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SOME RESULTS OF GENERALIZED HARDY-ROGER MAPPINGS IN RECTANGULAR b-METRIC SPACES

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Abstract. In this paper, we extend some fixed point theorems in rectangular b-metric spaces using subadditive altering distance and establishing the existence and uniqueness of fixed point for Hardy-Roger type mappings. Our result generalizes many known results in fixed point theory. Finally, we offer a example to illustrate our result.

1. Introduction

In 1922, Banach [2] developed what is now known as the *Banach's contraction principle*, which is the main finding in fixed point theory. In fact, Banach's contraction principle is very important to show the existence of solutions of some real world problems has been checked in various branches

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of mathematics, such as differential equations, integral equations, functional analysis, etc. and one has introduced solutions for this problems via fixed point theory. Additionally, applications of fixed point theory are not only found in mathematics but also in a number of other areas, including computer science, physics, chemistry, biology and economics. Particularly, a branch of economics that uses fixed point theory tools to solve various game theory. Many researchers have studied and expanded in various ways since Banach's contraction principle, including the following.

Theorem 1.1. ([2]) (Banach's contraction principle) *If* (X, d) *is a complete metric space and* $T : X \to X$ *be a mapping such that for some* $\alpha \in [0, 1)$*,*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

for each $x, y \in X$*, then* T *has a unique fixed point in* X*.*

It should be noted that while mappings T satisfying Theorem 1.1 is continuous, but the mappings T satisfying the following contractions conditions are not continuous. In 1968, Kannan's contraction [20]: for some $\beta \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \tag{1.2}$$

for each $x, y \in X$.

In 1971, Reich's contraction [20]: for some $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \tag{1.3}$$

for each $x, y \in X$.

In 1971, Ćirić's contraction [20]: for some $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha+\beta+\gamma+2\delta < 1$,

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta[d(x, Ty) + d(y, Tx)] \tag{1.4}$$

for each $x, y \in X$.

In 1972, Chatterjea's contraction [20]: for some $\beta \in [0, \frac{1}{2})$,

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)] \tag{1.5}$$

for each $x, y \in X$.

In 1972, Zamfirescu contractive conditions [20]: there exist real numbers $\alpha, \beta, \gamma, 0 \leq \alpha < 1, 0 \leq \beta < 1, \gamma < \frac{1}{2}$, such that for each $x, y \in X$, at least one of the following is true:

(i) $d(Tx, Ty) \leq \alpha d(x, y)$;
(ii) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$;
(iii) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

For each $x, y \in X, x \neq y$,

$$d(Tx,Ty) < \max\{d(x,y), [d(x,Tx)+d(y,Ty)]/2, [d(x,Ty)+d(y,Tx)]/2\}. \quad (1.6)$$

In 1973, Hardy and Rogers's contraction [20]: for some $\alpha, \beta, \gamma, \delta, \eta \geq 0$ with $\alpha + \beta + \gamma + \delta + \eta < 1$,

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + \eta d(y,Tx) \quad (1.7)$$

for each $x, y \in X$:

In addition, the Hardy and Rogers's contraction was studied and extended on various spaces by many researchers (see in [15, 22, 23, 24]).

Furthermore, Bakhtin [1] initially introduced the ideas of a b-metric spaces in 1998 by defining them as follows. Let X be a nonempty set and the mapping $d : X \times X \to [0, \infty)$ satisfies:

(bM1) $d(x, y) = 0$, if and only if $x = y$ for all $x, y \in X$;
(bM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(bM3) there exist a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a b-metric on X and (X, d) is called a b-metric space (in short bMS) with coefficient s.

The following is how Branciari [3] first described the ideas of a rectangle metric space in 2000. Let X be a nonempty set, $b \geq 1$ be a given real number and $d : X \times X \to [0, \infty]$ be a mapping such that for all $x, y \in X$:

(RM1) $d(x, y) = 0$, if and only if $x = y$;
(RM2) $d(x, y) = d(y, x)$;
(RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all distinct points $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular metric on X and (X, d) is called a rectangular metric space (in short RMS).

The concept of b-metric spaces was first introduced by Bakhtin [1] and Czerwik [4], in such a way that triangle inequality is replaced by the b-triangle inequality:

$$d(x, y) \leq b[d(x, z) + d(z, y)]$$

for all pairwise distinct points x, y, z and $b \geq 1$. Various fixed point results were established on such spaces, see in ([5, 6, 7, 11, 12, 13, 14, 16, 17, 18, 19]).

