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INEQUALITIES FOR COMPLEX POLYNOMIAL WITH RESTRICTED ZEROS

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Abstract. Let $p(z)$ be a polynomial of degree n and for any complex number β, let $D_\beta p(z) = np(z) + (\beta - z)p'(z)$ denote the polar derivative of the polynomial with respect to β. In this paper, we consider the class of polynomial

$$p(z) = (z - z_0)^s \left(a_0 + \sum_{\nu=0}^{n-s} a_\nu z^\nu \right)$$

of degree n having a zero of order s at z_0, $|z_0| < 1$ and the remaining $n - s$ zeros are outside $|z| < k$, $k \geq 1$ and establish upper bound estimates for the maximum of $|D_\beta p(z)|$ as well as $|p(Rz) - p(rz)|$, $R \geq r \geq 1$ on the unit disk.

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1. Introduction

The extremal problems of functions of complex variables and the results where some approaches to obtaining the classical inequalities are developed on using various methods of the geometric function theory are known for various constraints, and on various regions of the complex plane. A classical result due to Bernstein [5], that relates an estimate of the size of the derivative and the polynomial for the sup-norm on the unit circle states that: if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)| \leq n \max_{|z|=1}|p(z)|. \tag{1.1}$$

Inequality (1.1) was proved by Bernstein in 1912. And a simple deduction from the maximum modulus principle for $R \geq 1$ gives

$$\max_{|z|=1}|p(Rz)| \leq R^n \max_{|z|=1}|p(z)|\,. \tag{1.2}$$

Both inequalities (1.1) and (1.2) are sharp and the equality holds if and only if $p(z)$ has all its zeros at the origin.

Later in 1985, Frappier et al. [10] strengthened (1.1), by proving that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)| \leq n \max_{1\leq l\leq 2n}|p(e^{il\pi/n})|. \tag{1.3}$$

Clearly (1.3) represents a refinement of (1.1), since the maximum of $|p(z)|$ on $|z| = 1$ may be larger than the maximum of $|p(z)|$ taken over the $(2n)^{th}$ roots of unity, as is shown by the simple example $p(z) = z^n + ia, a > 0$. Following the approach of Frappier et al. [10] and Aziz [1] showed that the bounds in (1.3) and (1.2) can be considerably improved. In fact, Aziz proved that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{2}(M_\alpha + M_{\alpha+\pi}) \tag{1.4}$$

and for $R > 1$

$$\max_{|z|=1}|p(Rz) - p(z)| \leq \frac{R^n - 1}{2}(M_\alpha + M_{\alpha+\pi}), \tag{1.5}$$

where throughout the paper

$$M_\alpha = \max_{1\leq l\leq 2n}|p(e^{i(\alpha+2l\pi)/n})| \tag{1.6}$$

for all real α.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.7}$$

Inequality (1.7) was conjectured by Erdös and later proved by Lax [13]. Improvements and generalizations of inequality (1.7) were given by Aziz and Dawood [2], Malik [14], Govil [11], Dewan and Bidkham [9], Chan and Malik [6], Pukhta [25], Aziz and Zargar [4], Aziz and Shah [3], Qazi [26], Chanam and Dewan [7], Chanam et al. [8] etc.

Ideally, it is natural to look for improvements of (1.4) and (1.5) when $p(z)$ does not vanish in the unit disc, and accordingly Aziz [1] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every real number α,

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}} \tag{1.8}$$

and also improvement of inequality (1.5) for every real number α and $R > 1$

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{2} \left(M_\alpha^2 + M_{\alpha+\pi}^2\right)^{\frac{1}{2}}. \tag{1.9}$$

Recently, Nakprasit and Somsuwan [24] investigated $\max\limits_{|z|=1} |p'(z)|$ in terms of $\max\limits_{|z|=1} |p(z)|$ for a polynomial $p(z)$ of degree n having a zero of order s at z_0, where $|z_0| < 1$ and proved:

