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CONTROLLABILITY OF GENERALIZED FRACTIONAL DYNAMICAL SYSTEMS

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Abstract. This paper deals with the controllability of linear and nonlinear generalized fractional dynamical systems in finite dimensional spaces. The results are obtained by using fractional calculus, Mittag-Leffler function and Schauder's fixed point theorem. Observability of linear system is also discussed. Examples are given to illustrate the theory.

1. Introduction

Fractional differential equations are encountered in various fields of science and engineering. Several authors have contributed to the development of this field. Rich theory has been available for these type of differential equations in the literature [33, 37]. In fact fractional differential equations are alternative models to nonlinear differential equations. During the last decades different concepts of fractional derivatives and integrals are being introduced and used by various researchers, for example, Riemann-Lioville, Caputo, Grunwald-Letnikov, Riesz and Hadamard derivatives and integrals. It is known that fractional derivatives essentially capture the non-local nature of the dynamics. However, while dealing with increasingly complex systems, different types of non-locality arise and researchers try to fit in by generalizing the existing fractional derivatives.

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Almeida et al. [1] introduced a generalized fractional integral and derivative that interpolates the Caputo and Caputo-Hadamard fractional derivatives. Here the kernel is generalized using a parameter ρ that helps in capturing a variety of non-local phenomena and some direct applications found in the literature are image encryption and quantum mechanics related to chaos problems arising in fractional dynamical systems. Due to ever increasing importance of this field various kinds of problems are discussed by many researchers. Among them controllability is one of the important problems of fractional dynamical systems.

The problem of controllability of linear and nonlinear dynamical systems in finite dimensional space has been studied by many authors [3, 4, 34]. Recently several researchers have extended the controllability results to fractional dynamical systems [14, 15, 16, 17, 18, 31, 35, 36, 38, 39, 40, 41], fractional integrodifferential systems [5, 6, 7] and fractional delay dynamical systems [2, 26, 27, 28, 29, 30], neutral systems [8, 9, 24] and systems with delays in control variable [19, 21, 22]. Further the controllability of higher order systems [11, 12, 13] and numerical studies are discussed in [10, 25].

In this paper we study the controllability of linear and nonlinear fractional dynamical systems represented by generalized fractional differential equations in finite dimensional spaces. The results cover a large class of fractional dynamical systems.

2. Preliminaries

In this section we recall some basic facts about fractional integral and derivative. Let $\alpha > 0$, $\rho > 0$ and $[a, b]$ be an interval in $\mathbb{R}$.

Definition 2.1. ([32]) The generalized fractional integral of a function $f(t) \in L_1(a, b)$ is defined as

$$I_{0^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds.$$

Definition 2.2. ([1]) The generalized fractional derivative of a function $f(t)$ is defined as

$${}^C D_{0^+}^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t^\rho - s^\rho)^\alpha} f'(s) ds.$$

The fractional integral and derivative satisfy the following properties:

(i) $I_{0^+}^{\alpha,\rho}$ is linear and bounded,

(ii) ${}^C D_{0^+}^{\alpha,\rho} I_{0^+}^{\alpha,\rho} f(t) = f(t)$,

(iii) $I_{0^+}^{\alpha,\rho}\, {}^C D_{0^+}^{\alpha,\rho} f(t) = f(t) - f(0)$.

The two parameter Mittag-Leffler function is defined as [37]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \beta > 0.$$

Consider the following fractional differential equation

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho} x(t) = Ax(t) + f(t), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, A is an $n \times n$ matrix and $f(t)$ is continuous on $[0, T]$.

Now we want to find the solution representation of the equation (2.1). Multiplying $I_{0^+}^{\alpha,\rho}$ on both sides of (2.1) we get

$$x(t) = x(0) + AI_{0^+}^{\alpha,\rho} x(t) + I_{0^+}^{\alpha,\rho} f(t).$$

To obtain the solution we apply the successive approximation technique. For that define

$$\begin{cases} x_0(t) = x_0, \\ x_n(t) = x_0 + AI_{0^+}^{\alpha,\rho} x_{n-1}(t) + I_{0^+}^{\alpha,\rho} f(t). \end{cases}$$

