



## FIXED POINT THEOREMS IN $b$ -METRIC AND EXTENDED $b$ -METRIC SPACES

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**Abstract.** The first result of this paper is to give a revised proof of Sanatammappa et al.'s recent result in a  $b$ -metric space, under appropriate choice of constants without using the continuity of the  $b$ -metric. The second is to prove a fixed point theorem under a contraction type condition in an extended  $b$ -metric space.

### 1. INTRODUCTION

Let  $X$  be a nonempty set and  $\rho : X \times X \rightarrow \mathbb{R}$  be such that

- (m1)  $\rho(x, y) \geq 0$  for all  $x, y \in X$ ,
- (m2)  $\rho(x, y) = 0$  if and only if  $x = y$ ,
- (m3)  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in X$ ,
- (m4)  $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$  for all  $x, y, z \in X$ .

Then the pair  $(X, \rho)$  denotes a metric space with metric  $\rho$ . Let  $X = \mathbb{R}$ . Then the metric  $\rho(x, y) = |x - y|$  for all  $x, y \in X$  is called the *usual metric* and it

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gives the *distance* between the points  $x$  and  $y$  on the number line  $\mathbb{R}^1$ . Let  $X = \mathbb{R} \times \mathbb{R}$  and  $\rho(x, y) = |x - y|$  for all  $x, y \in X$ . Condition (m4) says that the length of one side in a triangle with vertices  $x, y$  and  $z$  never exceeds the sum of the lengths of other sides in it. Hence it is referred to as the *triangle inequality* of the metric  $\rho$ . The notion of metric space was due to Frechet in 1906.

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc., while the conditions imposed on the underlying mappings are usually metrical or compact type conditions. Further, new ambient algebraic structures were formulated to improve the results. One such was a  $b$ -metric, introduced by Bakhtin [1], by generalizing the triangle inequality (m4). For all the definitions of this section, one can refer to [2, 7, 8]:

**Definition 1.1.** ([2]) Let  $s \geq 1$ ,  $X$  be a nonempty set and  $\rho_s : X \times X \rightarrow [0, \infty)$  be such that

- (b1)  $\rho_s(x, y) = 0$  if and only if  $x = y$  for all  $x, y \in X$ ,
- (b2)  $\rho_s(x, y) = \rho_s(y, x)$  for all  $x, y \in X$ ,
- (b3)  $\rho_s(x, y) \leq s[\rho_s(x, z) + \rho_s(y, z)]$  for all  $x, y, z \in X$ .

Then  $\rho_s$  is a  $b$ -metric on  $X$ , and  $(X, \rho_s)$  denotes a  $b$ -metric space.

A  $b$ -metric space  $(X, \rho_s)$  reduces to a metric space  $(X, \rho)$ , if  $s = 1$ . However, a  $b$ -metric space is not necessarily a metric space. For instance, consider the pair  $(X, \rho_s)$ , where  $X = \mathbb{R}$  and  $\rho_s(x, y) = |x - y|^2$  for all  $x, y \in \mathbb{R}$ . Then the conditions (b1) and (b2) are obvious. Further,

$$\begin{aligned} \rho_s(x, y) &= |x - y|^2 = |x - z + z - y|^2 \\ &\leq 2(|x - z|^2 + |z - y|^2) \\ &= 2[\rho_s(x, z) + \rho_s(y, z)] \end{aligned}$$

for all  $x, y \in X$ . Thus  $(X = \mathbb{R}, \rho_s)$  is a  $b$ -metric space with  $s = 2$ . Since  $\rho_s(1, 3) + \rho_s(1, 0) = 4 + 1 = 5$  and  $\rho_s(0, 3) = 9$ , (m3) fails to hold good, showing that  $\rho_s$  is not a metric. Thus a  $b$ -metric space is not a metric space.

In view of the convexity of  $f(x) = x^p$ , where  $x > 0$  and  $1 < p < \infty$ , it follows that  $(\mathbb{R}, |x - y|^p)$  is a  $b$ -metric space, which is not a metric space. In other words, the class of  $b$ -metric spaces contains that of metric spaces.