Theorem 1.1. ([24]) *If $p(z) = (z - z_0)^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$ except a zero of order s, $0 \leq s < n$ at z_0, where $|z_0| < 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \left\{ \frac{s}{1 - |z_0|} + \frac{A}{(1 - |z_0|)^s} \right\} \max_{|z|=1} |p(z)| - \frac{A}{(k + |z_0|)^s} \min_{|z|=k} |p(z)|, \tag{1.10}$$

where

$$A = \frac{(1 + |z_0|)^{s+1}(n - s)}{(1 + k^\mu)(1 - |z_0|)}.$$

It is important to mention that different versions of the Bernstein type inequalities have appeared in the literature in more generalized forms in which the underlying polynomial is replaced by more general classes of functions. These inequalities have their own significance and importance in approximation theory. One of such generalizations is moving from the domain of ordinary derivative of polynomials to their polar derivative. Here, we remind that the polar derivative $D_\beta p(z)$ of $p(z)$ where $p(z)$ is a polynomial of degree n, with respect to the point β is defined as

$$D_\beta p(z) = np(z) + (\beta - z)p'(z).$$

Note that $D_\beta p(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $p(z)$ with respect to β. It generalizes the ordinary derivative in the sense that

$$\lim_{\beta\to\infty}\left\{\frac{D_\beta p(z)}{\beta}\right\} = p'(z),$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

One very important point worth to note is that derivative is a very general term applied to any function whereas the polar derivative mentioned above is defined for or referred to a polynomial only. Because in the definition of polar derivative of a polynomial, the degree of the polynomial plays a central role.

More information on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et. al. [16] and Rahmann and Schmeisser [27].

Over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of the largest root of $p(z)$, restrictions on the coefficients, using higher order derivatives, etc. Many of these generalizations involve the comparison of polar derivative $D_\beta p(z)$ with various choices of $p(z)$, β and other parameters. The latest research and development on this topic can be found in the papers([12, 15, 17, 18, 19, 21, 22, 23, 28]).

2. Lemmas

We shall need the following lemmas in order to prove the theorems. For a polynomial $p(z)$ of degree n, we will use $q(z) = z^n\overline{p\left(\frac{1}{\bar z}\right)}$.

Lemma 2.1. ([20]) *If $\{y_\nu\}$, $\nu = 1, 2, \ldots, n$ is a sequence of real numbers such that $y_\nu \ge 1$ for all $\nu \in \mathbb{N}$, then*

$$\sum_{\nu=1}^{n}\frac{y_\nu - 1}{y_\nu + 1} \ge \frac{\prod_{\nu=1}^{n} y_\nu - 1}{\prod_{\nu=1}^{n} y_\nu + 1} \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Lemma 2.2. *If $p(z) = (z - z_0)^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$, is a polynomial of degree n having no zero in $|z| < k$, $k \ge 1$, except a zero of multiplicity s at z_0, $|z_0| < 1$, then* *for each point z on $|z| = 1$ for which $p(z) \neq 0$, we have*

$$Re\left(\frac{zp'(z)}{p(z)}\right) \leq \frac{s}{1-|z_0|} + \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right). \tag{2.2}$$

Proof. Let $p(z) = (z - z_0)^s \phi(z)$, where $\phi(z) = \sum_{\nu=0}^{n-s} a_\nu z^\nu$ is a polynomial of degree $n - s$ having no zero in $|z| < k$, $k \geq 1$. Therefore, if $z_1, z_2, \ldots, z_{n-s}$ be the zeros of $\phi(z)$, then $|z_j| \geq k$, $k \geq 1$, $j = 1, 2, \ldots, n - s$. Hence, we have

$$\frac{zp'(z)}{p(z)} = \frac{sz}{z-z_0} + \sum_{j=1}^{n-s} \frac{z}{z-z_j}. \tag{2.3}$$

This, in particular, gives

$$Re\left(\frac{zp'(z)}{p(z)}\right) = Re\left(\frac{sz}{z-z_0}\right) + \sum_{j=1}^{n-s} Re\left(\frac{z}{z-z_j}\right).$$

For the points $e^{i\theta}$, $0 \leq \theta < 2\pi$ which are not the zeros of $p(z)$, we have

$$Re\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) = Re\left(\frac{se^{i\theta}}{e^{i\theta}-z_0}\right) + \sum_{j=1}^{n-s} Re\left(\frac{e^{i\theta}}{e^{i\theta}-z_j}\right).$$