It follows that

$$\begin{aligned} x_1(t) &= x_0 + AI_{0^+}^{\alpha,\rho} x_0 + I_{0^+}^{\alpha,\rho} f(t) \\ &= x_0 + \frac{Ax_0}{\Gamma(\alpha+1)} \left(\frac{t^\rho}{\rho}\right)^{\alpha} + I_{0^+}^{\alpha,\rho} f(t) \\ &= \sum_{k=0}^{1} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha} + I_{0^+}^{\alpha,\rho} f(t) \end{aligned}$$

and

$$x_2(t) = \sum_{k=0}^{2} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha} + AI_{0^+}^{2\alpha,\rho} f(t) + I_{0^+}^{\alpha,\rho} f(t)$$

and in general

$$\begin{aligned} x_n(t) &= \sum_{k=0}^{n} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha} + \sum_{k=0}^{n-1} A^k I_{0^+}^{(k+1)\alpha,\rho} f(t) \\ &= \sum_{k=0}^{n} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha} + \int_0^t \sum_{k=0}^{n-1} A^k \frac{\rho^{1-(k+1)\alpha}}{\Gamma(\alpha k+\alpha)} \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-\alpha(k+1)}} ds \end{aligned}$$

$$= \sum_{k=0}^{n} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha} + \int_0^t \sum_{k=0}^{n-1} \frac{A^k s^{\rho-1}}{\Gamma(\alpha k+\alpha)} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha(k+1)-1} f(s)ds$$

$$= \sum_{k=0}^{n} \frac{A^k x_0}{\Gamma(\alpha k+1)} \left(\frac{t^\rho}{\rho}\right)^{k\alpha}$$

$$+ \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \sum_{k=0}^{n-1} \frac{A^k}{\Gamma(\alpha k+\alpha)} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha k} f(s)ds.$$

Letting $n \to \infty$, we get

$$x(t) = E_\alpha \left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right) x_0 + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} f(s)ds. \tag{2.2}$$

The equation (2.2) is the solution representation of the generalized fractional differential equation (2.1).

3. Observability result

Consider the fractional order linear time invariant system

$$^{C}D_{0+}^{\alpha,\rho} x(t) = Ax(t), \ t \in [0,T], \tag{3.1}$$

where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix. Along with (3.1) we have a linear observation

$$y(t) = Hx(t), \tag{3.2}$$

where $y \in \mathbb{R}^m$ and H is an $m \times n$ matrix.

Definition 3.1. ([4]) The system (3.1), (3.2) is *observable* on an interval $[0,T]$ if

$$y(t) = Hx(t) = 0, \quad t \in [0,T],$$

implies

$$x(t) = 0, \qquad t \in [0,T].$$

Theorem 3.2. *The observed linear system* (3.1), (3.2) *is observable on* $[0,T]$ *if and only if the observability Grammian matrix*

$$M = \int_0^T E_\alpha \left(A^* \left(\frac{t^\rho}{\rho}\right)^\alpha\right) H^* H E_\alpha \left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \mathrm{d}t \tag{3.3}$$

is positive definite, where the $*$ *denotes the matrix transpose.*

Proof. The solution $x(t)$ of (3.1) corresponding to the initial condition $x(0) = x_0$ is given by

$$x(t) = E_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0$$

and we have, for $y(t) = Hx(t) = HE_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0$,

$$\begin{aligned}\|y\|^2 &= \int_0^T y^*(t)y(t)\mathrm{d}t\\ &= x_0^*\int_0^T E_\alpha\left(A^*\left(\frac{t^\rho}{\rho}\right)^\alpha\right)H^*HE_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\mathrm{d}t\ x_0\\ &= x_0^*\,M\,x_0,\end{aligned}$$

a quadratic form in x_0. Clearly M is an $n\times n$ symmetric matrix. If M is positive definite, then $y = 0$ implies $x_0^*\,M\,x_0 = 0$. Therefore $x_0 = 0$. Hence (3.1), (3.2) is observable on $[0,T]$. If M is not positive definite, then there is some $x_0 \neq 0$ such that $x_0^*\,M\,x_0 = 0$. Then $x(t) = E_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0 \neq 0$, for $t\in[0,T]$ but $\|y\|^2 = 0$, so $y = 0$ and we conclude that (3.1), (3.2) is not observable on $[0,T]$. □

