Nonlinear Functional Analysis and Applications
Vol. 28, No. 4 (2023), pp. 1127-1143
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.04.18
http://nfaa.kyungnam.ac.kr/journal-nfaa
Copyright © 2023 Kyungnam University Press

CONVERGENCE OF MODIFIED VISCOSITY INEXACT MANN ITERATION FOR A FAMILY OF NONLINEAR MAPPINGS FOR VARIATIONAL INEQUALITY IN $CAT(0)$ SPACES

Kyung Soo Kim

Department of Mathematics Education, Kyungnam University,
Changwon, Gyeongnam, 51767, Republic of Korea
e-mail: kksmj@kyungnam.ac.kr

Abstract. The purpose of this paper, we prove convergence theorems of the modified viscosity inexact Mann iteration process for a family of asymptotically quasi-nonexpansive type mappings in $CAT(0)$ spaces. We also show that the limit of the modified viscosity inexact Mann iteration $\{x_n\}$ solves the solution of some variational inequality.

1. Introduction

Let C be a nonempty subset of a metric space (X, d) and Y be a nonempty subset of C.

(1) The mapping $T : C \to C$ is said to be nonexpansive respect to Y if for each $x \in C$ and $y \in Y$,

$$d(Tx, Ty) \le d(x, y).$$

If $Y = C$, T is called nonexpansive and if $Y = F(T) = \{x \in C : Tx = x\}$, T is called quasi-nonexpansive.

(2) The mapping T is said to be asymptotically nonexpansive respect to Y if there exists a sequence $\{k_n\}$ of positive real numbers such that

[0]Received June 12, 2023. Revised September 4, 2023. Accepted September 7, 2023.
[0]2020 Mathematics Subject Classification: 47H09, 47H10, 47J20, 47J25.
[0]Keywords: $CAT(0)$ space, quasilinearization, asymptotically quasi-nonexpansive type mapping, modified viscosity inexact Mann iteration, $\triangle$-convergence, variational inequality, fixed point.

$k_n \to 1$ and for all $x \in C$ and $y \in Y$,

$$d(T^n x, T^n y) \leq k_n d(x, y).$$

If $Y = C$, T is called asymptotically nonexpansive and if $Y = F(T)$, T is called asymptotically quasi-nonexpansive.

(3) The mapping T is said to be asymptotically nonexpansive type respect to Y if

$$\limsup_{n\to\infty} \sup_{y\in Y}(d(T^n x, T^n y) - d(x, y)) \leq 0,$$

for all $x \in C$. If $Y = C$, T is called asymptotically nonexpansive type and if $Y = F(T)$, T is called asymptotically quasi-nonexpansive type.

It is clear that nonexpansive mappings(quasi-nonexpansive mappings) and asymptotically nonexpansive mappings(asymptotically quasi-nonexpansive mappings) are asymptotically nonexpansive type mappings (resp. asymptotically quasi-nonexpansive type mappings).

(4) The sequence $\{T_n\}$ of self mappings on C is called a family of asymptotically nonexpansive mappings respect to Y if for each T_i, there exists a sequence $\{k_{n,i}\}$ of positive real numbers such that $k_{n,i} \to 1$, as $n \to \infty$, and for all $x \in C$ and $y \in Y$,

$$d(T_i^n x, T_i^n y) \leq k_{n,i} d(x, y).$$

If $Y = C$, the sequence $\{T_n\}$ is called a family of asymptotically nonexpansive mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $\{T_n\}$ is called a family of asymptotically quasi-nonexpansive mappings.

(5) The sequence $\{T_n\}$ of self mappings on C is called a family of asymptotically nonexpansive type mappings respect to Y if each T_i satisfies

$$\limsup_{n\to\infty} \sup_{y\in Y}(d(T_i^n x, T_i^n y) - d(x, y)) \leq 0,$$

for all $x \in C$. If $Y = C$, the sequence $\{T_n\}$ is called a family of asymptotically nonexpansive type mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $\{T_n\}$ is called a family of asymptotically quasi-nonexpansive type mappings.

(6) A mapping $f : C \to C$ is called contractive respect to Y with coefficient $k \in (0, 1)$ if for each $x \in C$ and $y \in Y$,

$$d(f(x), f(y)) \leq kd(x, y).$$

If $Y = C$, f is called a contraction with coefficient $k \in (0, 1)$. f has a unique fixed point when C is a nonempty, closed, and subset of a complete metric space was guaranteed by Banach's contraction principle [2].

The existence theorems of fixed points and convergence theorems for various mappings in $CAT(0)$ spaces have been investigated by many authors [1, 8, 10, 12, 17], [19]-[24], [27], [29]-[34].

Let us to introduce the $CAT(0)$ spaces.