Now,

$$\begin{aligned}
\sum_{j=1}^{n-s} Re\left(\frac{e^{i\theta}}{e^{i\theta}-z_j}\right) &\leq \sum_{j=1}^{n-s} \frac{1}{1+|z_j|} \\
&= \frac{n-s}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n-s} \frac{|z_j|-k}{|z_j|+1} \\
&\leq \frac{n-s}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n-s} \frac{|z_j|-k}{|z_j|+k} \quad (\text{as } k \geq 1) \\
&= \frac{n-s}{1+k} - \frac{1}{1+k}\sum_{j=1}^{n-s} \frac{\frac{|z_j|}{k}-1}{\frac{|z_j|}{k}+1}.
\end{aligned}$$

Since $\frac{|z_j|}{k} \geq 1$, $j = 1, 2, \ldots, n$, we get on using Lemma 2.1 for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$ which are not the zeros of $p(z)$,

$$\begin{aligned}
\sum_{j=1}^{n-s} Re\left(\frac{e^{i\theta}}{e^{i\theta}-z_j}\right) &\leq \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{\prod_{\nu=1}^{n-s}\frac{|z_j|}{k}-1}{\prod_{\nu=1}^{n-s}\frac{|z_j|}{k}+1}\right) \\
&= \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)
\end{aligned}$$

and

$$Re\left(\frac{se^{i\theta}}{e^{i\theta}-z_0}\right)\le\frac{s}{1-|z_0|}.$$

Therefore,

$$Re\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right)\le\frac{s}{1-|z_0|}+\frac{n-s}{1+k}-\frac{1}{1+k}\left(\frac{|a_0|k^s-|a_{n-s}|k^n}{|a_0|k^s+|a_{n-s}|k^n}\right).$$

□

Lemma 2.3. ([1]) *If* $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ *is a polynomial of degree* n*, then for* $|z|=1$ *and for every real* α

$$\left|p'(z)\right|^2+\left|q'(z)\right|^2\le\frac{n^2}{2}\left(M_\alpha^2+M_{\alpha+\pi}^2\right). \tag{2.4}$$

3. Main results

The main purpose of this paper is to obtain upper bound estimates for $\max_{|z|=1}|D_\beta p(z)|$ as well as $\max_{|z|=1}|p(Rz)-p(rz)|$, $R\ge r\ge 1$, for the same class of polynomial considered in Theorem 1.1 by involving the coefficients of the polynomial and following the approach of Frappier et al. [10]. The obtained results sharpen as well as generalize some already known estimates as special case. We first prove the following result

Theorem 3.1. *If* $p(z)=(z-z_0)^s\sum_{\nu=0}^{n-s}a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z|<k$*,* $k\ge 1$*, except a zero of order* s*,* $0\le s<n$ *at* z_0*, where* $|z_0|<1$*, then for every complex number* β *with* $|\beta|\ge 1$*,*

$$\max_{|z|=1}|D_\beta p(z)|\le\frac{n}{2}\left[2\max_{|z|=1}|p(z)|+(|\beta|-1)\left\{M_\alpha^2+M_{\alpha+\pi}^2-2\left\{1-\frac{2}{n}\left\{\frac{s}{1-|z_0|}+\frac{n-s}{1+k}-\frac{1}{1+k}\left(\frac{|a_0|k^s-|a_n|k^n}{|a_0|k^s+|a_n|k^n}\right)\right\}\right\}|p(z)|^2\right\}^{\frac{1}{2}}\right]. \tag{3.1}$$

Proof. Since $p(z)$ has no zero in $|z|<k$, $k\ge 1$, except a zero of multiplicity s at z_0, $|z_0|<1$, $0\le s<n$. If $z_1,z_2,\ldots,z_{n-s}$ be the zeros of $p(z)$ such that $|z_j|\ge k$, $k\ge 1$, $j=1,2,\ldots,n-s$. Hence, we have by Lemma 2.2 for $|z|=1$

$$Re\left(\frac{zp'(z)}{p(z)}\right)\le\frac{s}{1-|z_0|}+\frac{n-s}{1+k}-\frac{1}{1+k}\left(\frac{|a_0|k^s-|a_{n-s}|k^n}{|a_0|k^s+|a_{n-s}|k^n}\right). \tag{3.2}$$