**Definition 1.2.** ([2]) A  $b$ -ball in a  $b$ -metric space  $(X, \rho_s)$  is defined by

$$B_{\rho_s}(x, r) = \{y \in X : \rho_s(x, y) < r\}.$$

The family of all  $b$ -balls forms a base topology, called the  *$b$ -metric topology*  $\tau(\rho_s)$  on  $X$ .

**Definition 1.3.** ([2]) Let  $(X, \rho_s)$  be a  $b$ -metric space with parameter  $s$ . A sequence  $\{x_n\}_{n=1}^\infty$  in  $X$  is said to be

- (a)  $b$ -convergent, with limit  $p \in X$ , if it converges to  $p$  in the  $b$ -metric topology  $\tau(\rho_s)$ ,
- (b)  $b$ -Cauchy, if  $\lim_{n,m \rightarrow \infty} \rho_s(x_n, x_m) = 0$ .

Like in a metric space, every  $b$ -convergent sequence has a unique limit, and is necessarily  $b$ -Cauchy.

**Definition 1.4.** ([2]) A  $b$ -metric space  $(X, \rho_s)$  is said to be  $b$ -complete, if every  $b$ -Cauchy sequence in  $X$  is  $b$ -convergent in it.

**Remark 1.5.** A  $b$ -metric is not jointly continuous in general in its coordinate variables  $x$  and  $y$ , though a metric  $d$  is known to be continuous (See Example 2.13, [9]).

**Lemma 1.6.** ([4]) Let  $(X, \rho_s)$  be a  $b$ -metric space with parameter  $s$ . Suppose that  $\{x_n\}_{n=1}^\infty$  is  $b$ -convergent with limit  $x$  and  $\{y_n\}_{n=1}^\infty$  is  $b$ -convergent with limit  $y$  in  $X$ . Then

$$\frac{1}{s^2} \rho_s(x, y) \leq \liminf_{n \rightarrow \infty} \rho_s(x_n, y_n) \leq \limsup_{n \rightarrow \infty} \rho_s(x_n, y_n) \leq s^2 \rho_s(x, y). \quad (1.1)$$

In particular,  $x = y$ , then  $\lim_{n \rightarrow \infty} \rho_s(x_n, y_n) = 0$ . Further, for each  $z \in X$ , we have

$$\frac{1}{s} \rho_s(x, z) \leq \liminf_{n \rightarrow \infty} \rho_s(x_n, z) \leq \limsup_{n \rightarrow \infty} \rho_s(x_n, z) \leq s \rho_s(x, z). \quad (1.2)$$

## 2. A MODIFIED PROOF OF SANATAMMAPPA ET AL.'S RESULT

Sanatammappa et al. [6] recently proved the following result:

**Theorem 2.1.** Let  $s \geq 1$  and  $(X, \rho_s)$  be a complete  $b$ -metric space. If a self-map  $f$  on  $X$  is such that

$$\begin{aligned} \rho_s(fx, fy) \leq & a_1 \rho_s(x, y) + a_2 \rho_s(x, fx) + a_3 \rho_s(y, fy) + a_4 \rho_s(x, fy) \\ & + a_5 \rho_s(y, fx) + a_6 [\rho_s(y, fx) + \rho_s(x, fy)], \quad \forall x, y \in X, \end{aligned} \quad (2.1)$$

where  $a_j$ ,  $1 \leq j \leq 6$  are non-negative real numbers, not all zero, with

$$a_1 + a_2 + a_3 + 2a_4 + a_5 + 2a_6 < 1. \quad (2.2)$$

Then the sequence  $\{x_n\}_{n=1}^\infty$  defined by

$$x_n = fx_{n-1} = f^n x_0, \quad n \geq 1, \quad (2.3)$$

converges to a point  $p \in X$ , which is a unique fixed point of  $f$ .

In the proof of Theorem 2.1, the authors obtained that for each  $m > n$ ,

$$\rho_s(x_n, x_m) \leq \omega^n \rho_s(x_0, f x_0), \quad \forall n, \quad (2.4)$$

where

$$\omega = \frac{a_1 + a_2 + sa_4 + sa_6}{1 - a_3 - sa_4 - sa_6}. \quad (2.5)$$

Employing the limit as  $n \rightarrow \infty$  in (2.4), they concluded that  $\rho_s(x_n, x_m) \rightarrow 0$ .