Let (X,d) be a metric space. A *geodesic path* joining $p_1 \in X$ to $p_2 \in X$ (or, a *geodesic* from p_1 to p_2) is a mapping g from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $g(0) = p_1, g(l) = p_2$, and

$$d(g(t), g(t')) = |t - t'|, \quad \forall\, t, t' \in [0,l].$$

In particular, g is an isometry and $d(p_1,p_2) = l$. The image α of g is said to be a *geodesic segment* (or, *metric segment*) joining p_1 and p_2. When it is unique, this geodesic segment is denoted by $[p_1,p_2]$. The space (X,d) is called a *geodesic space* if every two points of X are joined by a geodesic segment, and X is called *uniquely geodesic segment* if there is exactly one geodesic segment joining p_1 and p_2 for each $p_1,p_2 \in X$. A subset $Y \subseteq X$ is called *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(p_1,p_2,p_3)$ is a geodesic metric space (X,d) consists of three *vertices* of $\triangle$(the points $p_1,p_2,p_3 \in X$) and the *edges* of $\triangle$(a geodesic segment between each pair of vertices). A *comparison triangle* for the geodesic triangle $\triangle(p_1,p_2,p_3)$ in (X,d) is a triangle $\bar{\triangle}(p_1,p_2,p_3) = \triangle(\bar{p_1},\bar{p_2},\bar{p_3})$ in $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p_i},\bar{p_j}) = d(p_i,p_j), \quad i,j \in \{1,2,3\}.$$

A comparison triangle for the geodesic triangle always exists(see, [4], [30]).

A geodesic metric space is called a $CAT(0)$ *space*(this term is due to Gromov [15] and it is an acronym for Cartan, Aleksandrov and Toponogov) if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in (X,d) and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all vertices $p_1,p_2 \in \triangle$ and all comparison points $\bar{p_1},\bar{p_2} \in \bar{\triangle}$,
>
> $$d(p_1,p_2) \le d_{\mathbb{R}^2}(\bar{p_1},\bar{p_2}).$$

Let p,p_1,p_2 are points in $CAT(0)$ space, if p_0 is the midpoint of the segment $[p_1,p_2]$, which we will denote by $\frac{p_1 \oplus p_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(p, \frac{p_1 \oplus p_2}{2}\right) = d^2(p,p_0) \le \frac{1}{2}d^2(p,p_1) + \frac{1}{2}d^2(p,p_2) - \frac{1}{4}d^2(p_1,p_2).$$

This inequality is called the *(CN) inequality* ([6]).

Remark 1.1. A geodesic metric space (X, d) is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [4, p.163]).

The above (CN) inequality has been extended as

$$d^2(p, \alpha p_1 \oplus (1-\alpha)p_2) \leq \alpha d^2(p, p_1) + (1-\alpha) d^2(p, p_2) - \alpha(1-\alpha) d^2(p_1, p_2), \quad \forall\, p, p_1, p_2 \in X \tag{CN*}$$

for all $0 \leq \alpha \leq 1$ ([12]).

In the recent years, $CAT(0)$ spaces have attracted many researchers as they treated a very important role in different directions of geometry and mathematics (see [4], [5], [7], [14], [23]). Some examples of $CAT(0)$ spaces are pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [23]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [14]), Hadamard manifolds and many others. Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [23]).

It is well known that a normed linear space satisfies the (CN) inequality if and only if it satisfies the parallelogram identity, that is, it is a pre-Hilbert space ([4]). Hence it is not so unusual to have an inner product-like notion in Hadamard spaces. In [3], they introduced the concept of quasilinearization as follows:

Let us usually denote a pair $(x, y) \in X^2 = X \times X$ by $\overrightarrow{xy}$ and call it a *vector*. Then *quasilinearization* is defined as a mapping $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$ by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \frac{1}{2}(d^2(x, v) + d^2(y, u) - d^2(x, u) - d^2(y, v)), \quad \forall\, x, y, u, v \in X.$$

It is easily seen that

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle, \quad \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$$

and

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{xw}, \overrightarrow{uv} \rangle + \langle \overrightarrow{wy}, \overrightarrow{uv} \rangle$$

for all $x, y, u, v, w \in X$. We say that X satisfies the *Cauchy-Schwarz inequality* if

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \leq d(x, y) d(u, v), \quad \forall\, x, y, u, v \in X. \tag{1.1}$$

Remark 1.2. A geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality([3, Corollary 3]).

In [10], they introduced the concept of duality mapping in $CAT(0)$ spaces, by using the concept of quasilinearization, and studied its relation with subdifferential. Also they proved a characterization of metric projection in $CAT(0)$ spaces as follows.

Theorem 1.3. ([10, Theorem 2.4]) *Let C be a nonempty convex subset of a complete $CAT(0)$ space X. Then*

$$p = P_C x \quad \Leftrightarrow \quad \langle \overrightarrow{yp}, \overrightarrow{px} \rangle \geq 0, \ \ \forall\, y \in C$$

for all $x \in X$ and $p \in C$.

