.1. □

By setting $\mathcal{S} = \mathcal{T}$ in Theorem 3.1, one deduces the following:

Corollary 3.2. *Let $(\mathcal{X}, d)$ be a complete complex valued metric space and let the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy:*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &\precsim \alpha d(x, y) + \frac{\beta d(y, \mathcal{T}y) d(x, \mathcal{T}x)}{1 + d(x, y)} + \gamma[d(x, \mathcal{T}x) + d(y, \mathcal{T}y)] \\ &\quad + \delta[d(x, \mathcal{T}y) + d(y, \mathcal{T}x)] \end{aligned}$$

for all $x, y \in \mathcal{X}$ where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha + \beta + 2\gamma + 2\delta < 1$. Then $\mathcal{T}$ has a unique fixed point.

As an application of Theorem 3.1, we prove the following theorem for two finite families of mappings.

Theorem 3.3. *If $\{\mathcal{T}_i\}_1^m$ and $\{\mathcal{S}_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space $(\mathcal{X}, d)$ such that the mappings $\mathcal{S}$ and $\mathcal{T}$ (with $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2...\mathcal{T}_m$ and $\mathcal{S} = \mathcal{S}_1\mathcal{S}_2...\mathcal{S}_n$) satisfy the condition* (3.1), *then the component maps of the two families $\{\mathcal{T}_i\}_1^m$ and $\{\mathcal{S}_i\}_1^n$ have a unique common fixed point.*

Proof. In view of Theorem 3.1, one can infer that $\mathcal{T}$ and $\mathcal{S}$ have a unique common fixed point ℓ i.e., $\mathcal{T}\ell = \mathcal{S}\ell = \ell$. Now we are required to show that ℓ is common fixed point of all the components maps of both the families. In view of pairwise commutativity of the families $\{\mathcal{T}_i\}_1^m$ and $\{\mathcal{S}_i\}_1^n$, (for every $1 \leq k \leq m$) we can write

$$\mathcal{T}_k\ell = \mathcal{T}_k\mathcal{S}\ell = \mathcal{S}\mathcal{T}_k\ell \quad \text{and} \quad \mathcal{T}_k\ell = \mathcal{T}_k\mathcal{T}\ell = \mathcal{T}\mathcal{T}_k\ell$$

which show that $\mathcal{T}_k\ell$ (for every k) is also a common fixed point of $\mathcal{T}$ and $\mathcal{S}$. By using the uniqueness of common fixed point, we can write $\mathcal{T}_k\ell = \ell$ (for every k) which shows that ℓ is a common fixed point of the family $\{\mathcal{T}_i\}_1^m$. Using the foregoing arguments, one can also show that (for every $1 \leq k \leq n$) $\mathcal{S}_k\ell = \ell$. This completes the proof of the Theorem 3.3. □

By setting $\mathcal{T}_1 = \mathcal{T}_2 = \cdots = \mathcal{T}_m = \mathcal{F}$ and $\mathcal{S}_1 = \mathcal{S}_2 = \cdots = \mathcal{S}_n = \mathcal{G}$, in Theorem 3.3, we derive the following common fixed point theorem involving iterates of mappings.

Corollary 3.4. *If $\mathcal{F}$ and $\mathcal{G}$ are two commuting self-mappings defined on a complete complex valued metric space $(\mathcal{X}, d)$ satisfying the condition*

$$\begin{aligned} d(\mathcal{F}^m x, \mathcal{G}^n y) \quad \precsim \quad & \alpha d(x,y) + \frac{\beta d(x, \mathcal{F}^m x) d(y, \mathcal{G}^n y)}{1 + d(x,y)} + \gamma[d(x, \mathcal{G}^n x) + d(y, \mathcal{F}^m y)] \\ & + \delta[d(x, \mathcal{F}^m y) + d(y, \mathcal{G}^n x)] \end{aligned}$$

for all $x, y \in \mathcal{X}$ where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha + \beta + 2\gamma + 2\delta < 1$, then $\mathcal{F}$ and $\mathcal{G}$ have a unique common fixed point.