We assert that  $\rho_s(f^n x_0, f^m x_0) \rightarrow 0$  as  $n \rightarrow \infty$  holds good only if  $\omega < 1$ . But the choice (2.2) does not guarantee to give  $\omega < 1$ . Also, the authors used the continuity of the  $b$ -metric  $\rho_s$  in Theorem 2.1, without mentioning it. However, in view of Remark 1.5,  $\rho_s$  is not continuous. Therefore, we restate Theorem 2.1 as follows:

**Theorem 2.2.** *Let  $s \geq 1$ ,  $(X, \rho_s)$  be a complete  $b$ -metric space, and  $f : X \rightarrow X$  satisfy the inequality (2.1), where  $a_j$ ,  $1 \leq j \leq 6$  are nonnegative real numbers, not all zero, such that*

$$a_1 + a_2 + a_3 + 2sa_4 + a_5 + 2sa_6 < 1. \quad (2.6)$$

*Then the sequence  $\{x_n\}_{n=1}^{\infty}$  defined by (2.3) converges to a point  $p \in X$ , which is a unique fixed point of  $f$ .*

*Proof.* Given  $x_0 \in X$ , consider the sequence  $\{x_n\}_{n=1}^{\infty}$  with the choice (2.3). Writing  $x = x_{n-1}$  and  $y = x_n$  in (2.1) and then using (2.3) and (b3),

$$\begin{aligned} \rho_s(x_n, x_{n+1}) &= \rho_s(f x_{n-1}, f x_n) \\ &\leq a_1 \rho_s(x_{n-1}, x_n) + a_2 \rho_s(x_{n-1}, f x_{n-1}) + a_3 \rho_s(x_n, f x_n) \\ &\quad + a_4 \rho_s(x_{n-1}, f x_n) + a_5 \rho_s(x_n, f x_{n-1}) \\ &\quad + a_6 [\rho_s(x_n, f x_{n-1}) + \rho_s(x_{n-1}, f x_n)] \\ &= a_1 \rho_s(x_{n-1}, x_n) + a_2 \rho_s(x_{n-1}, x_n) + a_3 \rho_s(x_n, x_{n+1}) \\ &\quad + a_4 \rho_s(x_{n-1}, x_{n+1}) + a_5 \rho_s(x_n, x_n) \\ &\quad + a_6 [\rho_s(x_n, x_n) + \rho_s(x_{n-1}, x_{n+1})] \\ &\leq (a_1 + a_2) \rho_s(x_{n-1}, x_n) + a_3 \rho_s(x_n, x_{n+1}) \\ &\quad + a_4 \cdot s [\rho_s(x_{n-1}, x_n) + \rho_s(x_n, x_{n+1})] + a_5 \cdot 0 \\ &\quad + a_6 \cdot s [\rho_s(x_{n-1}, x_n) + \rho_s(x_n, x_{n+1})]. \end{aligned}$$

Rearranging the terms and simplifying, this gives

$$(1 - a_3 - sa_4 - sa_6) \rho_s(x_n, x_{n+1}) \leq (a_1 + a_2 + sa_4 + sa_6) \rho_s(x_{n-1}, x_n)$$

or

$$\rho_s(x_n, x_{n+1}) \leq \omega \rho_s(x_{n-1}, x_n), \quad \forall n,$$

where  $\omega$  is given by (2.5). By induction,

$$\rho_s(x_n, x_{n+1}) \leq \omega^n \rho_s(x_0, x_1) \quad \text{for all } n \geq 1. \tag{2.7}$$

Therefore, for all  $m > n$ , repeatedly using (b3) and (2.7), we have

$$\begin{aligned} \rho_s(x_n, x_m) &\leq s [\rho_s(x_n, x_{n+1}) + \rho_s(x_{n+1}, x_m)] \\ &\leq s \rho_s(x_n, x_{n+1}) + s^2 [\rho_s(x_{n+1}, x_{n+2}) + \rho_s(x_{n+2}, x_m)] \\ &\vdots \\ &\leq \underbrace{s \rho_s(x_n, x_{n+1}) + s^2 \rho_s(x_{n+1}, x_{n+2}) + \dots + s^{m-n} \rho_s(x_{m-1}, x_m)}_{m-n \text{ terms}} \\ &\leq [s\omega^n + s^2\omega^{n+1} + \dots + s^{m-n}\omega^{m-1}] \rho_s(x_0, x_1) \\ &= s\omega^n (1 + s\omega + \dots + s^{m-n-1}\omega^{m-n-1}) \rho_s(x_0, x_1) \\ &\leq \frac{s\omega^n}{1 - s\omega} \cdot \rho_s(x_0, x_1) \quad \text{for all } n. \end{aligned}$$