Let C be a nonempty closed subset of a $CAT(0)$ space X and let $T : C \to C$ be an asymptotically nonexpansive type mapping. The Krasnoselski-Mann iteration starting from $x_1 \in C$ is defined by

$$x_{n+1} = \alpha_n T^n(x_n) \oplus (1 - \alpha_n)x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$is a sequence in $[0, 1]$. In 2011, Zhang and Cui [35] consider the convergence of the above iteration (1.2) for continuous mappings of asymptotically nonexpansive mappings.

In 2016, Ranjbar and Khatibzadeh [29] extended the results of Zhang and Cui [35] to a family of asymptotically quasi-nonexpansive mappings in the setting of complete $CAT(0)$ spaces. They consider the sequence given by the modified inexact Mann iteration

$$x_{n+1} = \alpha_n P y_n \oplus (1 - \alpha_n)T_n^n P y_n, \quad d(y_n, x_n) \leq e_n, \quad x_0 \in C, \tag{1.3}$$

where $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on a closed and convex subset C of a complete $CAT(0)$ space X, $\{\alpha_n\} \subset [0, 1]$, $\{e_n\} \subset \mathbb{R}$ and P is the nearest point projection on C. They prove $\triangle$-convergence of the sequence given by (1.3) to be a common fixed point of the sequence $\{T_n\}$ under appropriate assumptions on $\{\alpha_n\}$ and $\{e_n\}$ in complete $CAT(0)$ spaces.

In 2015, using the concept of quasilinearization, Wangkeeree *et al.* [34] proved the strong convergence theorems of the following Moudafi's viscosity iterations for an asymptotically nonexpansive mapping T: for given a contraction mapping f defined on C and $0 < \alpha_n < 1$, let $x_n \in C$ be the unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1 - \alpha_n)T^n x$, that is,

$$x_n = \alpha_n f(x_n) \oplus (1 - \alpha_n)T^n x_n, \ \ \forall\, n \geq 1 \tag{1.4}$$

and $x_1 \in C$ is arbitrary chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T^n x_n, \ \ \forall\, n \geq 1. \tag{1.5}$$

They proved the iterative schemes $\{x_n\}$ defined by (1.4) and (1.5) strongly convergent to the same point $\bar{x} \in F(T)$ with $\bar{x} = P_{F(T)} f(\bar{x})$ which is the unique solution of the variational inequality

$$\langle \overrightarrow{\bar{x} f(\bar{x})}, x\bar{x} \rangle \geq 0, \ \ x \in F(T),$$

where $F(T) = \{x : Tx = x\}$.

The purpose of this paper is to prove convergence theorems of the modified viscosity inexact Mann iteration process

$$\begin{aligned} x_{n+1} &= \alpha_n f(Py_n) \oplus (1-\alpha_n)T_n^n(Py_n), \\ d(y_n, x_n) &\le e_n, \\ x_0 &\in C \end{aligned} \tag{1.6}$$

for a family of asymptotically quasi-nonexpansive type mappings $\{T_n\}$ in $CAT(0)$ spaces, where f is given contraction mapping and P is the nearest point projection on C. We also show that the limit of the modified viscosity inexact Mann iteration $\{x_n\}$ generated by (1.6) solves the solution of some variational inequality.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let C be a nonempty subset of a metric space (X, d). $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n) = \{x : T_n x = x\}$ denotes the set of fixed points of T_n.

We write $(1-t)p_1 \oplus tp_2$ for the unique point p in the geodesic segment joining from p_1 to p_2 such that

$$d(p, p_1) = td(p_1, p_2) \quad \text{and} \quad d(p, p_2) = (1-t)d(p_1, p_2).$$

We also denote by $[p_1, p_2]$ the geodesic segment joining from p_1 to p_2, that is, $[p_1, p_2] = \{(1-t)p_1 \oplus tp_2 : t \in [0,1]\}$. A subset C of a $CAT(0)$ space is convex if $[p_1, p_2] \subset C$ for all $p_1, p_2 \in C$.

Now, we give the concept of $\triangle$-convergence and its some basic properties.

Kirk and Panyanak [24] insisted the concept of $\triangle$-convergence in $CAT(0)$ spaces that was introduced by Lim [25] in 1976 is very similar to the weak convergence in a Banach space setting.

Let $\{x_n\}$ be a bounded sequence in $CAT(0)$ spaces X. For $p \in X$, we set

$$r(p, \{x_n\}) = \limsup_{n\to\infty} d(p, x_n).$$

The asymptotic radius $A_r(\{x_n\})$ of $\{x_n\}$ is given by

$$A_r(\{x_n\}) = \inf \{r(p, \{x_n\}) : p \in X\}$$

and the asymptotic center $A_c(\{x_n\})$ of $\{x_n\}$ is the set

$$A_c(\{x_n\}) = \{p \in X : r(p, \{x_n\}) = A_r(\{x_n\})\}.$$

It is well known that asymptotic center $A_c(\{x_n\})$ consists of exactly one point (see, e.g., [11, Proposition 7, p.767]) in a complete $CAT(0)$ space.