Recently, Rossafi and Massit [21] extended some fixed point theorems in rectangular b-metric spaces using subadditive altering distance and establishing the existence and uniqueness of fixed point for Kannan type mappings as follows:

Theorem 1.2. ([21]) *Let (X,d) be a complete rectangular b-metric spaces with coefficient $b \geq 1$ and $T: X \to X$ be a mapping such that there exist $p < \frac{1}{2b+1}$ satisfying*

$$\varphi(d(Tx,Ty)) \leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) + \varphi(d(y,Ty))]$$

for all $x,y \in X$. Then, T has a unique fixed point $u \in X$, the sequence $\{T^n x\}$ converges to u and for $q = \frac{pb+2p}{1-pb-p} < 1$, we have

$$\varphi(d(T^{n+1}x, T^n x)) \leq q^n d(x,Tx)), n = 0,1,2,\ldots .$$

Motivated and inspired by (1.7) and Theorem 1.2, we extend some fixed point theorems in rectangular b-metric spaces using subadditive altering distance and establishing the existence and uniqueness of fixed point for Hardy-Roger type mappings. Moreover, we give some non-trivial examples for support our results.

2. Preliminaries

Some definitions and concepts used in the primary results are presented in this section. The following is a definition provided by George et al. [10] in 2015 for rectangular b-metric spaces (see [25]).

Definition 2.1. ([10]) Let X be a nonempty set, $b \geq 1$ be a given real number, and let $d: X \times X \to [0,\infty]$ be a mapping such that for all $x,y \in X$ and all distinct points $u,v \in X$, each distinct from x and y satisfies::

(RbM1) $d(x,y) = 0$, if and only if $x = y$;
(RbM2) $d(x,y) = d(y,x)$;
(RbM3) $d(x,y) \leq b[d(x,u) + d(u,v) + d(v,y)]$ b-rectangular inequality.

Then (X,d) is said to be a rectangular b-metric spaces.

Example 2.2. Let $X = N$, define $d: X \times X \to X$ by

$$d(x,y) = \begin{cases} 0, & \text{if} \quad x = y, \\ 9a, & \text{if} \quad x,y \in \{3,5\}, \\ 2a, & \text{otherwise}, \end{cases}$$

where $a > 0$ is a constant. Then (X,d) is a rectangular b-metric space with coefficient $b = 3 > 1$.

The definition of a subadditive altering distance function is then provided as follows.

Definition 2.3. A function $\varphi: [0,\infty) \to [0,\infty)$ is called a subadditive altering distance function if

(i) φ is an altering distance function (that is, φ is continuous, strictly increasing and $\varphi(0) = 0$);
(ii) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in [0, \infty)$.

Example 2.4. The functions $\varphi_1(x) = 3\sqrt{x}$, $\varphi_2(x) = \frac{2x}{5}$ and $\varphi_3(x) = ln(x+1)$ are subadditive altering distance functions.

We note that, if φ is subadditive, then for any nonnegative real number $k < 1$, $\varphi(d(x, y)) \leq k\varphi(d(a, b))$ implies $d(x, y) \leq k'd(a, b)$ for some $k' < 1$.

A bounded compact metric space [8] is a metric space X in which every bounded sequence in X has a convergent subsequence. The same notion may be defined in the case of rectangular b-metric spaces. The class of bounded compact rectangular b-metric spaces is larger than that of sequentially compact spaces as the rectangular b-metric space R of real numbers with the usual metric is not sequentially compact but bounded compact.

Garai et al [9] defined T-orbitally compact metric spaces and derived a fixed point result for the same. The definition of T-orbitally compactness can be extended to rectangular b-metric spaces as follows.

Definition 2.5. Let (X, d) be a rectangular b-metric space and T be a self-mapping on X. The orbit of T at $x \in X$ is defined as

$$O_x(T) = \{x, Tx, T^2x, T^3x, ...\}.$$

If every sequence in $O_x(T)$ has a convergent subsequence for all $x \in X$, X is said to be T-orbitally compact.

3. Main results

Motivated and inspired by (1.7) and Theorem 1.2, we prove the existence of a Hardy-Rogers mapping in a complete rectangular b-metric spaces as follows.

Theorem 3.1. *Let (X, d) be a complete rectangular b-metric spaces with coefficient $b \geq 1$ and $T : X \to X$ be a mapping such that there exist $p < \frac{1}{3b+3}$ satisfying*

$$\begin{aligned} \varphi(d(Tx, Ty)) \quad \leq \quad & p[\varphi(d(x, y)) + \varphi(d(x, Tx)) \\ & +\varphi(d(y, Ty)) + \varphi(d(x, Ty)) + \varphi(d(Tx, y))] \end{aligned} \tag{3.1}$$

for all $x, y \in X$. Then, T has a unique fixed point $u \in X$, the sequence $\{T^n x\}$ converges to u and for $q = \frac{pb+2p}{1-pb-p} < 1$, we have

