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APPROXIMATING FIXED POINTS FOR GENERALIZED α-NONEXPANSIVE MAPPING IN $CAT(0)$ SPACE VIA NEW ITERATIVE ALGORITHM

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Abstract. In this paper, we provide certain fixed point results for a generalized α-nonexpansive mapping, as well as a new iterative algorithm called SRJ-iteration for approximating the fixed point of this class of mappings in the setting of $CAT(0)$ spaces. Furthermore, we establish strong and Δ-convergence theorem for generalized α-nonexpansive mapping in $CAT(0)$ space. Finally, we present a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature. Our results obtained in this paper improve, extend and unify results of Abbas et al. [10], Thakur et al. [22] and Piri et al. [19].

1. Introduction

Let C be a nonempty subset of a metric space (X,d) and $T\colon C\to C$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T)=\{x\in C\colon x=Tx\}$.

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Remember that a self-map T on a subset C of a metric space X is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C. \tag{1.1}$$

Kirk [13, 14] was the first to introduce fixed point theory of nonexpansive operators in the context of nonlinear $CAT(0)$ spaces. Suzuki [21] made a significant breakthrough in 2008 by introducing a weak notion of nonexpansive operators. It is worth noting that a self-map T on a subset C is said to satisfy Condition (C) (also known as Suzuki map) if for any $x, y \in C$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y). \tag{1.2}$$

In 2011, Aoyama and Kohsaka [2] proposed the class of α-nonexpansive mapping, a mapping $T\colon C \to C$ is said to be α-nonexpansive if there exists $\alpha \in [0, 1)$ for any $x, y \in C$ such that

$$d(Tx, Ty)^2 \leq \alpha d(x, Ty)^2 + \alpha d(y, Tx)^2 + (1 - 2\alpha)d(x, y)^2. \tag{1.3}$$

Recently in 2017, Pant and Shukla [18] introduced the class of generalized α-nonexpansive mappings, a mapping $T\colon C \to C$ is said to be generalized α-nonexpansive if there exists $\alpha \in [0, 1)$ such that

$$\begin{aligned} &\frac{1}{2}d(x, Ty) \leq d(x, y) \\ &\implies d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(Ty, x) + (1 - 2\alpha)d(x, y) \end{aligned} \tag{1.4}$$

for all $x, y \in C$. They established some existence and convergence theorems for the newly introduced class of mappings. One can easily see that every mapping satisfying the Condition (C) is a generalized α-nonexpansive mapping.

Over the last few years many iterative processes have been obtained in different domains to approximate fixed points of various classes of mappings. Mann iteration [17], Ishikawa iteration [9], Halpern iteration [8] and Piri iteration [19] are the few basic iteration processes.

Question: Is it possible to develop an iteration process whose rate of convergence is even faster than the iteration processes defined above?

To answer this, we introduce the new iteration process called SRJ-iteration.

Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X and $T\colon C \to C$ be a mapping. Let $x_1 \in C$ be arbitrary and the sequence

$\{x_n\}$ generated by

$$\begin{cases} x_1 \in C, \\ z_n = T((1-\alpha_n)x_n \oplus \alpha_n T x_n), \\ y_n = T((1-\beta_n)z_n \oplus \beta_n T z_n), \\ x_{n+1} = T((1-\gamma_n)y_n \oplus \gamma_n T y_n), \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1) and $n \in \mathbb{N}$.

In this paper, we introduce a new iteration algorithm to approximate fixed point of generalized α-nonexpansive mapping and then prove numerically our iteration process is faster than well known iteration process Abbas et al. [10], Thakur et al. [22] and Piri et al. [19]. We also established strong and Δ-convergence theorem for same mapping in $CAT(0)$ space and generalizes the results of Akkasriworn et al. [1], Kim et al. [11] and Kim [12].

2. Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a mapping C from a closed interval $[0,r] \subset \mathbb{R}$ to X such that

$$c(0) = x,\ c(r) = y,\ d(c(t), c(s)) = |t-s|$$

for all $s,t \in [0,r]$. In particular, C is an isometry and $d(x,y) = r$. The image of C is call a geodesic segment (or metric segment) joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. We denote the point $w \in [x,y]$ such that $d(x,w) = \alpha d(x,y)$ by $w = (1-\alpha)x \oplus \alpha y$, where $\alpha \in [0,1]$.