4. Controllability results

Consider the linear fractional dynamical system of the form

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho}x(t) = Ax(t) + Bu(t), \quad t\in J = [0,T],\\ x(0) = x_0,\end{cases} \tag{4.1}$$

where $0<\alpha<1, 0<\rho\le 1, x(t), x_0\in\mathbb{R}^n$ and A is an $n\times n$ matrix, $u\in\mathbb{R}^m$ and B is an $n\times m$ matrix. The solution of (4.1) is given by [23]

$$\begin{aligned} x(t) &= E_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0\\ &\quad + \int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)s^{\rho-1}Bu(s)ds. \end{aligned} \tag{4.2}$$

Definition 4.1. ([4, 34]) System (4.1) is said to be controllable on J, if for every $x_0, x_1\in\mathbb{R}^n$, there exists a control $u(t)$ such that the solution $x(t)$ of (4.1) satisfies the conditions $x(0) = x_0$ and $x(T) = x_1$.

Define the controllability Grammian matrix as

$$W(0,t) = \int_0^t\left[E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)B\right]\left[E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right)B\right]^* ds, \tag{4.3}$$

where the $*$ denotes the matrix transpose.

Theorem 4.2. *The linear system* (4.1) *is controllable if and only if the controllability Grammian W is positive definite.*

Proof. If W is nonsingular then the control

$$u(t) = \left(\frac{T^\rho - t^\rho}{\rho}\right)^{1-\alpha} t^{1-\rho}\left[E_{\alpha,\alpha}\left(A\left(\frac{T^\rho - t^\rho}{\rho}\right)^\alpha\right)B\right]^* \times W(0,T)^{-1}\left[x_1 - E_\alpha\left(A\left(\frac{T^\rho}{\rho}\right)^\alpha\right)x_0\right] \tag{4.4}$$

steers the system (4.1) from $x(0) = x_0$ to $x(T) = x_1$. Thus (4.1) is controllable on J. Conversely assume that (4.1) is controllable and prove that W is nonsingular. This can be established by using contra positive argument as in observability case. □

Consider the nonlinear fractional dynamical system of the form

$$\begin{cases} {}^C D_{0^+}^{\alpha;\rho} x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0,T], \\ x(0) = x_0, \end{cases} \tag{4.5}$$

where A, B, x, u are as before and the function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. The solution of (4.5) is given by [23]

$$\begin{aligned} x(t) = {} & E_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0 \\ & + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} Bu(s)ds \\ & + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} f(s, x(s))ds. \end{aligned} \tag{4.6}$$

Theorem 4.3. *If the linear system* (4.1) *is controllable and the nonlinear function f satisfies the condition*

$$\lim_{|x|\to\infty} \frac{|f(t,x)|}{|x|} = 0,$$

then the nonlinear system (4.5) *is controllable on J.*

Proof. (Sketch of the proof) Define the nonlinear operator $P : C(J;\mathbb{R}^n) \to C(J;\mathbb{R}^n)$ as

$$\begin{aligned} Px(t) = & E_\alpha\left(A\left(\frac{t^\rho}{\rho}\right)^\alpha\right)x_0 \\ & + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} Bu(s)ds \\ & + \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} E_{\alpha,\alpha}\left(A\left(\frac{t^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} f(s,x(s))ds, \quad (4.7) \end{aligned}$$

where the control $u(t)$ is given by

$$\begin{aligned} u(t) \;=\; & \left(\frac{T^\rho - t^\rho}{\rho}\right)^{1-\alpha} t^{1-\rho}\Big[E_{\alpha,\alpha}\left(A\left(\frac{T^\rho - t^\rho}{\rho}\right)^\alpha\right)B\Big]^* W(0,T)^{-1} \\ & \times\Bigg[x_1 - E_\alpha\left(A\left(\frac{T^\rho}{\rho}\right)^\alpha\right)x_0 - \int_0^T \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} \\ & \times E_{\alpha,\alpha}\left(A\left(\frac{T^\rho - s^\rho}{\rho}\right)^\alpha\right) s^{\rho-1} f(s,x(s))ds\Bigg]. \quad (4.8) \end{aligned}$$

Substituting (4.8) in to (4.7) we observe that $Px(T) = x_1$. If we are able to prove that the operator P has a fixed point then the nonlinear system (4.5) is controllable. That is, if $Q(r) = \{x \in C(J;\mathbb{R}^n) : \|x\| \leq r\}$ (r depends on the bounds of $E_\alpha, x_0, E_{\alpha,\alpha}, f$) then it is easy to prove that P maps $Q(r)$ into itself.