By setting $m = n$ and $\mathcal{F} = \mathcal{G} = \mathcal{T}$ in Corollary 3.4, we deduce the following corollary.

Corollary 3.5. *Let $(\mathcal{X}, d)$ be a complete complex valued metric space and let the mappings $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfy (for some fixed n):*

$$\begin{aligned} d(\mathcal{T}^m x, \mathcal{T}^n y) \quad \precsim \quad & \alpha d(x,y) + \frac{\beta d(x, \mathcal{T}^m x) d(y, \mathcal{T}^n y)}{1 + d(x,y)} + \gamma[d(x, \mathcal{T}^n x) + d(y, \mathcal{T}^m y)] \\ & + \delta[d(x, \mathcal{T}^m y) + d(y, \mathcal{T}^n x)] \end{aligned}$$

for all $x, y \in \mathcal{X}$ where $\alpha, \beta, \gamma, \delta$ are nonnegative reals with $\alpha + \beta + 2\gamma + 2\delta < 1$. Then $\mathcal{T}$ has a unique fixed point.

Proof. By Corollary 3.2, we obtain $v \in \mathcal{X}$ such that $\mathcal{T}^n v = v$. The result then follows from the fact that

$$\begin{aligned}
d(\mathcal{T}v, v) &= d(\mathcal{T}\mathcal{T}^n v, \mathcal{T}^n v) = d(\mathcal{T}^n \mathcal{T}v, \mathcal{T}^n v) \\
&\precsim \alpha d(\mathcal{T}v, v) + \frac{\beta d(\mathcal{T}v, \mathcal{T}^n \mathcal{T}v) d(v, \mathcal{T}^n v)}{1 + d(\mathcal{T}v, v)} \\
&\quad + \gamma[d(\mathcal{T}v, \mathcal{T}^n \mathcal{T}v) + d(v, \mathcal{T}^n v)] + \delta[d(\mathcal{T}v, \mathcal{T}^n v) + d(v, \mathcal{T}^n \mathcal{T}v)] \\
&\precsim \alpha d(\mathcal{T}v, v) + \frac{\beta d(\mathcal{T}v, \mathcal{T}\mathcal{T}^n v) d(v, v)}{1 + d(\mathcal{T}v, v)} \\
&\quad + \gamma[d(\mathcal{T}v, \mathcal{T}\mathcal{T}^n v) + d(v, v)] + \delta[d(\mathcal{T}v, v) + d(v, \mathcal{T}\mathcal{T}^n v)] \\
&= (\alpha + 2\delta) d(\mathcal{T}v, v).
\end{aligned}$$

□

We conclude this paper with an illustrative example which one demonstrates Theorem 3.1.

Example 3.6. *Consider*

$$\mathcal{X}_1 = \{z \in \mathbb{C} : Re(z) \geq 0, Im(z) = 0\},$$

$$\mathcal{X}_2 = \{z \in \mathbb{C} : Im(z) \geq 0, Re(z) = 0\}$$

and write $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$. *Define a mapping* $d : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ *as :*

$$d(z_1, z_2) = \begin{cases} \max\{x_1, x_2\} - i\max\{x_1, x_2\}, & z_1, z_2 \in \mathcal{X}_1, \\ \max\{y_1, y_2\} - i\max\{y_1, y_2\}, & z_1, z_2 \in \mathcal{X}_2, \\ (x_1 + y_2) - i(x_1 + y_2), & z_1 \in \mathcal{X}_1, z_2 \in \mathcal{X}_2, \\ (x_2 + y_1) - i(x_2 + y_1), & z_1 \in \mathcal{X}_2, z_2 \in \mathcal{X}_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. *By a simple calculation, one can easily verify that* $(\mathcal{X}, d)$ *is a complete complex valued metric space.*
Set $\mathcal{T} = \mathcal{S}$ *and define a self-mapping* $\mathcal{T}$ *on* $\mathcal{X}$ *(with* $z = (x, y)$*) as*

$$\mathcal{T}z = \begin{cases} \left(\frac{x}{3}, 0\right), & z \in \mathcal{X}_1, \\ \left(0, \frac{y}{3}\right), & z \in \mathcal{X}_2. \end{cases}$$