The space (X,d) is called a geodesic space if any two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $D \subseteq X$ is said to be convex if D includes geodesic segment joining every two points of itself. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consist of three points (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for geodesic triangle (or $\Delta(x_1,x_2,x_3)$) in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) = \Delta(\bar{x_1},\bar{x_2},\bar{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x_i},\bar{x_j}) = d(x_i,x_j)$$

for $i,j \in \{1,2,3\}$. A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangle of appropriate size satisfy the following $CAT(0)$ comparison axiom:

Let Δ be a geodesic triangle in C and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x,y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality impels

$$d^2(x, \frac{y_1 \oplus y_2}{2}) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2),$$

this inequality is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a $CAT(0)$ space. For other examples, Euclidean buildings [4], Pre-Hilbert spaces, $\mathbb{R}$-trees [3], the complex Hilbert ball with a hyperbolic metric [7] is a $CAT(0)$ space. Further, complete $CAT(0)$ spaces are called Hadamard spaces.

Now, we give some elementary properties about $CAT(0)$ spaces as follows:

Lemma 2.1. ([6]) *Let X be a CAT(0) space, $x, y, z \in X$ and $t \in [0, 1]$. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z).$$

Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space X. For $x \in X$ set:

$$r(x, \{x_n\}) = \limsup_{n \to \infty} \; d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) \colon x \in C\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in C \colon r(x, \{x_n\}) = r(\{x_n\})\}.$$

Remark 2.2. The cardinality of the set $A(\{x_n\})$ in any $CAT(0)$ space is always equal to one, (see e.g., [6]).

The [6, Proposition 2.1] tells us that in the setting of $CAT(0)$ spaces, for every bounded sequence, namely, $\{x_n\} \subset C$, the set $A(\{x_n\})$ is essentially the subset of C provided that C is convex and bounded. It is well known that $\{x_n\}$ has a subsequence which Δ-converges to some point provided that the sequence is bounded.

Definition 2.3. ([6]) A sequence $\{x_n\}$ in $CAT(0)$ space is said to be Δ-convergent to $x \in C$ if x is the unique asymptotic center for every subsequence $\{a_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_n x_n = x$ and read as x is the $\Delta -$ limit of $\{x_n\}$.

Notice that a bounded sequence $\{x_n\}$ in a $CAT(0)$ space is known as regular if and only if for every subsequence, namely, $\{a_n\}$ of $\{x_n\}$ one has $r(\{x_n\}) = r\{a_n\}$. It is well known that, in the setting of $CAT(0)$ spaces each regular sequence is Δ-convergent and consequently each bounded sequence has a Δ-convergent subsequence.

Definition 2.4. ([20]) Let T be a self-map on a subset C of a given $CAT(0)$ space and f be a self-map of $[0, \infty)$. We say that T has condition (I) if the following holds:

(1) $f(g) = 0$ if and only if $g = 0$.
(2) $f(g) > 0$ for every $g > 0$.
(3) $d(x, Tx) \geq f(d(x, F(T)))$.

Definition 2.5. Suppose C is a nonempty subset of a given $CAT(0)$ space. If $T : C \to C$ is a generalized α-nonexpansive mapping, then for every fixed point p of T, we have

$$d(p, Tx) \leq d(p, x) \tag{2.1}$$

for each $x \in C$.

Lemma 2.6. ([14]) *Suppose C is nonempty closed convex subset of a given $CAT(0)$ space. If $T : C \to C$ is a generalized α-nonexpansive mapping and the sequence $\{x_n\} \subseteq C$ satisfy* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ *and* $\Delta - \lim_n x_n = p$, *then $p = Tp$.*

Lemma 2.7. ([14]) *Let C be a nonempty subset of a given $CAT(0)$ space. If $T : C \to C$ is a generalized α-nonexpansive mapping, then the set $F(T)$ always closed. Moreover, if X is strictly convex and C is convex then $F(T)$ is also convex.*

Lemma 2.8. ([15]) *Let C be a nonempty subset of a given $CAT(0)$ space and $T : C \to C$ be a generalized α-nonexpansive mapping. Then for all $x, y \in C$,*

(1) $d(Tx, T^2x) \leq d(x, Tx)$,
(2) *either* $\frac{1}{2}d(x, Tx) \leq d(x, y)$ *or* $\frac{1}{2}d(Tx, T^2x) \leq d(Tx, y)$,
(3) *either* $d(Tx, Ty) \leq \alpha d(Tx, y) + \alpha d(x, Ty) + (1 - 2\alpha)d(x, y)$ *or* $d(T^2x, Ty) \leq \alpha d(Tx, Ty) + \alpha d(T^2x, y) + (1 - 2\alpha)d(Tx, y)$.