Also, we have for $|z| = 1$,

$$\left|q'(z)\right| = \left|np(z) - zp'(z)\right|.$$

This gives for $|z| = 1$

$$\begin{aligned}\left|\frac{zq'(z)}{p(z)}\right|^2 &= \left|n - z\frac{p'(z)}{p(z)}\right|^2 \\ &= n^2 + \left|z\frac{p'(z)}{p(z)}\right|^2 - 2nRe\left(\frac{zp'(z)}{p(z)}\right),\end{aligned}$$

which on using Lemma 2.2 gives

$$\begin{aligned}\left|\frac{zq'(z)}{p(z)}\right|^2 \geq n^2 + \left|z\frac{p'(z)}{p(z)}\right|^2 - 2n\Bigg\{&\frac{s}{1-|z_0|} + \frac{n-s}{1+k} \\ &- \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}.\end{aligned}$$

Equivalently

$$\begin{aligned}\left|q'(z)\right|^2 \geq n^2\left|p(z)\right|^2 + \left|zp'(z)\right|^2 - 2n|p(z)|^2\Bigg\{&\frac{s}{1-|z_0|} \\ &+ \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}.\end{aligned}$$

Therefore

$$\begin{aligned}2\left|p'(z)\right|^2 \leq \left|p'(z)\right|^2 + \left|q'(z)\right|^2 - \Bigg\{n^2 - 2n\Bigg\{&\frac{s}{1-|z_0|} + \frac{n-s}{1+k} \\ &- \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2. \qquad (3.3)\end{aligned}$$

Applying Lemma 2.3 to (3.3), we get

$$\begin{aligned}2\left|p'(z)\right|^2 \leq \frac{n^2}{2}\left(M_\alpha^2 + M_{\alpha+\pi}^2\right) - \Bigg\{n^2 - 2n\Bigg\{&\frac{s}{1-|z_0|} \\ &+ \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2,\end{aligned}$$

which implies

$$\left|p'(z)\right| \leq \frac{n}{2}\left[M_\alpha^2 + M_{\alpha+\pi}^2 - 2\left\{1 - \frac{2}{n}\left\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\right\}\right\}|p(z)|^2\right]^{\frac{1}{2}}. \tag{3.4}$$

Using the definition of polar derivative of a polynomial with respect to the complex number β with $|\beta| \geq 1$, we have

$$\begin{aligned} |D_\beta p(z)| &= \left|np(z) + (\beta - z)p'(z)\right| \\ &\leq \left|np(z) + zp'(z)\right| + |\beta|\left|p'(z)\right| \\ &= \left|q'(z)\right| + |\beta|\left|p'(z)\right| \\ &\leq n\max_{|z|=1}|p(z)| + (|\beta|-1)\left|p'(z)\right|. \end{aligned} \tag{3.5}$$

Inequality (3.5) in conjunction with inequality (3.4) gives,

$$\max_{|z|=1}|D_\beta p(z)| \leq \frac{n}{2}\left[2\max_{|z|=1}|p(z)| + (|\beta|-1)\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\left\{1 - \frac{2}{n}\left\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\right\}\right\}|p(z)|^2\right\}^{\frac{1}{2}}\right].$$

This completes the proof of Theorem 3.1. □

Taking $z_0 = 0$ in Theorem 3.1, we get the following result for the polynomial having $s-$fold zeros at the origin.

Corollary 3.2. *If $p(z) = z^s\sum_{\nu=0}^{n-s} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, except a zero of order s, $0 \leq s < n$ at origin, then for every complex number β with $|\beta| \geq 1$,*

$$\max_{|z|=1}|D_\beta p(z)| \leq \frac{n}{2}\left[2\max_{|z|=1}|p(z)| + (|\beta|-1)\left\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\left\{1 - \frac{2}{n}\left\{\frac{n+sk}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\right\}\right\}|p(z)|^2\right\}^{\frac{1}{2}}\right]. \tag{3.6}$$

Setting $s = 0$ to Theorem 3.1, we get the following result due to Mir and Hussian [20].