$$\varphi(d(T^{n+1}x, T^n x)) \leq q^n d(x, Tx)), \quad n = 0, 1, 2, ...$$

Proof. Let $z = Tx$ for an arbitrary element $x \in X$. Then

$$\begin{aligned}\varphi(d(z,Tz)) &= \varphi(d(Tx,Tz)) \\ &\leq p[\varphi(d(x,z)) + \varphi(d(x,Tx)) + \varphi(d(z,Tz)) \\ &\quad + \varphi(d(x,Tz)) + \varphi(d(Tx,z))] \\ &\leq p[\varphi(d(x,Tx)) + \varphi(d(x,Tx)) + \varphi(d(Tx,Tz)) \\ &\quad + \varphi(d(x,Tz)) + \varphi(d(Tx,Tx))] \\ &\leq p\varphi(d(x,Tx)) + p\varphi(d(x,Tx)) \\ &\quad + p\varphi(d(Tx,Tz)) + p\varphi(d(x,Tz)) \\ &\leq 2p\varphi(d(x,Tx)) + p\varphi(d(Tx,Tz)) \\ &\quad + p\varphi([bd(x,Tx) + bd(Tx,Tz) + bd(Tz,Tz)]) \\ &\leq 2p\varphi(d(x,Tx)) + p\varphi(d(Tx,Tz)) \\ &\quad + pb\varphi(d(x,Tx) + pb\varphi(d(Tx,Tz)).\end{aligned}$$

Hence we have

$$\varphi(d(Tx,Tz)) \leq q\varphi(d(x,Tx)),$$

where $q = \frac{pb+2p}{1-pb-p} < 1$, it implies that

$$d(Tx,Tz) \leq q' d(x,Tx) \tag{3.2}$$

for $q' < 1$.

Now we assume that $q' = q$. Let $x_0 \in X$, consider the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_n = Tx_n$. Then x_n is a fixed point of T and the proof is finished. Hence, let $x_n \neq Tx_n$ for all $n \in \mathbb{N}$. Then for $m \geq 1$ and $r \geq 1$ it follows that

$$\begin{aligned}d(x_{m+r},x_m) &\leq b[d(x_{m+r},x_{m+r-1}) + d(x_{m+r-1},x_{m+r-2}) + d(x_{m+r-2},x_m)] \\ &\leq bd(x_{m+r},x_{m+r-1}) + bd(x_{m+r-1},x_{m+r-2}) \\ &\quad + b[b[d(x_{m+r-2},x_{m+r-3}) + d(x_{m+r-3},x_{m+r-4}) \\ &\quad + d(x_{m+r-4},x_m)]] \\ &= bd(x_{m+r},x_{m+r-1}) + bd(x_{m+r-1},x_{m+r-2}) \\ &\quad + b^2 d(x_{m+r-2},x_{m+r-3}) + b^2 d(x_{m+r-3},x_{m+r-4}) \\ &\quad + b^2 d(x_{m+r-4},x_m)\end{aligned}$$

$$\begin{aligned}
&\leq bd(x_{m+r}, x_{m+r-1}) + bd(x_{m+r-1}, x_{m+r-2}) \\
&\quad + b^2 d(x_{m+r-2}, x_{m+r-3}) + b^2 d(x_{m+r-3}, x_{m+r-4}) \\
&\quad + b^2 d(x_{m+r-4}, x_{m+r-5}) + ... + b^{\frac{r-1}{2}} d(x_{m-3}, x_{m-2}) \\
&\quad + b^{\frac{r-1}{2}} d(x_{m-2}, x_{m-1}) + b^{\frac{r-1}{2}} d(x_{m-1}, x_m) \\
&\leq bq^{m+r-1} d(x_1, x_0) + bq^{m+r-2} d(x_1, x_0) + b^2 q^{m+r-3} d(x_1, x_0) \\
&\quad + b^2 q^{m+r-4} d(x_1, x_0) + b^2 q^{m+r-5} d(x_1, x_0) \\
&\quad + ... + b^{\frac{r-1}{2}} q^{m+2} d(x_1, x_0) + b^{\frac{r-1}{2}} q^{m+1} d(x_1, x_0) + b^{\frac{r-1}{2}} q^m d(x_1, x_0) \\
&= d(x_1, x_0)[bq^{m+r-1} + b^2 q^{m+r-3} + b^2 q^{m+r-5} + ... + b^{\frac{r-1}{2}} q^{m+2}] \\
&\quad + d(x_1, x_0)[bq^{m+r-2} + b^2 q^{m+r-4} + ... + b^{\frac{r-1}{2}} q^{m+1}] \\
&\quad + d(x_1, x_0) b^{\frac{r-1}{2}} q^m \\
&\leq d(x_1, x_0)[\Sigma_{i=1}^{\frac{r-1}{2}} b^i q^{m+r-(2i-1)} + \Sigma_{i=1}^{\frac{r-1}{2}} b^i q^{m+r-(2i)} + b^{\frac{r-1}{2}} q^m] \\
&\to 0 \ \ as\ m \to \infty.
\end{aligned}$$

So the sequence $\{x_n\}$ is Cauchy. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x.$$