Lemma 2.9. ([15]) *Let C be a nonempty subset of a given CAT(0) space and $T : C \to C$ be a generalized α-nonexpansive mapping. Then for all $x, y \in C$,*

$$d(x, Ty) \leq \frac{3 + \alpha}{1 - \alpha} d(x, Tx) + d(x, y). \tag{2.2}$$

Lemma 2.10. ([16]) *Let X be a $CAT(0)$ space and $\{a_n\}$ be any real sequence such that $0 < a \leq a_n \leq b < 1$ for $n \geq 1$. Let $\{y_n\}$ and $\{z_n\}$ be any two sequences of X such that $\limsup_{n\to\infty} d(y_n, x) \leq q$, $\limsup_{n\to\infty} d(z_n, x) \leq q$ and $\limsup_{n\to\infty} d(a_n y_n \oplus (1-a_n)z_n, x) = q$ hold for some $q \geq 0$. Then $\lim_{n\to\infty} d(y_n, z_n) = 0$.*

3. Main results

In this section, we prove strong and Δ-convergence theorems for generalized α-nonexpansive mapping for our results that are generalized the results of Abbas et al. [10], Thakur et al. [22] and Piri et al. [19] in a $CAT(0)$ space.

Theorem 3.1. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined as (1.5), then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

Proof. For any $p \in F(T)$, by Definition 2.5, we have

$$\begin{aligned} d(z_n, p) &= d(T((1-\alpha_n)x_n \oplus \alpha_n Tx_n), p) \\ &\leq ((1-\alpha_n))d(x_n, p) + \alpha_n d(Tx_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.1}$$

Using (1.4), (3.1) and Definition 2.5, we have

$$\begin{aligned} d(y_n, p) &= d(T((1-\beta_n)z_n \oplus \beta_n Tz_n), p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.2}$$

Hence, by using (3.1), (3.2) and Definition 2.5, we get

$$\begin{aligned} d(x_{n+1}, p) &= d(T((1-\gamma_n)y_n \oplus \gamma_n Ty_n), p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(y_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.3}$$

Thus, $\{d(x_n, p)\}$ is a non-increasing sequence which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. $\square$

Theorem 3.2. *Let C be a nonempty closed convex subset of complete CAT(0) space X and $T\colon C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined as in* (1.5)*, then* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof. Suppose that $F(T) \neq \emptyset$ and $p \in F(T)$. Then by Theorem 3.1, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists and $\{x_n\}$ is bounded. Put, for $c \geq 0$,

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.4}$$

If $c \neq 0$, then by (3.1)

$$\limsup_{n\to\infty}\ d(z_n, p) \leq \limsup_{n\to\infty}\ d(x_n, p) = c. \tag{3.5}$$

By using Definition 2.5, we have

$$\limsup_{n\to\infty}\ d(Tx_n, p) \leq \limsup_{n\to\infty}\ d(x_n, p) = c. \tag{3.6}$$

Again by the proof of Theorem 3.1, we have $d(y_n, p) \leq d(x_n, p)$. Therefore,

$$\begin{aligned} d(x_{n+1}, p) &= d(T((1-\gamma_n)y_n \oplus \gamma_n Ty_n), p) \\ &\leq (1-\gamma_n)d(y_n, p) + \gamma_n d(Ty_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(y_n, p). \end{aligned}$$

It follows that

$$\begin{aligned} d(x_{n+1}, p) - d(x_n, p) &\leq \frac{d(x_{n+1}, p) - d(x_n, p)}{\gamma_n} \\ &\leq d(y_n, p) - d(x_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) - d(x_n, p) \\ &\leq d(z_n, p) - d(x_n, p). \end{aligned}$$