As  $m, n \rightarrow \infty$ , this implies that  $\rho_s(x_m, x_n) \rightarrow 0$ . Thus  $\{x_n\}_{n=1}^\infty$  is a  $b$ -Cauchy sequence in  $X$ . Since  $X$  is  $b$ -complete, there exists a point  $z \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f^n x_0 = z. \tag{2.8}$$

Now we show that  $z$  is a fixed point of  $f$ . In fact, writing  $x = x_n$  and  $y = z$  in (2.1), the we have

$$\begin{aligned} \rho_s(fx_n, fz) &\leq a_1 \rho_s(x_n, z) + a_2 \rho_s(x_n, fx_n) + a_3 \rho_s(z, fz) \\ &\quad + a_4 \rho_s(x_n, fz) + a_5 \rho_s(z, fx_n) \\ &\quad + a_6 [\rho_s(z, fx_n) + \rho_s(x_n, fz)] \end{aligned}$$

or

$$\begin{aligned} \rho_s(x_{n+1}, fz) &\leq a_1 \rho_s(x_n, z) + a_2 \rho_s(x_n, x_{n+1}) + a_3 \rho_s(z, fz) \\ &\quad + a_4 \rho_s(x_n, fz) + a_5 \rho_s(z, x_{n+1}) \\ &\quad + a_6 [\rho_s(z, x_{n+1}) + \rho_s(x_n, fz)] \quad \text{for all } n. \end{aligned}$$

Employing the limit superior as  $n \rightarrow \infty$  in this and using Lemma 1.6, this gives

$$\begin{aligned} \frac{1}{s^2} \rho_s(z, fz) &\leq \limsup_{n \rightarrow \infty} \rho_s(x_{n+1}, fz) \\ &\leq a_3 \rho_s(z, fz) + (a_4 + a_6) \limsup_{n \rightarrow \infty} \rho_s(x_{n+1}, fz) \\ &\leq a_3 \rho_s(z, fz) + (a_4 + a_6) s \rho_s(z, fz). \end{aligned}$$

If  $\rho_s(z, fz) > 0$ , this would give a contradiction that

$$0 < \frac{1}{s^2} \rho_s(z, fz) \leq (a_3 + sa_4 + sa_6) \rho_s(z, fz) < \rho_s(z, fz).$$

Therefore,  $\rho_s(z, fz) = 0$ . The uniqueness of the fixed point is easily established.  $\square$

Writing  $a_2 = a_3 = a_6 = 0$  in Theorem 2.2, we have:

**Corollary 2.3.** *Let  $s \geq 1$ ,  $(X, \rho_s)$  be a complete  $b$ -metric space, and  $f : X \rightarrow X$  satisfy the inequality*

$$\rho_s(fx, fy) \leq \alpha \rho_s(x, y) + \beta \rho_s(x, fy) + \gamma \rho_s(y, fx) \text{ for all } x, y \in X, \quad (2.9)$$

where  $0 < \alpha + \beta s + \gamma < 1$ . Then  $f$  has a unique fixed point.

### 3. RESULT IN EXTENDED $b$ -METRIC SPACE

Kamran et al. ([3]) generalized a  $b$ -metric space as an extended  $b$ -metric space as follows:

**Definition 3.1.** Let  $s \geq 1$ ,  $X$  be a nonempty set and  $\theta : X \times X \rightarrow [1, \infty)$ . Consider  $\rho_\theta : X \times X \rightarrow [0, \infty)$  such that

- (eb1)  $\rho_\theta(x, y) = 0$  for all  $x, y \in X$ ,
- (eb2)  $\rho_\theta(x, y) = 0$  implies that  $x = y$  for all  $x, y \in X$ ,
- (eb3)  $\rho_\theta(x, y) = \rho_\theta(y, x)$  for all  $x, y \in X$ ,
- (eb4)  $\rho_\theta(x, y) \leq \theta(x, y)[\rho_\theta(x, z) + \rho_\theta(z, y)]$  for all  $x, y, z \in X$ .