Since

$$\begin{aligned}
d(Tx, x) &\leq b[d(Tx, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, x)] \\
&= bd(Tx, Tx_n) + bd(Tx_n, Tx_{n+1}) + bd(Tx_{n+1}, x),
\end{aligned}$$

it implies that

$$\begin{aligned}
\varphi(d(Tx, x)) &\leq b\varphi(d(Tx, Tx_n)) + b\varphi(d(Tx_n, Tx_{n+1})) + b\varphi(d(Tx_{n+1}, x)) \\
&\leq bp[\varphi(d(x, x_n)) + \varphi(d(Tx, x_n)) + \varphi(d(x, Tx)) \\
&\quad + \varphi(d(x, Tx_n)) + \varphi(d(Tx_n, x_n))] + bp[\varphi(d(x_n, x_{n+1})) \\
&\quad + \varphi(d(x_n, Tx_{n+1})) + \varphi(d(Tx_n, x_{n+1})) + \varphi(d(x_n, Tx_n)) \\
&\quad + \varphi(d(x_{n+1}, Tx_{n+1}))] + b\varphi(d(Tx_{n+1}, x)) \\
&= bp\varphi(d(x, x_n)) + bp\varphi(d(x, x_{n+1})) + bp\varphi(d(x, Tx)) \\
&\quad + bp\varphi(d(Tx, x_n)) + bp\varphi(d(x_{n+1}, x_n)) + bp\varphi(d(x_n, x_{n+1})) \\
&\quad + bp\varphi(d(x_n, x_{n+2})) + bp\varphi(d(x_{n+1}, x_{n+1})) + bp\varphi(d(x_n, x_{n+1})) \\
&\quad + bp\varphi(d(x_{n+1}, x_{n+2})) + b\varphi(d(Tx_{n+1}, x))
\end{aligned}$$

$$\begin{aligned}
&\leq bp\varphi(d(x,x_n)) + bp\varphi(d(x,x_{n+1})) + bp\varphi(d(x,Tx)) \\
&\quad + b^2p\varphi(d(Tx,x)) + b^2p\varphi(d(x,x_{n+1})) + b^2p\varphi(d(x_{n+1},x_n)) \\
&\quad + bp\varphi(d(x_{n+1},x_n)) + bp\varphi(d(x_n,x_{n+1})) + bp\varphi(d(x_n,x_{n+2})) \\
&\quad + bp\varphi(d(x_{n+1},x_{n+1})) + bp\varphi(d(x_n,x_{n+1})) \\
&\quad + bp\varphi(d(x_{n+1},x_{n+2})) + b\varphi(d(Tx_{n+1},x)).
\end{aligned}$$

Then

$$\begin{aligned}
(1-bp-b^2p)\varphi(d(x,Tx)) &\leq bp\varphi(d(x,x_n)) + bp\varphi(d(x,x_{n+1})) \\
&\quad + b^2p\varphi(d(x,x_{n+1})) + b^2p\varphi(d(x_{n+1},x_n)) \\
&\quad + bp\varphi(d(x_{n+1},x_n)) + bp\varphi(d(x_n,x_{n+1})) \\
&\quad + bp\varphi(d(x_n,x_{n+2})) + bp\varphi(d(x_{n+1},x_{n+1})) \\
&\quad + bp\varphi(d(x_n,x_{n+1})) + bp\varphi(d(x_{n+1},x_{n+2})) \\
&\quad + b\varphi(d(Tx_{n+1},x)) \\
&\to 0 \ \ as\ n \to \infty.
\end{aligned}$$

This implies that $Tx = x$.

Now if y is an another fixed point of T, then

$$\begin{aligned}
\varphi(d(Tx,Ty)) &= \varphi(d(x,y)) \\
&\leq p[\varphi(d(x,y)) + \varphi(d(x,Tx)) \\
&\quad + \varphi(d(y,Ty)) + \varphi(d(x,Ty)) + \varphi(d(Tx,y))],
\end{aligned}$$

this implies that

$$\varphi(d(x,y)) \leq 3p\varphi(d(x,y)).$$

Since $p < \frac{1}{3b+3}$ and φ is strictly increasing, so $d(x,y) = 0$. Therefore the fixed point of T is unique. By (3.2) we have

$$\begin{aligned}
\varphi(d(T^{n+1}x,T^nx)) &\leq q\varphi(d(T^{n-1}x,T^nx)) \\
&\leq q^2\varphi(d(T^{n-2}x,T^{n-1}x)) \\
&\leq q^3\varphi(d(T^{n-3}x,T^{n-2}x)) \\
&\ \ \vdots \\
&\leq q^n d(x,Tx)),\ n = 0,1,2,...,
\end{aligned}$$

where $q = \frac{pb+2p}{1-pb-p} < 1$. This completes the proof. □

Example 3.2. Consider the complete rectangular b-metric space (X,d), where $X = R$ and $d(x,y) = |x-y|$ for all $x,y \in X$. Define the self-mapping