Definition 2.1. ([24]) A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$, that is, $A_c(\{u_n\}) = \{x\}$. In this case one can write

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

The concept of $\triangle$-convergence has been studied by many authors and extend the notion of weak convergence of Hilbert space to $CAT(0)$ spaces.

Lemma 2.2. ([12]) *Let X be a $CAT(0)$ space, $p_1, p_2, z \in X$ and $t \in [0, 1]$. Then*

$$d^2(tp_1 \oplus (1-t)p_2, z) \leq td^2(p_1, z) + (1-t)d^2(p_2, z) - t(1-t)d^2(p_1, p_2).$$

Lemma 2.3. ([28]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\lambda_n\}$ are nonnegative sequences such that*

$$a_{n+1} \leq (1+\lambda_n)a_n + b_n, \quad n \geq 1$$

with $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim \alpha_n$ exists. Moreover, if $\liminf_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. ([24]) *Every bounded sequence in a complete $CAT(0)$ space always has a $\triangle$-convergent subsequence.*

Remark 2.5. In a $CAT(0)$ space, strong convergence in the metric implies $\triangle$-convergence (see, [17, 19]).

Lemma 2.6. ([17, Theorem 2.6]) *Let X be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x if and only if*

$$\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0, \quad \forall\, y \in X.$$

The following two useful lemmas can be found in [34].

Lemma 2.7. ([34]) *Let X be a complete $CAT(0)$ space. Then the following inequality holds*

$$d^2(p, r) \leq d^2(q, r) + 2\langle \overrightarrow{pq}, \overrightarrow{pr} \rangle, \quad \forall\, p, q, r \in X.$$

Lemma 2.8. ([34]) *Let X be a $CAT(0)$ space. For any $l \in (0, 1)$ and $x, y \in X$, let*

$$x_l = lx \oplus (1-l)y.$$

Then, for all $u, v \in X$,

(i) $\langle \overrightarrow{x_l u}, \overrightarrow{x_l v} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{x_l v} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{x_l v} \rangle$,

(ii) $\langle \overrightarrow{x_l u}, \overrightarrow{xv} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{xv} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{xv} \rangle$ *and*
$\langle \overrightarrow{x_l u}, \overrightarrow{yv} \rangle \leq l \langle \overrightarrow{xu}, \overrightarrow{yv} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{yv} \rangle$.

3. Main results

In this section, we prove the convergence of the modified viscosity inexact Mann iteration $\{x_n\}$ generated by (1.6) such that the family $\{T_n\}$ of asymptotically (quasi-)nonexpansive type self-mappings on subset C in a $CAT(0)$ space (X, d) satisfies the following condition:

For subsequence $\{T_{n_j}\}$ of $\{T_n\}$ and $\{x_{n_j}\} \subset C$ such that

$$x_{n_j} \xrightarrow{\triangle} x \quad \text{and} \quad d(x_{n_j}, T_{n_j}^{n_j} x_{n_j}) \to 0.$$

$$\text{Then } x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.1}$$

Theorem 3.1. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X, d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset [0, \infty)$ and $\{y_n\} \subset X$ be sequences such that the modified viscosity inexact Mann iteration $\{x_n\}$ is generated by (1.6). Suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $\{\alpha_n\} \subset [a, b]$ with $a, b \in (0, 1)$. Then we have the following statements.*

(i) *Let $\{T_n\}$ be a sequence of asymptotically nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y \in C}(d(T_i^n x, T_i^n y) - d(x, y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$;

(ii) *Let $e_n \equiv 0$ and $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x, p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$. Moreover,

$$x^* = P_{\mathcal{F}} f(x^*),$$

which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \geq 0, \ \ \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.2}$$

Proof. Let $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings. Suppose $q \in \mathcal{F} \subset C$. Then

$$\begin{aligned} d(x_{n+1}, q) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n (Py_n), q) \\ &\le \alpha_n d(f(Py_n), f(q)) + \alpha_n d(f(q), q) + (1-\alpha_n) d(T_n^n(Py_n), q) \\ &\le \alpha\alpha_n d(Py_n, q) + \alpha_n d(f(q), q) + (1-\alpha_n)(c_{nn} + d(Py_n, q)) \\ &\le d(y_n, q) + \alpha_n d(f(q), q) + (1-\alpha_n) c_{nn} \\ &\le d(x_n, q) + e_n + \alpha_n d(f(q), q) + c_{nn}, \end{aligned}$$

so, by the assumption and Lemma 2.3, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \mathcal{F}$ and $\{x_n\}$, $\{y_n\}$ and $\{Py_n\}$ are bounded. Also are $\{f(Py_n)\}$ and $\{T_n(Py_n)\}$.