Further since f is continuous, this implies that the operator is continuous, and hence is completely continuous by the application of the Arzela-Ascoli theorem. Since $Q(r)$ is closed, bounded and convex, the Schauder fixed point theorem guarantees that P has a fixed point $x \in Q(r)$ such that $Px = x$. This x satisfies the required conditions $x(0) = x_0$ and $x(T) = x_1$. Hence the system (4.5) is controllable [11, 20]. □

Remark 4.4. The above results can be extended to the following larger class of fractional dynamical systems

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho} x(t) = Ax(t) + Bu(t) + f(t, x(t), u(t)), \quad t \in J = [0,T], \\ x(0) = x_0, \end{cases} \quad (4.9)$$

where A, B, x, u are as before and the function $f : J \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and the fractional integrodifferential systems

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho} x(t) = Ax(t) + Bu(t) + f(t, x(t), \int_0^t h(t,s,x(s))ds, u(t)), \\ x(0) = x_0, \end{cases} \quad (4.10)$$

where the functions $h : J\times J\times\mathbb{R}^n \to \mathbb{R}^n$ and $f : J\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ are continuous. Further if we change $Bu(t)$ as the time varying delays $\sum_{i=0}^{M} B_i u(r_i(t))$ and the distributed delays $\int_{-h}^{0} d_s B(t,s)u(t+s)$ with appropriate conditions on the delays in the equations (4.9) and (4.10) then the results of Theorem (4.2) and (4.3) can be proved with similar arguments.

5. Examples

Example 5.1. Consider the linear fractional control system

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho} x(t) = Ax(t) + Bu(t), \ \ t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{5.1}$$

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $0 < \alpha < 1,\ \ 0 < \rho \le 1$.

Take $\rho = 1$ and the fractional control system becomes

$$\begin{cases} {}^C D_{0^+}^{\alpha,1} x(t) = {}^C D_{0^+}^{\alpha} x(t) = Ax(t) + Bu(t), \ \ t \in [0,T], \\ x(0) = x_0. \end{cases} \tag{5.2}$$

For this the Mittag-Leffler matrix function of A is

$$E_{\alpha,\alpha}(At^\alpha) = \begin{bmatrix} E_{2\alpha,\alpha}(t^{2\alpha}) & t^\alpha E_{2\alpha,2\alpha}(t^{2\alpha}) \\ t^\alpha E_{2\alpha,2\alpha}(t^{2\alpha}) & E_{2\alpha,\alpha}(t^{2\alpha}) \end{bmatrix}.$$

The controllability Grammian of this system is

$$W = \int_0^T E_{\alpha,\alpha}(A(T-\tau)^\alpha) BB^* E_{\alpha,\alpha}(A^*(T-\tau)^\alpha) \mathrm{d}\tau.$$

Using matrix calculations, we can show that W is positive definite for $T > 0$. So the linear system (5.2) is controllable on $[0,T]$. By tedious matrix calculation the case $0 < \rho < 1$ can be established for the system (5.1). Further if $B = 0$ in (5.2) then the linear system is observable with observer $y(t) = x(t)$ by Theorem (3.2).

Example 5.2. Consider the nonlinear fractional control system

$$\begin{cases} {}^C D_{0^+}^{\alpha,\rho} x(t) = Ax(t) + Bu(t) + f(t,x), \ \ t \in [0,T], \\ x(0) = x_0, \end{cases} \tag{5.3}$$

where A, B are as above, $0 < \alpha < 1,\ \ \rho = 1$ and $f(t,x) = \begin{pmatrix} \sin x_1 \\ \sin x_2 \end{pmatrix}$.

Since the controllability Grammian W is positive definite for $T > 0$ and so the linear part of the system is controllable on $[0, T]$. Further the nonlinear function f satisfies the hypothesis of the Theorem (4.3). Observe that the control defined by

$$\begin{aligned} u(t) &= (T-t)^{1-\alpha} B^* E_{\alpha,\alpha}(A^*(T-t)^\alpha) W^{-1} \Big[x_1 - E_\alpha(A(T)^\alpha) x_0 \\ &\quad - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(A^*(T-t)^\alpha) f(s, x(s), u(s)) \mathrm{d}s \Big] \end{aligned}$$

steers the system (5.3) from x_0 to x_1 and hence the nonlinear fractional control system (5.3) is controllable on $[0, T]$.

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