$T : X \to X$ by

$$T(x) = \begin{cases} 0, & \text{if } \quad x \leq 2, \\ -\frac{1}{5}, & \text{if } \quad x > 2. \end{cases}$$

Then T is not continuous at 2. For $\varphi(x) = \frac{x}{5}$, we have

$$\begin{aligned} \varphi(d(Tx,Ty)) &= \frac{1}{5}d(Tx,Ty) \\ &\leq \frac{p}{5}[\varphi(d(x,y)) + \varphi(d(x,Tx)) \\ &\quad +\varphi(d(y,Ty)) + \varphi(d(x,Ty)) + \varphi(d(Tx,y))]. \end{aligned}$$

For $x \leq 2$ and $y \leq 2$, we obtain

$$\begin{aligned} d(Tx,Ty) &= 0 \\ &\leq p[|x-y| + 2|x| + 2|y|] \\ &\leq p[b(|x-0| + |0-0| + |0-y|) + 2|x| + 2|y|] \\ &= p(b+2)[|x| + |y|] \end{aligned}$$

and $\varphi(d(Tx,Ty)) \leq p(b+2)[\varphi(|x|) + \varphi(|y|)]$.
For $x > 2$ and $y > 2$, we obtain

$$\begin{aligned} d(Tx,Ty) &= 0 \\ &\leq p[|x-y| + 2|x + \frac{1}{5}| + 2|y + \frac{1}{5}|] \\ &\leq p[b(|x - (-\frac{1}{5})| + |-\frac{1}{5} - (-\frac{1}{5})| + |-\frac{1}{5} - y|) \\ &\quad +2|x + \frac{1}{5}| + 2|y + \frac{1}{5}|] \\ &= p(b+2)[|x + \frac{1}{5}| + |y + \frac{1}{5}|] \\ &\leq p(b+2)(x + y + \frac{2}{5}) \end{aligned}$$

and $\varphi(d(Tx,Ty)) \leq \frac{p(b+2)}{5}(x + y + \frac{2}{5})$. Thus, T satisfies (3.1). Therefore, T has a unique fixed point $x = 0$.

From Theorem 3.1, if $\varphi(\Delta) = \Delta$, we get the result as follows.

Corollary 3.3. *Let (X,d) be a complete rectangular b-metric spaces and $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(Tx,y), \tag{3.3}$$

where $p < \frac{1}{3b+3}$. Then T has a fixed point in X.

Proof. Follow the proof of Theorem 3.1, when we take $\varphi(\Delta) = \Delta$, we get the result. □

From Theorem 3.1, if $d(x, y) = d(x, Ty) = d(Tx, y) = 0$, we get the result as follows.

Corollary 3.4. *Let (X, d) be a complete rectangular b-metric spaces with coefficient $b \geq 1$ and $T : X \to X$ be a mapping such that there exist $p < \frac{1}{2b+1}$ satisfying*

$$\varphi(d(Tx, Ty)) \leq p[\varphi(d(x, y)) + \varphi(d(x, Tx)) + \varphi(d(y, Ty))] \quad (3.4)$$

for all $x, y \in X$. Then, T has a unique fixed point $u \in X$, the sequence $\{T^n x\}$ converges to u and for $q = \frac{2p}{1-p} < 1$ we have

$$\varphi(d(T^{n+1}x, T^n x)) \leq q^n d(x, Tx)), \quad n = 0, 1, 2, ...$$

Proof. See main result in [21]. □

Theorem 3.5. *Let (X, d) be a complete rectangular b-metric spaces with coefficient $b \geq 1$ and $T : X \to X$ be a mapping such that there exist $p_1 + p_2 + p_3 + p_4 + p_5 < 1$ and $p_5 < 1$ satisfying*

$$\begin{aligned} \varphi(d(Tx, Ty)) \leq\ & p_1\varphi(d(x, y)) + p_2\varphi(d(x, Tx)) + p_3\varphi(d(y, Ty)) \\ & + p_4\varphi(d(x, Ty)) + p_5\varphi(d(Tx, y))] \end{aligned} \quad (3.5)$$

for all $x, y \in X$. Then, T has a unique fixed point $u \in X$, the sequence $\{T^n x\}$ converges to u and for $q = \frac{p_1+p_2+p_4 b}{1-p_3-p_4 b} < 1$, we have

$$\varphi(d(T^{n+1}x, T^n x)) \leq q^n d(x, Tx)), \quad n = 0, 1, 2, ...$$

Proof. Follow the proof of Theorem 3.1, when we consider a metric space (X, d) and $\varphi(\Delta) = \Delta$. □

In the next theorem, p is independent of the coefficient b of the rectangular b-metric space.