I. We claim that $\lim_{n\to\infty} d(x_n, T_n^n x_n) = 0$.

We have

$$\begin{aligned} d(x_{n+1}, T_n^n(Py_n)) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n), T_n^n(Py_n)) \\ &\le \alpha_n d(f(Py_n), T_n^n(Py_n)). \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} d(f(Py_n), T_n^n(Py_n)) &\le d(f(Py_n), x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) \\ &= d(f(Py_n), \alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n)) \\ &\quad + d(x_{n+1}, T_n^n(Py_n)) \\ &\le (1-\alpha_n) d(f(Py_n), T_n^n(Py_n)) + d(x_{n+1}, T_n^n(Py_n)), \end{aligned}$$

we obtain

$$\alpha_n d(f(Py_n), T_n^n(Py_n)) \le d(x_{n+1}, T_n^n(Py_n)). \tag{3.4}$$

Since $\sum_{n=1}^{\infty} \alpha_n < \infty$, from (3.3) and (3.4), we have

$$\begin{aligned} \lim_{n\to\infty} \alpha_n d(f(Py_n), T_n^n(Py_n)) &= \lim_{n\to\infty} d(x_{n+1}, T_n^n(Py_n)) \\ &= 0. \end{aligned} \tag{3.5}$$

From (3.5), we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n), x_n) \\ &\le \alpha_n d(f(Py_n), x_n) + (1-\alpha_n) d(T_n^n(Py_n), x_n) \\ &\le \alpha_n [d(f(Py_n), T_n^n(Py_n)) + d(T_n^n(Py_n), x_n)] \\ &\quad + (1-\alpha_n) d(T_n^n(Py_n), x_n) \\ &= \alpha_n d(f(Py_n), T_n^n(Py_n)) + d(x_n, T_n^n(Py_n)) \\ &\to 0. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} d(x_n, T_n^n x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + d(T_n^n(Py_n), T_n^n x_n) \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + d(Py_n, x_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + d(y_n, x_n) \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + e_n, \end{aligned}$$

from (3.5) and (3.6), we obtain

$$\lim_{n\to\infty} d(x_n, T_n^n x_n) = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $\triangle$-converges to x^*. Therefore, from (3.7), the condition (3.1) guaranties that $x^* \in \mathcal{F}$.

II. Next, we will show that $\{x_n\}$ contains a subsequence converging strongly to x^* such that $x^* = P_{\mathcal{F}} f(x^*)$, which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \ge 0, \ \ \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n).$$

II-1. It follows from Lemma 2.8 (i) that

$$\begin{aligned} d^2(x_{n_j}, x^*) &= \langle \overrightarrow{x_{n_j} x^*}, \overrightarrow{x_{n_j} x^*} \rangle \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) \langle \overrightarrow{T_{n_j}^{n_j}(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) d(T_{n_j}^{n_j}(Py_{n_j}), x^*) d(x_{n_j}, x^*) \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) c_{n_j n_j} d(x_{n_j}, x^*) \\ &\quad + (1-\alpha_{n_j}) d(Py_{n_j}, x^*) d(x_{n_j}, x^*) \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) c_{n_j n_j} d(x_{n_j}, x^*) \\ &\quad + (1-\alpha_{n_j}) (d(x_{n_j}, x^*) + e_{n_j}) d(x_{n_j}, x^*), \end{aligned}$$

thus

$$\alpha_{n_j} d^2(x_{n_j}, x^*) \le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j})(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*),$$

$$
\begin{aligned}
d^2(x_{n_j}, x^*) &\leq \langle \overrightarrow{f(Py_{n_j})x^*}, \overrightarrow{x_{n_j}x^*} \rangle + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&= \langle \overrightarrow{f(Py_{n_j})f(x^*)}, \overrightarrow{x_{n_j}x^*} \rangle + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&\leq d(f(Py_{n_j}), f(x^*)) d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&\leq \alpha(d(x_{n_j}, x^*) + e_{n_j}) d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*)
\end{aligned}
$$

and

$$
\begin{aligned}
(1-\alpha) d^2(x_{n_j}, x^*) &\leq \alpha e_{n_j} d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d^2(x_{n_j}, x^*) &\leq \left(\frac{\alpha}{1-\alpha} e_{n_j} + \frac{1-\alpha_{n_j}}{1-\alpha} \cdot \frac{c_{n_j n_j} + e_{n_j}}{\alpha_{n_j}} \right) d(x_{n_j}, x^*) \\
&\quad + \frac{1}{1-\alpha} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle. \qquad (3.8)
\end{aligned}
$$

Since $\{x_{n_j}\}$ is $\triangle$-convergent to x^*, by Lemma 2.6, we have

$$
\limsup_{n \to \infty} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \leq 0.
$$

It follows from (3.8) that $\{x_{n_j}\}$ converges strongly to x^*. Since

$$
d(y_{n_j}, x^*) \leq d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x^*) = e_{n_j} + d(x_{n_j}, x^*),
$$

$\{y_{n_j}\}$ converges strongly to x^*.