Corollary 3.3. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then for every complex number* β *with* $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta p(z)| \leq \frac{n}{2}\Bigg[2\max_{|z|=1}|p(z)| + (|\beta|-1)\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{1+k}\Bigg\{(k-1) + \frac{2}{n}\left(\frac{|a_0| - |a_n|k^n}{|a_0| + |a_n|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg\}^{\frac{1}{2}}\Bigg].$$

Clearly, Theorem 3.1 and Corollary 3.2 are extensions of Corollary 3.3.

Dividing both side of inequality (3.1) by $|\beta|$ and taking limit $|\beta| \to \infty$, we have the following corollary.

Corollary 3.4. *If* $p(z) = (z - z_0)^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *except a zero of order* s, $0 \leq s < n$ *at* z_0, *where* $|z_0| < 1$, *then*

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2}\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\Bigg\{1 - \frac{2}{n}\Bigg\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg\}^{\frac{1}{2}}. \tag{3.7}$$

Taking $z_0 = 0$ in Corollary 3.4, we get the following result.

Corollary 3.5. *If* $p(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *except a zero of order* s, $0 \leq s < n$ *at origin, then*

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2}\Bigg\{M_\alpha^2 + M_{\alpha+\pi}^2 - 2\Bigg\{1 - \frac{2}{n}\Bigg\{\frac{n+sk}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg\}^{\frac{1}{2}}. \tag{3.8}$$

Setting $s = 0$ to Corollary 3.4, we get the following extension as well as generalization of inequality (1.8).

Corollary 3.6. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2}\left\{ M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{1+k}\left[(k-1) + \frac{2}{n}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\right]|p(z)|^2 \right\}^{\frac{1}{2}}. \tag{3.9}$$

Clearly, Corollaries 3.4 and 3.5 are extensions of Corollary 3.6. Further, as an application of Corollary 3.4, we obtain the following extension of inequality (1.9) to the class of polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, except a zero of order s, $0 \leq s < n$ at z_0, where $|z_0| < 1$.

Theorem 3.7. *If* $p(z) = (z - z_0)^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *except a zero of order* s, $0 \leq s < n$ *at* z_0, *where* $|z_0| < 1$, *then for every real* α *and* $R \geq r \geq 1$,

$$\begin{aligned}\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{2}\Bigg[& M_\alpha^2 + M_{\alpha+\pi}^2 \\ & - 2\left\{1 - \frac{2}{n}\left\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right.\right. \\ & \left.\left. - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\right\}\right\}|p(z)|^2\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{3.10}$$

Proof. Applying inequality (1.2) to the polynomial $p'(z)$, which is of degree $n-1$, we obtain for $t \geq 1$ and $0 \leq 0 < 2\pi$

$$\left|p'\left(te^{i\theta}\right)\right| \leq t^{n-1} \max_{|z|=1} |p'(z)|.$$

Using Corollary 3.4, we have

$$\begin{aligned}\left|p'\left(te^{i\theta}\right)\right| \leq \frac{nt^{n-1}}{2}\Bigg[& \left(M_\alpha^2 + M_{\alpha+\pi}^2\right) - 2\left\{1 - \frac{2}{n}\left\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right.\right. \\ & \left.\left. - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\right\}\right\}|p(z)|^2\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{3.11}$$

Hence for each θ, $0 \leq \theta < 2\pi$ and $R \geq r \geq 1$, we have

$$\begin{aligned} \left|p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right)\right| &= \left|\int_r^R e^{i\theta} p'\left(te^{i\theta}\right) dt\right| \\ &\leq \int_r^R \left|p'\left(te^{i\theta}\right)\right| dt \end{aligned} \tag{3.12}$$

and from inequality (3.11), we have

$$\begin{aligned} \left|p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right)\right| \leq \frac{1}{2}\Bigg[&(M_\alpha^2 + M_{\alpha+\pi}^2) - 2\Bigg\{1 - \frac{2}{n}\Bigg\{\frac{s}{1-|z_0|} + \frac{n-s}{1+k} \\ &- \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg]^{\frac{1}{2}} \int_r^R nt^{n-1} dt, \end{aligned}$$

that is,

$$\begin{aligned} \left|p\left(Re^{i\theta}\right) - p\left(re^{i\theta}\right)\right| \leq \frac{R^n - r^n}{2}\Bigg[&(M_\alpha^2 + M_{\alpha+\pi}^2) - 2\Bigg\{1 - \frac{2}{n}\Bigg\{\frac{s}{1-|z_0|} \\ &+ \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_{n-s}|k^n}{|a_0|k^s + |a_{n-s}|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg]^{\frac{1}{2}}. \end{aligned}$$