Theorem 3.6. *Let (X, d) be a bounded compact rectangular b-metric space and $T : X \to X$ be a continuous mapping satisfying (3.1) for some $0 \leq p < \frac{1}{5}$. Then T has a unique fixed point $u \in X$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to u.*

Proof. Let $x_0 \in X$ be an arbitrary point. Consider a sequence $\{x_n\}$, where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Then by (3.1) we have

$$\begin{aligned}
\varphi(d(x_n, x_{n+1})) &= \varphi(d(T^n x_0, T^{n+1} x_0)) \\
&= \varphi(d(T(T^{n-1}x_0), T(T^n x_0))) \\
&\leq p[\varphi(d(T^{n-1}x_0, T^n x_0)) + \varphi(d(T^{n-1}x_0, T(T^{n-1}x_0))) \\
&\quad + \varphi(d(T^n x_0, T(T^n x_0))) + \varphi(d(T^{n-1}x_0, T(T^n x_0))) \\
&\quad + \varphi(d(T(T^{n-1}x_0), T^n x_0))] \\
&= p[\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_{n+1})) \\
&\quad + \varphi(d(x_{n-1}, x_{n+1})) + \varphi(d(x_n, x_n))] \\
&= 2p\varphi(d(x_{n-1}, x_n)) + p\varphi(d(x_n, x_{n+1})) + p\varphi(d(x_{n-1}, x_{n+1})) \\
&\leq 2p\varphi(d(x_{n-1}, x_n)) + p\varphi(d(x_n, x_{n+1})) \\
&\quad + pb[\varphi(d(x_{n-1}, x_n)) + \varphi(d(x_n, x_n)) + \varphi(d(x_n, x_{n+1}))] \\
&= (2p + pb)\varphi(d(x_{n-1}, x_n)) + (2p + pb)\varphi(d(x_n, x_{n+1})).
\end{aligned}$$

It implies that

$$(1 - 2p - pb)\varphi(d(x_n, x_{n+1})) \leq (2p + pb)\varphi(d(x_{n-1}, x_n)), \quad \forall n \in \mathbb{N}.$$

Since $1 - 2p - pb \geq 2p + pb$, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}.$$

Hence, the sequence $\{d(x_n, x_{n+1})\}_{n\in N}$ is decreasing and bounded below, thus, there exists $t \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = t.$$

For $m, n \in \mathbb{N}$ with $n < m$, we have

$$\varphi(d(x_m, x_n)) \leq \varphi(d(x_{m-1}, x_m)) + \varphi(d(x_{n-1}, x_n))$$

and hence $\varphi(d(x_m, x_n)) \leq \varphi(t)$ as $m, n \to \infty$. This implies that $d(x_m, x_n) \leq t$ as $m, n \to \infty$, so, $\{x_n\}$ is a bounded sequence. Hence, $\{x_n\}$ has a subsequence which converges to u, that is, $\lim_{k\to\infty} x_{n_k} = u$. By the continuity of the mapping T we obtain

$$Tu = T(\lim_{k\to\infty} x_{n_k}) = T(\lim_{k\to\infty} T^{n_k+1}x_0) = T(\lim_{k\to\infty} T^{n_k+2}x_0) = u,$$

so, u is a fixed point of T.

Finally, we show the uniqueness of the fixed point of T. Let z be an another fixed point of T. Then

$$\begin{aligned}\varphi(d(Tz,Tu)) &\le p[\varphi(d(z,u))+\varphi(d(z,Tz))+\varphi(d(u,Tu))\\ &\quad+\varphi(d(z,Tu))+\varphi(d(u,Tz))]\\ &= p[\varphi(d(z,u))+\varphi(d(z,z))+\varphi(d(u,u))+\varphi(d(z,u))+\varphi(d(u,z))]\\ &= 3p\varphi(d(z,u)),\end{aligned}$$

it implies that

$$(1-3p)\varphi(d(Tz,Tu))\le 0,$$

which is a contradiction. Hence, $u=z$. This completes the proof. □

Example 3.7. Let (X,d) be a bounded compact rectangular b-metric space, where $X=[0,\infty)$ and

$$d(x,y)=\begin{cases}|x+y|, & x\ne y,\\ 0, & x=y.\end{cases}$$

Define $T:X\to X$ by

$$T(x)=\begin{cases}\frac{1}{5}, & \text{if } 0\le x\le 2,\\ \frac{1}{x}, & \text{if } x>2.\end{cases}$$

Then, for $\varphi(\Delta)=\frac{1}{5}\Delta$, we have

$$d(Tx,Ty)<\frac{1}{5}[d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)].$$

For $x\ne y$ and $x,y>2$, we have

$$d(Tx,Ty)=|\frac{1}{x}+\frac{1}{y}|<1$$

and

$$\begin{aligned}&\frac{1}{5}[d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)]\\ &=\frac{1}{5}[|x+y|+|x+\frac{1}{x}|+|y+\frac{1}{y}|+|x+\frac{1}{y}|+|y+\frac{1}{x}|]\\ &>1.\end{aligned}$$

Similarly, for $0\le x\le 2$ and $y>2$, we have

$$d(Tx,Ty)=|\frac{1}{5}+\frac{1}{y}|$$

and

$$\begin{aligned}&\frac{1}{5}[d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)]\\&=\frac{1}{5}[|x+y|+|x+\frac{1}{5}|+|y+\frac{1}{y}|+|x+\frac{1}{y}|+|y+\frac{1}{5}|]\\&>|\frac{1}{5}+\frac{1}{y}|.\end{aligned}$$

Thus, T has a unique fixed point $x=\frac{1}{5}$.