II-2. Next, we show that x^* solves the variational inequality (3.2).
Let $q \in \mathcal{F}$. Since

$$\begin{aligned}
d^2(Py_{n_j}, q) &\le d^2(y_{n_j}, q) \\
&\le (d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x_{n_j+1}) + d(x_{n_j+1}, q))^2 \\
&\le 2d^2(y_{n_j}, x_{n_j}) + 2d^2(x_{n_j}, x_{n_j+1}) + d^2(x_{n_j+1}, q) \\
&\quad + 2\{d(y_{n_j}, x_{n_j})d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1})d(x_{n_j+1}, q) \\
&\quad + d(x_{n_j+1}, q)d(y_{n_j}, x_{n_j})\},
\end{aligned} \tag{3.9}$$

applying Lemma 2.2 and (3.9),

$$\begin{aligned}
d^2(x_{n_j+1}, q) &= d^2(\alpha_{n_j} f(Py_{n_j}) \oplus (1-\alpha_{n_j})T_{n_j}^{n_j}(Py_{n_j}), q) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) + (1-\alpha_{n_j}) d^2(T_{n_j}^{n_j}(Py_{n_j}), q) \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j}) d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) \\
&\quad + (1-\alpha_{n_j})[c_{n_j n_j}^2 + d^2(Py_{n_j}, q) + 2c_{n_j n_j} d(Py_{n_j}, q)] \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j}) d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) \\
&\quad + (1-\alpha_{n_j})[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) + d^2(x_{n_j+1}, q) \\
&\quad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1}) d(x_{n_j+1}, q) \\
&\quad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)] \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j}) d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})).
\end{aligned}$$

So,

$$\begin{aligned}
&\alpha_{n_j}(1-\alpha_{n_j}) d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) + \alpha_{n_j} d^2(x_{n_j+1}, q) \\
&\quad \le \alpha_{n_j} d^2(f(Py_{n_j}), q) + (1-\alpha_{n_j})[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) \\
&\qquad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1}) d(x_{n_j+1}, q) \\
&\qquad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)],
\end{aligned}$$

we have

$$\begin{aligned}
&(1-\alpha_{n_j}) d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) + d^2(x_{n_j+1}, q) \\
&\quad \le d^2(f(Py_{n_j}), q) + \frac{1-\alpha_{n_j}}{\alpha_{n_j}}[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) \\
&\qquad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1}) d(x_{n_j+1}, q) \\
&\qquad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)].
\end{aligned} \tag{3.10}$$

Since $x_{n_j} \to x^*$ and (3.5), we have $T_{n_j}^{n_j}(Py_{n_j}) \to x^*$. Take limit on both sides in (3.10), from assumptions and continuity of the metric distance d, we obtain

$$d^2(f(x^*), x^*) + d^2(x^*, q) \leq d^2(f(x^*), q).$$

Hence

$$\begin{aligned} 0 &\leq \frac{1}{2}[d^2(x^*, x^*) + d^2(f(x^*), q) - d^2(x^*, q) - d^2(f(x^*), x^*)] \\ &= \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{qx^*} \rangle, \quad \forall\, q \in \mathcal{F}, \end{aligned}$$

that is, x^* solves the inequality (3.2).

III. Finally, we will show the uniqueness of the solution of the variational inequality of Equation (3.2). Assume there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $\triangle$-converges to ω by the same argument. We know that $\omega \in \mathcal{F}$ and solves the variational inequality of Equation (3.2), that is,

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* \omega} \rangle \leq 0 \tag{3.11}$$

and

$$\langle \overrightarrow{\omega f(\omega)}, \overrightarrow{\omega x^*} \rangle \leq 0. \tag{3.12}$$

From (3.11) and (3.12), we can obtain

$$\begin{aligned} 0 &\geq \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{\omega f(\omega)}, \overrightarrow{x^* \omega} \rangle \\ &= \langle \overrightarrow{x^* f(\omega)}, \overrightarrow{x^* \omega} \rangle + \langle \overrightarrow{f(\omega) f(x^*)}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{\omega x^*}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{x^* f(\omega)}, \overrightarrow{x^* \omega} \rangle \\ &= \langle \overrightarrow{x^* \omega}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{f(\omega) f(x^*)}, \overrightarrow{\omega x^*} \rangle \\ &\geq \langle \overrightarrow{x^* \omega}, \overrightarrow{x^* \omega} \rangle - d(f(\omega), f(x^*)) d(\omega, x^*) \\ &\geq d^2(x^*, \omega) - \alpha d^2(\omega, x^*) \\ &= (1 - \alpha) d^2(x^*, \omega). \end{aligned}$$

Since $0 < \alpha < 1$, we have

$$d(x^*, \omega) = 0,$$

so

$$x^* = \omega.$$

Hence $\{x_n\}$ converges strongly to x^*, which solves the variational inequality of Equation (3.2). □

If we have $P = I$ (Identity mapping), we get the following result.