This implies that, for $|z| = 1$ and $R \geq r \geq 1$,

$$\begin{aligned} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{2}\Bigg[&M_\alpha^2 + M_{\alpha+\pi}^2 - 2\Bigg\{1 - \frac{2}{n}\Bigg\{\frac{s}{1-|z_0|} \\ &+ \frac{n-s}{1+k} - \frac{1}{1+k}\left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n}\right)\Bigg\}\Bigg\}|p(z)|^2\Bigg]^{\frac{1}{2}}. \end{aligned}$$

This completes the proof of Theorem 3.7. □

Putting $z_0 = 0$ to Theorem 3.7, we have the following result.

Corollary 3.8. *If $p(z) = z^s \sum_{\nu=0}^{n-s} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, except a zero of order s, $0 \leq s < n$ at origin, then for* *every real* α *and* $R \geq r \geq 1$,

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{2} \Bigg\{ M_\alpha^2 + M_{\alpha+\pi}^2 - 2\Bigg\{ 1 - \frac{2}{n} \Bigg\{ \frac{n+sk}{1+k} - \frac{1}{1+k} \left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n} \right) \Bigg\} \Bigg\} |p(z)|^2 \Bigg\}^{\frac{1}{2}}. \tag{3.13}$$

Setting $s = 0$ to Theorem 3.7, we get the following result.

Corollary 3.9. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < k$, $k \geq 1$, *then for every real* α *and* $R \geq r \geq 1$,

$$\max_{|z|=1} |p(Rz) - p(rz)| \leq \frac{R^n - r^n}{2} \Bigg\{ M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{1+k} \Bigg[(k-1) + \frac{2}{n} \left(\frac{|a_0|k^s - |a_n|k^n}{|a_0|k^s + |a_n|k^n} \right) \Bigg] |p(z)|^2 \Bigg\}^{\frac{1}{2}}. \tag{3.14}$$

It is easy to verify that Corollay 3.9 generalizes as well as sharpens inequality (1.9). Taking $k = 1$, $r = 1$ in Corollary 3.9, we get the following result.

Corollary 3.10. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for every real* α *and* $R \geq 1$,

$$\max_{|z|=1} |p(Rz) - p(z)| \leq \frac{R^n - 1}{2} \Bigg\{ M_\alpha^2 + M_{\alpha+\pi}^2 - \frac{2}{n} \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|} \right) |p(z)|^2 \Bigg\}^{\frac{1}{2}}. \tag{3.15}$$

Remark 3.11. The bound obtained in Corollary 3.10 is always sharpen than the bound obtained from inequality (1.9), for this it needs to show that

$$\frac{|a_0| - |a_n|}{|a_0| + |a_n|} \geq 0,$$

which is equivalent to

$$|a_0| \geq |a_n|,$$

which is true as $p(z) \neq 0$ in $|z| < 1$.

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References

[1] A. Aziz, *A refinement of an inequality of S. Bernstein*, J. Math. Anal. Appl., **114** (1989), 226-235, https://doi.org/10.1016/0022-247X(89)90370-3.

[2] A. Aziz and Q.M. Dawood, *Inequalities for a polynomial and its derivative,* J. Approx. Theory, **54** (1988), 306-313.

[3] A. Aziz and W.M. Shah, *Inequalities for a polynomial and its derivative,* Math. Ineq. Appl., **7**(3) (2004), 379-391.

[4] A. Aziz and B.A. Zargar, *Inequalities for a polynomial and its derivative,* Math. Ineq. Appl., **1**(4) (1998), 543-550.

[5] S. Bernstein, *Sur lordre de la meilleure approximation des functions continues par des polynomes de degr donn,* Mem. Acad. R. Belg., **4** (1912), 1-103.