Then  $\rho_\theta$  is called an extended  $b$ -metric on  $X$ , and  $(X, \rho_\theta)$  is an extended  $b$ -metric space.

If  $\theta(x, y) = s \geq 1$  for all  $x, y \in X$ , then  $\rho_\theta$  reduces to a  $b$ -metric  $\rho_s$ . In this paper, we denote an extended  $b$ -metric space by  $(X, \rho_\theta)$ .

The notions of convergence and completeness in an extended  $b$ -metric space are similar to that in a  $b$ -metric space.

**Definition 3.2.** A sequence  $\{x_n\}_{n=1}^\infty \subset X$  is said to be convergent to  $z \in X$ , written as  $\lim_{n \rightarrow \infty} x_n = z$ , if for every  $\epsilon > 0$  there exists a natural number  $N$  such that  $\rho_\theta(x_n, z) < \epsilon$  for all  $n \geq N$ .

If  $\rho_\theta$  is continuous, then every convergent sequence in  $X$  has a unique limit in it.

**Definition 3.3.** A sequence  $\{x_n\}_{n=1}^\infty \subset X$  is said to be Cauchy, if for every  $\epsilon > 0$  there exists a natural number  $N$  such that  $\rho_\theta(x_n, x_m) < \epsilon$  for all  $m, n \geq N$ .

A Cauchy sequence in  $X$  need not be convergent in it. But, if Cauchy sequence in  $X$  is convergent in it, then we say that  $X$  is complete. Banach contraction mapping theorem in an extended  $b$ -metric space was proved in [3]. We establish fixed point theorems for some contraction types other than Banach's, in an extended  $b$ -metric space. In this sequel, we employ the following notion:

**Definition 3.4.** Let  $f$  be a self-map on an extended  $b$ -metric space  $(X, \rho_\theta)$  and  $x_0 \in X$ . Then the orbit  $O_f(x_0)$  at  $x_0$  is the sequence of  $f$ -iterates  $x_0, fx_0, \dots, f^n x_0, \dots$ .

We need the following lemmas:

**Lemma 3.5.** ([5, Theorem 3.22, p. 59]) The infinite series  $\sum_{n=1}^\infty u_n$  of positive terms converges if and only if, given  $\epsilon > 0$ , there is a natural number  $n_0$  such that  $\sum_{j=n}^m u_n \leq \epsilon$  for all  $m \geq n \geq n_0$ .

**Lemma 3.6.** ([5, Theorem 3.34, p. 66]) The infinite series  $\sum_{n=1}^\infty u_n$  of positive terms converges, provided  $\limsup_{n \rightarrow \infty} (u_{n+1}/u_n) < 1$ .

Our main result in extended  $b$ -metric space  $(X, \rho_\theta)$ , where  $\rho_\theta$  is continuous, is the following:

**Theorem 3.7.** Let  $(X, \rho_\theta)$  be a complete extended  $b$ -metric space, where  $\rho_\theta$  is continuous. Suppose that  $f : X \rightarrow X$  satisfies the condition

$$\rho_\theta(fx, fy) \leq \alpha\rho_\theta(x, y) + \beta\rho_\theta(x, fy) + \gamma\rho_\theta(y, fx) \text{ for all } x, y \in X, \tag{3.1}$$

where  $0 < \alpha + 2\beta + \gamma < 1$  is such that for each  $x_0 \in X$ ,

$$\lim_{n, m \rightarrow \infty} \left\{ \frac{\alpha + \beta\theta(f^n x_0, f^{n+2} x_0)}{1 - \beta\theta(f^n x_0, f^{n+2} x_0)} \right\} \theta(f^{n+1} x_0, f^m x_0) < 1. \tag{3.2}$$

Then  $f$  has a unique fixed point.