Corollary 3.2. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X,d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset [0,\infty)$ and $\{y_n\} \subset X$ be sequences such that the modified viscosity inexact Mann iteration $\{x_n\}$ is generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n f(y_n) \oplus (1-\alpha_n) T_n^n(y_n), \\ d(y_n, x_n) &\leq e_n, \\ x_0 &\in C, \end{aligned}$$

where f is given contraction mapping. Suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $\{\alpha_n\} \subset [a,b]$ with $a, b \in (0,1)$. Then we have the following statements.

(i) *Let $\{T_n\}$ be a sequence of asymptotically nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y\in C}(d(T_i^n x, T_i^n y) - d(x,y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$;

(ii) *let $e_n \equiv 0$ and $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x,p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$. Moreover,

$$x^* = \mathcal{F} \cap F(f),$$

which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \geq 0, \quad \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n), \tag{3.13}$$

where $F(f) = \{x^* \in C : x^* = f(x^*)\}$.

If we have $f = I$(: Identity mapping), we get the following result.

Corollary 3.3. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X,d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset [0,\infty)$ and $\{y_n\} \subset X$ be sequences such that the modified inexact Mann* *iteration* $\{x_n\}$ *is generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n P(y_n) \oplus (1-\alpha_n) T_n^n(Py_n), \\ d(y_n, x_n) &\leq e_n, \\ x_0 &\in C, \end{aligned}$$

where f *is given contraction mapping. Suppose* $\sum_{n=1}^{\infty} e_n < \infty$ *and* $\{\alpha_n\} \subset [a,b]$ *with* $a, b \in (0,1)$*. Then we have the following statements.*

(i) *Let* $\{T_n\}$ *be a sequence of asymptotically nonexpansive type mappings such that the condition* (3.1) *is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y\in C}(d(T_i^n x, T_i^n y) - d(x,y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ *and* $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, *then* $\{x_n\}$ *is convergent to* $q \in \mathcal{F}$*;*

(ii) *Let* $e_n \equiv 0$ *and* $\{T_n\}$ *be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition* (3.1) *is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x,p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ *and* $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, *then* $\{x_n\}$ *is convergent to* $q \in \mathcal{F}$.

Remark 3.4. Corollary 3.2 and 3.3 are generalization and improvement of the results of [34] and [29], respectively.

4. Open problem

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ.

For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces (see, [4], [13], [22]).

Open Problem. It will be interesting to obtain a generalization of both Theorem 3.1, Corollary 3.2 and Corollary 3.3 to $CAT(\kappa)$ space.

Acknowledgments: The author would like to thank the referees for their valuable comments and suggestions which improved the presentation of this paper. This work was supported by Kyungnam University Foundation Grant, 2023.

References

[1] M. Asadi, Sh. Ghasemzadehdibagi, S. Haghayeghi and N. Ahmad, *Fixed point theorems for (α, p)-nonexpansive mappings in $CAT(0)$ spaces*, Nonlinear Funct. Anal. Appl., **26**(5) (2021), 1045–1057.

[2] S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., **3** (1922), 133–181.

[3] I.D. Berg and I.G. Nikoleav, *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedic., **133** (2008), 195–218.

[4] M. Bridson and A. Haefliger, Metric spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg, 1999.

[5] K.S. Brown, Building, Springer, New York, 1989.

[6] F. Bruhat and J. Tits, *Groups réductifss sur un corps local. I. Données radicielles valuées*, Publ. Math. Inst. Hautes Études Sci., **41** (1972), 5–251.

[7] D. Burago, Y. Burago and S. Ivanov, A course in metric Geometry, in:Graduate studies in Math., **33**, Amer. Math. Soc., Providence, Rhode Island, 2001.

[8] P. Chaoha and A. Phon-on, *A note on fixed point sets in $CAT(0)$ spaces*, J. Math. Anal. Appl., **320** (2006), 983–987.

[9] C.E. Chidume, E.U. Ofoedu and H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **280** (2003), 364–374.

[10] H. Dehghan and J. Rooin, *A characterization of metric projection in $CAT(0)$ spaces*, In: Proceedings of International Conference on Functional Equation, Geometric Functions and Applications(ICFGA 2012), Payame Noor University, Tabriz, Iran, 10-12 May 2012, (2012), 41–43.