It is easy to see that every compact rectangular b-metric space is T-orbitally compact. Also the bounded compactness and T-orbitally compactness are totally independent. Moreover, T-orbitally compactness of X does not give to be complete.

Theorem 3.8. *Let (X,d) be a T-orbitally compact rectangular b-metric space and T satisfying (3.1) with $p<\frac{1}{5}$ and $bp<1$. Then T has a unique fixed point u and*

$$\lim_{n\to\infty} T^n x = u, \quad \forall x\in X.$$

Proof. Let $x_0\in X$ be arbitrary but fixed, and consider the iterative sequence $\{x_n\}$, where $x_n=T^n x_0$ for each $n\in\mathbb{N}$. Now, we denote $d_n=d(x_n,x_{n+1})$ for $n\in\mathbb{N}$. Then by (3.1) we have

$$\begin{aligned}\varphi(d_n) &\le 2p\varphi(d_{n-1})+p\varphi(d_n)+p\varphi(d(x_{n-1},x_{n+1}))\\&\le (2p+pb)\varphi(d_{n-1})+(2p+pb)\varphi(d_n),\end{aligned}$$

it implies that

$$(1-2p-pb)\varphi(d_n)\le(2p+pb)\varphi(d_{n-1}).$$

Since $1-2p-pb\ge 2p+pb$, $p<\frac{1}{5}$ and φ is strictly increasing, we get $d_n<d_{n-1}$, this show that $\{d_n\}$ is a strictly decreasing sequence of non negative real numbers and convergent. Since X is T-orbitally compact, so $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty}x_{n_k}=u$

$$\begin{aligned}\lim_{k\to\infty} d_{n_k} &= \lim_{k\to\infty} d(x_{n_k},x_{n_k+1})\\&= d(\lim_{k\to\infty}x_{n_k},\lim_{k\to\infty}x_{n_k+1})\\&= d(u,u)\\&= 0.\end{aligned}$$

So, $\lim_{n\to\infty} d_n=0$. We have for $n,m\in\mathbb{N}$,

$$\varphi(d(x_n,x_m))\le(pb+pb^2)(\varphi(d_{n-1})+\varphi(d_{m-1}))+(pb+pb^2)\varphi(d(x_n,x_m),$$

it implies that

$$\begin{aligned}(1-pb-pb^2)\varphi(d(x_n,x_m) &\leq (pb+pb^2)(\varphi(d_{n-1})+\varphi(d_{m-1}))\\ &\to 0 \quad \text{as } m,n\to\infty.\end{aligned}$$

This means that the sequence $\{x_n\}$ is Cauchy and $x_n \to u$ as $n \to \infty$. Also we have

$$\begin{aligned}\varphi(d(u,Tu) \leq (2pb+pb^2)\varphi(d(u,x_n)) &+ (2pb+pb^2)\varphi(d(x_{n-1},x_n))\\ &+ (2pb+pb^2)\varphi(d(x_n,x_{n+1})) + pb\varphi(d(u,x_{n+1}))\\ &+ (pb+pb^2)\varphi(d(u,Tu)).\end{aligned}$$

This implies that

$$\begin{aligned}(1-pb-pb^2)\varphi(d(u,Tu) &\leq (2pb+pb^2)[\varphi(d(u,x_n))+\varphi(d(x_{n-1},x_n))\\ &\quad + \varphi(d(x_n,x_{n+1}))] + pb\varphi(d(u,x_{n+1}))\\ &\to 0 \quad \text{as } m,n\to\infty.\end{aligned}$$

Hence, $Tu = u$.

Next, let u^* be an another fixed point of T. Then, we have

$$\begin{aligned}\varphi(d(u,u^*) &= \varphi(d(Tu,Tu^*)\\ &< \varphi(d(u,u^*)+\varphi(d(u,Tu)+\varphi(d(u^*,Tu^*)\\ &\quad + \varphi(d(u,Tu^*)+\varphi(d(u^*,Tu)\\ &= 3p\varphi(d(u,u^*),\end{aligned}$$

so, $\varphi(d(u,u^*)) < 0$ implies $d(u,u^*) < 0$ which is contradiction. Hence, T has a unique fixed point. □

To find a solution, we assume that T is an asymptotically regular mapping, that is, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in \mathbb{N}$.