*Proof.* Let  $x_0 \in X$  be arbitrary. Define  $\{x_n\}_{n=1}^\infty \subset X$  by

$$x_n = fx_{n-1} \text{ for } n \geq 1. \tag{3.3}$$

By induction, (3.3) gives

$$x_1 = fx_0, x_2 = f^2x_0, \dots, x_n = f^n x_0 \text{ for } n \geq 1. \tag{3.4}$$

Now writing  $x = x_{n-1}$  and  $y = x_n$  in (3.1) and using (3.4) and (eb4), we find that

$$\begin{aligned} \rho_\theta(x_n, x_{n+1}) &= \rho_\theta(fx_{n-1}, fx_n) \\ &\leq \alpha\rho_\theta(x_{n-1}, x_n) + \beta\rho_\theta(x_{n-1}, fx_n) + \gamma\rho_\theta(x_n, fx_{n-1}) \\ &= \alpha\rho_\theta(x_{n-1}, x_n) + \beta\rho_\theta(x_{n-1}, x_{n+1}) \\ &\leq \alpha\rho_\theta(x_{n-1}, x_n) + \beta\theta(x_{n-1}, x_{n+1})[\rho_\theta(x_{n-1}, x_n) + \rho_\theta(x_n, x_{n+1})]. \end{aligned}$$

Rearranging and simplifying, this gives

$$\rho_\theta(x_n, x_{n+1}) \leq \frac{\alpha + \beta\theta(x_{n-1}, x_{n+1})}{1 - \beta\theta(x_{n-1}, x_{n+1})} \cdot \rho_\theta(x_{n-1}, x_n).$$

By induction, it follows that

$$\rho_\theta(x_n, x_{n+1}) \leq \psi_n \cdot \rho_\theta(x_0, x_1), \quad n = 1, 2 \dots, \tag{3.5}$$

where

$$\psi_n = \prod_{j=1}^n \left\{ \frac{\alpha + \beta\theta(x_{j-1}, x_{j+1})}{1 - \beta\theta(x_{j-1}, x_{j+1})} \right\} \text{ for all } n. \tag{3.6}$$

Now for  $m > n$ , by using (eb4) repeatedly and (3.6), we obtain

$$\begin{aligned} \rho_\theta(x_n, x_m) &\leq \theta(x_n, x_m) [\rho_\theta(x_n, x_{n+1}) + \rho_\theta(x_{n+1}, x_m)] \\ &\leq \theta(x_n, x_m) [\psi_n \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+1}, x_m)] \\ &\vdots \\ &\leq \rho_\theta(x_0, x_1) \theta(x_n, x_m) [\psi_n + \psi_{n+1} \cdot \theta(x_{n+1}, x_m) \\ &\quad + \dots + \psi_{m-1} \cdot \theta(x_{n+1}, x_m) \theta(x_{n+2}, x_m) \dots \theta(x_{m-1}, x_m)]. \end{aligned}$$

Since,  $\theta(x, y) \geq 1$  for all  $x$  and  $y$ , this can be written as

$$\begin{aligned} \rho_\theta(x_n, x_m) &\leq \rho_\theta(x_0, x_1) [\psi_n \cdot \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_n, x_m) \\ &\quad + \psi_{n+1} \cdot \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_n, x_m) \theta(x_{n+1}, x_m) \\ &\quad + \dots + \psi_{m-1} \cdot \theta(x_1, x_m) \theta(x_2, x_m) \dots \theta(x_n, x_m) \\ &\quad \times \theta(x_{n+1}, x_m) \theta(x_{n+2}, x_m) \dots \theta(x_{m-1}, x_m)]. \end{aligned} \tag{3.7}$$

Consider the series  $P = \sum_{n=1}^\infty \psi_n \cdot \prod_{i=1}^n \theta(x_i, x_m)$  for each  $m \geq 1$ . Write  $v_n = \psi_n \cdot \prod_{i=1}^n \theta(x_i, x_m)$  for each  $m$  and  $n \geq 1$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{v_{n+1}}{v_n} &= \lim_{n \rightarrow \infty} \frac{\psi_{n+1} \cdot \prod_{i=1}^{n+1} \theta(x_i, x_m)}{\psi_n \cdot \prod_{i=1}^n \theta(x_i, x_m)} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{\alpha + \beta\theta(x_n, x_{n+2})}{1 - \beta\theta(x_n, x_{n+2})} \right\} \theta(x_{n+1}, x_m) \text{ for each } m. \end{aligned} \tag{3.8}$$