[11] S. Dhompongsa, W.A. Kirk and B. Sims, *Fixed point of uniformly Lipschitzian mappings*, Nonlinear Anal., **65**(4) (2006), 762–772.

[12] S. Dhompongsa and B. Panyanak, *On $\triangle$-convergence theorems in $CAT(0)$ spaces*, Comput. Math. Anal., **56** (2008), 2572–2579.

[13] R. Espínola and A. Fernández-León, *$CAT(\kappa)$-spaces, weak convergence and fixed points*, J. Math. Anal. Appl., **353**(1) (2009), 410–427.

[14] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Series of Monographs and Textbooks in Pure and Applied Mathematics, **83**, Dekker, New York, 1984.

[15] M. Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ. **8**, Springer, New York, 1987.

[16] N. Hussain and M.A. Khamsi, *On asymptotic pointwise contraction in metric space*, Nonlinear Anal., **71** (2009), 4423–4429.

[17] B.A. Kakavandi, *Weak topologies in complete $CAT(0)$ metric spaces*, Proc. Amer. Math. Soc., **141** (2013), 1029–1039.

[18] K.S. Kim, *Approximation of common fixed points for an implicit iteration with errors of finite family of nonself nonexpansive mappings in Banach spaces*, Dyn. Contin. Discrete Impulse. Syst. Ser. B(Appl. Algorithms), **15**(6) (2008), 807–815.

[19] K.S. Kim, *Equivalence between some iterations in $CAT(0)$ spaces*, J. Comput. Anal. Appl., **24**(3) (2018), 474–485.

[20] K.S. Kim, *Existence theorem of a fixed point for asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces*, Nonlinear Funct. Anal. Appl., **25**(2) (2020), 355–362.

[21] J.K. Kim, R.P. Pathak, S. Dashputre, S.D. Diwan and R. Gupta, *Demiclosedness principle and convergence theorems for Lipshitzian type nonself-mappings in $CAT(0)$ spaces*, Nonlinear Funct. Anal. Appl., **23**(1) (2018), 73–95.
[22] Y. Kimura and K. Satô, *Convergence of subsets of a complete geodesic space with curvature bounded above*, Nonlinear Anal., **75**(13) (2012), 5079–5085.
[23] W.A. Kirk, *A fixed point theorem in $CAT(0)$ spaces and $\mathbb{R}$-trees*, Fixed Point Theory Appl., **2004**(4) (2004), 309–316.
[24] W.A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal., **68**(12) (2008), 3689–3696.
[25] T.C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., **60** (1976), 179–182.
[26] L.S. Liu *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194** (1995), 114–125.
[27] G.A. Okeke, M. Abbas and M. de la Sen, *Fixed point theorems for convex minimization problems in complex valued $CAT(0)$ spaces*, Nonlinear Funct. Anal. Appl., **25**(4) (2020), 671–696.
[28] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling, **32** (2000), 1181–1191.
[29] S. Ranjbar and H. Khatibzadeh, *$\triangle$-convergence and W-convergence of the modified Mann iteration for a family of asymptotically nonexpansive type mappings in complete $CAT(0)$ spaces*, Fixed Point Theory, **17**(1) (2016), 151–158.
[30] G.S. Saluja and J.K. Kim, *On the convergence of modified S-iteration process for asymptotically quasi-nonexpansive type mappings in a $CAT(0)$ space*, Nonlinear Funct. Anal. Appl., **19**(3) (2014), 329–339.
[31] G.S. Saluja, J.K. Kim and H.G. Hyun, *Demiclosed principle and convergence theorems for generalized asymptotically nonexpansive mappings in $CAT(k)$ spaces*, Nonlinear Funct. Anal. Appl., **22**(3) (2017), 555–571.
[32] G.S. Saluja and H.K. Nashine, *Demiclosed principle and $\triangle$-convergence theorems for asymptotically nonexpansive type mappings in $CAT(0)$ spaces*, Indian J. Math., **58**(2) (2016), 193–219.
[33] L.Y. Shi, R.D. Chen and Y.j. Wu, *$\triangle$-convergence problems for asymptotically nonexpansive mappings in $CAT(0)$ spaces*, Abstr. Appl. Anal., **2013** (2013), Article ID 251705.
[34] R. Wangkeeree, U. Boonkong and P. Preechasilp, *Viscosity approximation methods for asymptotically nonexpansive mapping in $CAT(0)$ spaces*, Fixed Point Theory Appl., **2015**(23) (2015), 15 pages.
[35] J. Zhang and Y. Cui, *Existence and convergence of fixed points for mappings of asymptotically nonexpansive type in uniformly convex W-hyperbolic spaces*, Fixed Point Theory Appl., **2011**(39) (2011), 12 pages.