Theorem 3.9. *Let (X,d) be a complete rectangular b-metric space and $T : X \to X$ be an asymptotically regular mapping satisfying (3.1) for some p with $3bp < 1$. Then T has a unique fixed point.*

Proof. Let $x \in X$ and define the sequence $x_n = T^n x$, $n \in \mathbb{N}$. Since T is an asymptotically regular mapping, we get for $m > n$,

$$\begin{aligned}\varphi(d(T^{n+1}x,T^{m+1}x)) &= \varphi(d(T(T^n x),T(T^m x))\\ &\leq (2pb+p)\varphi(d(T^n x,T^{n+1}x)\\ &\quad + (2pb+p)\varphi(d(T^m x,T^{m+1}x)+3pb\varphi(d(T^{n+1}x,T^{m+1}x),\end{aligned}$$

it implies that

$$(1-3pb)\varphi(d(T^{n+1}x,T^{m+1}x) \leq (2pb+p)[\varphi(d(T^nx,T^{n+1}x)+\varphi(d(T^mx,T^{m+1}x)] \to 0 \quad \text{as } m,n\to\infty.$$

So, $d(T^{n+1}x,T^{m+1}x)\to 0$ as $m,n\to\infty$. Thus $\{x_n\}$ is a Cauchy sequence and convergent in X with $\lim_{n\to\infty}x_n=u$. Hence, we have

$$\begin{aligned}
\varphi(d(u,Tu)) &\leq \varphi(b[d(u,T^nx)+d(T^nx,T^{n+1}x)+d(T^{n+1}x,Tu)]) \\
&\leq b\varphi(d(u,T^nx))+b\varphi(d(T(T^{n-1}x),T(T^nx)))+b\varphi(d(T(T^nx),Tu)) \\
&\leq b\varphi(d(u,T^nx))+bp\varphi(d(T^{n-1}x,T^nx))+bp\varphi(d(T^{n-1}x,T^nx)) \\
&\quad +bp\varphi(d(T^nx,T^{n+1}x))+bp\varphi(d(T^{n-1}x,T^{n+1}x)) \\
&\quad +bp\varphi(d(T^nx,T^nx))+bp\varphi(d(T^nx,u))) \\
&\quad +bp\varphi(d(T^nx,T^{n+1}x)))+bp\varphi(d(u,Tu))) \\
&\quad +bp\varphi(d(T^nx,Tu)))+bp\varphi(d(T^{n+1}x,u))) \\
&\leq b\varphi(d(u,T^nx))+bp\varphi(d(T^{n-1}x,T^nx)) \\
&\quad +bp\varphi(d(T^{n-1}x,T^nx))+bp\varphi(d(T^nx,T^{n+1}x)) \\
&\quad +b^2p\varphi(d(T^{n-1}x,T^nx))+b^2p\varphi(d(T^nx,T^{n+1}x)) \\
&\quad +bp\varphi(d(T^nx,u))+bp\varphi(d(T^nx,T^{n+1}x))+bp\varphi(d(u,Tu)) \\
&\quad +b^2p\varphi(d(T^nx,u))+b^2p\varphi(d(u,Tu))+bp\varphi(d(T^{n+1}x,u)),
\end{aligned}$$

it implies that

$$\begin{aligned}
(1-bp-b^2p)\varphi(d(u,Tu)) &\leq (b+2bp+b^2p)\varphi(d(u,T^nx)) \\
&\quad +(2bp+b^2p)\varphi(d(T^{n-1}x,T^nx)) \\
&\quad +(2bp+b^2p)\varphi(d(T^nx,T^{n+1}x)) \\
&\quad +(bp+b^2p)\varphi(d(T^nx,u))+bp\varphi(d(T^{n+1}x,u))
\end{aligned}$$

as $n\to\infty$, we obtain $d(u,Tu)=0$. Therefore, u is a fixed point of T. Let u^* be an another fixed point of T. Then

$$\begin{aligned}
\varphi(d(u,u^*) &= \varphi(d(Tu,Tu^*) \\
&< \varphi(d(u,u^*)+\varphi(d(u,Tu) \\
&\quad +\varphi(d(u^*,Tu^*)+\varphi(d(u,Tu^*)+\varphi(d(u^*,Tu) \\
&= 3p\varphi(d(u,u^*),
\end{aligned}$$

so, $\varphi(d(u,u^*))<0$ implies $d(u,u^*)<0$ which is contradiction. Hence, T has a unique fixed point. □

Example 3.10. Let (X, d) be a complete rectangular b-metric space and $T : X \to X$ be an asymptotically regular mapping satisfying $Tx = \frac{x}{5}$ for all $x \in X$ and $d(x, y) = |x + y|$, $b = 2$ and $p < \frac{1}{6}$. Then for $\varphi(\Delta) = \sqrt{\Delta}$, we have

$$\varphi(\sqrt{|x+y|^2}) = |x+y| < 6(|x| + |y|).$$

Therefore, T has a unique fixed point $x = 0$.

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