Now, from (3.2), we find that  $\lim_{n \rightarrow \infty} \frac{v_{n+1}}{v_n} < 1$ , and hence in view of Lemma 3.6, the series  $P$  converges. Also, the partial sums of  $P$ , given by

$$P_n = \sum_{j=1}^n \psi_j \cdot \prod_{i=1}^j \theta(x_i, x_m), \quad n = 1, 2, \dots, \text{ for each } m \tag{3.9}$$

are bounded. Using (3.9) in (3.7), it follows that

$$\rho_\theta(x_n, x_m) \leq \rho_\theta(x_0, x_1) (P_{m-1} - P_n) \text{ for } m > n. \tag{3.10}$$



Given  $\epsilon > 0$ , using the convergence of  $P$  and Lemma 3.5, (3.10) implies that

$$\rho_\theta(x_n, x_m) \leq \rho_\theta(x_0, x_1)\epsilon \quad \text{for } m > n \geq n_0, \tag{3.11}$$

for some natural number  $n_0$ . Thus  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence in  $X$ . Since  $X$  is complete, we can find a point  $z \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} f x_{n-1} = z. \tag{3.12}$$

Now we establish that  $z$  is a fixed point of  $f$ . In fact, writing  $x = x_{n-1}$  and  $y = z$ , the inequality (3.1) gives

$$\rho_\theta(f x_{n-1}, f z) \leq \alpha \rho_\theta(x_{n-1}, z) + \beta \rho_\theta(x_{n-1}, f z) + \gamma \rho_\theta(z, f x_{n-1})$$

or

$$\rho_\theta(x_n, f z) \leq \alpha \rho_\theta(x_{n-1}, z) + \beta \rho_\theta(x_{n-1}, f z) + \gamma \rho_\theta(z, x_n).$$

Applying the limit as  $n \rightarrow \infty$  and using (3.12) and the continuity of  $\rho_\theta$ , we obtain

$$\rho_\theta(z, f z) \leq \beta \rho_\theta(z, f z)$$

or  $\rho_\theta(z, f z) = 0$ . That is,  $f z = z$ .

To establish the uniqueness of the fixed point, let  $q \neq z$  be also a fixed point of  $f$ . Then with  $x = z$  and  $y = z$  in (3.1),

$$\begin{aligned} 0 < \rho_\theta(z, q) &= \rho_\theta(f z, f q) \leq \alpha \rho_\theta(z, q) + \beta \rho_\theta(z, f q) + \gamma \rho_\theta(q, f z) \\ &= (\alpha + \beta + \gamma) \rho_\theta(q, z) < \rho_\theta(z, f q), \end{aligned}$$

which is a contradiction. Hence  $z = q$ , and the fixed point is unique. □

Writing  $\beta = 0$  and  $\gamma = 0$  in Theorem 3.7, we get the following version of Banach's contraction mapping theorem in extended  $b$ -metric space, proved in [3]:

**Corollary 3.8.** *Let  $(X, \rho_\theta)$  be a complete extended  $b$ -metric space, where  $\rho_\theta$  is continuous. Suppose that  $f : X \rightarrow X$  satisfies the condition*

$$\rho_\theta(f x, f y) \leq \alpha \rho_\theta(x, y) \quad \text{for all } x, y \in X, \tag{3.13}$$

where  $0 < \alpha < 1$  is such that for each  $x_0 \in X$ ,

$$\lim_{n, m \rightarrow \infty} \alpha \theta(f^{n+1} x_0, f^m x_0) < 1. \tag{3.14}$$

Then  $f$  has a unique fixed point.

Now, writing  $\theta(x, y) = s \geq 1$  in Theorem 3.7, we obtain Corollary 2.3. It may be noted from the proof of Theorem 2.2 that the continuity of  $\rho_s(x, y)$  is not needed to obtain a unique fixed point.

## CONCLUSION

In the first result of this paper, a revised proof has been given to a recent theorem of Sanatammappa et al. in a  $b$ -metric space, under appropriate choice of constants, without using the continuity of the  $b$ -metric. As a second result, a general fixed point theorem has been proved in an extended  $b$ -metric space under a contraction type condition.

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