Now, we show that $\mathcal{T}$ *satisfies condition* (3.1). *We distinguish the following cases: Before discussing different cases, one needs to notice that*

$$0 \precsim d(\mathcal{S}z_1, \mathcal{T}z_2), d(z_1, z_2), \frac{d(z_1, \mathcal{S}z_1) d(z_2, \mathcal{T}z_2)}{1 + d(z_1, z_2)}, d(z_1, \mathcal{T}z_2) + d(z_2, \mathcal{S}z_1), \\ d(z_1, \mathcal{S}z_1) + d(z_2, \mathcal{T}z_2).$$

- *If* $z_1, z_2 \in \mathcal{X}_1$*, then we have*

$$\begin{aligned} d(\mathcal{S}z_1, \mathcal{T}z_2) &= d\left((\frac{x_1}{3}, 0), (\frac{x_2}{3}, 0)\right) = \max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\} - i\max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\} \\ &= \max\left\{\frac{x_1}{3}, \frac{x_2}{3}\right\}(1-i) = \frac{1}{3}\max\{x_1, x_2\}(1-i) \\ &\precsim \frac{1}{3}d(z_1, z_2). \end{aligned}$$

- *If* $z_1, z_2 \in \mathcal{X}_2$*, then we have*

$$\begin{aligned} d(\mathcal{S}z_1, \mathcal{T}z_2) &= d\left((0, \frac{y_1}{3}), (0, \frac{y_2}{3})\right) = \max\left\{\frac{y_1}{3}, \frac{y_2}{3}\right\} - i\max\left\{\frac{y_1}{3}, \frac{y_2}{3}\right\} \\ &= \max\left\{\frac{y_1}{3}, \frac{y_2}{3}\right\}(1+i) = \frac{1}{3}\max\{y_1, y_2\}(1-i) \\ &\precsim \frac{1}{3}d(z_1, z_2). \end{aligned}$$

- *If* $z_1 \in \mathcal{X}_1, z_2 \in \mathcal{X}_2$*, then we have*

$$\begin{aligned} d(\mathcal{S}z_1, \mathcal{T}z_2) &= d\left((\frac{x_1}{3}, 0), (0, \frac{y_2}{3})\right) = \left[\frac{x_1}{3} + \frac{y_2}{3}\right](1-i) \\ &= \frac{1}{3}[x_1 + y_2](1-i) = \frac{1}{3}[x_1 + y_2](1-i) \\ &\precsim \frac{1}{3}d(z_1, z_2). \end{aligned}$$

- *If* $z_2 \in \mathcal{X}_1, z_1 \in \mathcal{X}_2$*, then we have*

$$\begin{aligned} d(\mathcal{S}z_1, \mathcal{T}z_2) &= d\left((0, \frac{y_1}{3}), (\frac{x_2}{3}, 0)\right) = \left[\frac{y_1}{3} + \frac{x_2}{3}\right](1-i) \\ &= \frac{1}{3}[y_1 + x_2](1-i) = \frac{1}{3}[y_1 + x_2](1-i) \\ &\precsim \frac{1}{3}d(z_1, z_2). \end{aligned}$$

Thus, condition (3.1) *is satisfied with* $\alpha = \frac{1}{3}$ *and* $0 < \beta + 2\gamma + 2\delta < \frac{2}{3}$ *and, in all, conditions of Theorem* 3.1 *are satisfied. Notice that the point* $0 \in \mathcal{X}$ *remains fixed under* $\mathcal{T}$ *and is indeed unique. Thus, in all, this example substantiates the genuineness of our results proved in this paper.*

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