[6] T.N. Chan and M.A. Malik, *On Erdös-Lax theorem*, Proc. Indian Acad. Sci., **92**(3) (1983), 191-193.

[7] B. Chanam and K.K. Dewan, *Inequalities for a polynomial and its derivative,* J. Math. Anal. Appl., **336** (2007), 171-179.

[8] B. Chanam, N. Reingachan, K. B. Devi, M. T. Devi and K. Krishnadas, *Some L^q inequalities for polynomial*, Nonlinear Funct. Anal. Appl., **26**(2)(2021), 331-345.

[9] K.K. Dewan and M. Bidkham, *Inequalities for a polynomial and its derivative,* J. Math. Anal. Appl., **166** (1992), 319-324.

[10] C. Frappier, Q.I. Rahman and S. Ruscheweyh, *New inequalities for polynomials*, Trans. Amer. Math. Soc., **288** (1985), 6999, https://doi.org/10.1090/S0002-9947-1985-0773048-1.

[11] N.K. Govil, *Some inequalities for derivatives of polynomials,* J. Approx. Theory, **66** (1991), 29-35.

[12] A. Hussain, A. Mir and A. Ahmad, *On Bernstein-type inequalities for polynomials involving the polar derivative*, J. Class. Anal., **16** (2020), 9-15, https://doi.org/10.7153/jca-2020-16-02.

[13] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.

[14] M.A. Malik, *On the derivative of a polynomial,* J. London Math. Soc., **1** (1969), 57-60.

[15] G.V. Milovanović and A. Mir, *Generalizations of Zygmund-type integral inequalities for the polar derivative of a complex polynomial,* J. Ineq. Appl., **2020** (2020), 1-12, https://doi.org/10.1186/s13660-020-02404-x.

[16] G.V. Milovanović, D.S. Mitrinović and D.S. Rassias, *Topics in Polynomials: Extremal Properties,* Inequalities, Zeros, World Scientific Publishing Co. Singapore; 1994.

[17] A. Mir, *On an operator preserving inequalities between polynomials*, Ukrainian Math. J., **69** (2018), 1234-1247, https://doi.org/10.1007/s11253-017-1427-2.

[18] A. Mir, *Some sharp upper bound estimates for the maximal modulus of polar derivative of a polynomial,* Annali Dell Universitá Di Ferrara. **65** (2019), 327-336, https://doi.org/10.1007/s11565-019-00317-2.

[19] A. Mir and B. Dar, *On the polar derivative of a polynomial,* J. Ramanujan Math. Soc., **29** (2014), 403-412.

[20] A. Mir and A. Hussain, *Generalizations of some Bernstein-type inequalities for the polar derivative of a polynomial*, Kragujevac J. Math., **40**(1) (2021), 31–41.

[21] A. Mir and I. Hussain, *On the Erds-Lax inequality concerning polynomials,* C. R. Math. Acad. Sci. Paris, **355** (2017), 1055-1062, https://doi.org/10.1016/j.crma.2017.09.017.

[22] A. Mir and M.A. Malik, *Inequalities concerning the rate of growth of polynomials involving the polar derivative,* Ann. Univ. Mariae Curie-Sk Lodowska Sect. A, **74** (2020), 67-75, http:dx.doi.org/10.17951/a.2020.74.1.67-75.

[23] A. Mir and A. Wani, *A note on two recent results about polynomials with restricted zeros,* J. Math. Ineq., **14** (2020), 45-50, https://doi.org/10.7153/jmi-2020-14-04.

[24] K.M. Nakprasit and J. Somsuwan, *An upper bound of a derivative for some class of polynomials,* J. Math. Ineq., **11** (2017), 143-150.

[25] M.S. Pukhta, *Extremal problems for polynomials and on location of zeros of polynomials,* Ph.D. Thesis of the Jamia Millia Islamia, New Delhi; 1995.

[26] Q.A. Qazi, *On the maximum modulus of polynomials,* Proc. Amer. Math. Soc., **115** (1992), 337-343.

[27] Q.I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, New York, 2002.

[28] A. Zireh and A.M. Bidkham, *Inequalities for the polar derivative of a polynomial with restricted zeros,* Kragjevac J. Math., **40** (2016), 113-124, https://doi.org/10.5937/KgJMath1601113Z.