So, we can get $d(x_{n+1}, p) \leq d(z_n, p)$ and from (3.4), we have

$$c \leq \lim_{n\to\infty} inf\ d(z_n, p). \tag{3.7}$$

Hence, from (3.5) and (3.7), we obtain

$$c = \lim_{n\to\infty} d(z_n, p). \tag{3.8}$$

Therefore, from (3.8), we have

$$\begin{aligned} c &= \lim_{n\to\infty} d(z_n, p) \\ &= \lim_{n\to\infty} d(T((1-\alpha_n)x_n \oplus \alpha_n Tx_n), p) \\ &\leq \lim_{n\to\infty} [(1-\alpha_n)d(x_n, p) + \alpha_n d(Tx_n, p)] \\ &\leq \lim_{n\to\infty} (1-\alpha_n)d(x_n, p) + \lim_{n\to\infty} \alpha_n d(Tx_n, p) \\ &\leq c. \end{aligned} \tag{3.9}$$

Hence,

$$\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T x_n, p) = c. \tag{3.10}$$

Now, from (3.5), (3.6), (3.10) and Lemma 2.10, we conclude that,

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $p \in A(\{x_n\})$. By Lemma 2.9, we have

$$d(x_n, Tp) \leq (\frac{3+\alpha}{1-\alpha}) d(x_n, Tp) + d(x_n, p). \tag{3.11}$$

This implies that

$$\begin{aligned} r(x_n, Tp) &= \limsup_{n\to\infty}\ d(x_n, Tp) \\ &\leq (\frac{3+\alpha}{1-\alpha}) \limsup_{n\to\infty}\ d(x_n, p) \\ &\leq \limsup_{n\to\infty}\ d(x_n, p) = r(x_n, p). \end{aligned} \tag{3.12}$$

So $Tp \in A(\{x_n\})$. By the uniqueness of asymptotic center, one can conclude that $Tp = p$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T\colon C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence generated by* (1.5)*, then $\{x_n\}$ is Δ-convergent to a fixed point of T.*

Proof. By Theorem 3.2, the sequence $\{x_n\}$ is bounded. Hence one can take $A(\{x_n\}) = \{c\}$ for some $c \in X$. We are going to prove $A(\{x_n\}) = \{c\}$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $A(\{x_n\}) = \{c\}$. Since $\{x_{n_k}\}$ is bounded, one can find a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_j}\}$ is Δ-convergent to p for some $p \in C$. By Theorem 3.2, Lemma 2.6 one has $p \in F(T)$ and hence $\limsup_{n\to\infty}\ d(x_n, p)$ exists. If $p \neq x$, then the singletoness of the cardinality of the asymptotic center allows us the following:

$$\begin{aligned} \limsup_{n\to\infty}\ d(x_n, p) &= \limsup_{j\to\infty}\ d(x_{n_j}, p) \\ &< \limsup_{j\to\infty}\ d(x_{n_j}, x) \\ &\leq \limsup_{k\to\infty}\ d(x_{n_k}, x) \\ &< \limsup_{k\to\infty}\ d(x_{n_k}, p) \\ &= \limsup_{n\to\infty}\ d(x_n, p), \end{aligned} \tag{3.13}$$

which is a contradiction. Therefore, $x = p \in F(T)$. Suppose that $x \neq c$. Then

$$\begin{aligned}\limsup_{n\to\infty}\ d(x_n, x) &= \limsup_{k\to\infty}\ d(x_{n_k}, x)\\ &\leq \limsup_{k\to\infty}\ d(x_{n_k}, c)\\ &\leq \limsup_{n\to\infty}\ d(x_m, c)\\ &< \limsup_{n\to\infty}\ d(x_m, x)\\ &= \limsup_{n\to\infty}\ d(x_n, x),\end{aligned} \tag{3.14}$$

which is also a contraction. Hence $\{x_n\}$ is Δ-convergent to an element $c \in F(T)$. □

Note that the strong convergence of our scheme on a non-compact domain is valid by following theorem.

Theorem 3.4. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T\colon C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by (1.5), then $\{x_n\}$ strongly converges to a fixed point of T if and only if $\lim_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. If the sequence $\{x_n\}$ converges to a point $p \in F(T)$, then

$$\liminf_{n\to\infty} d(x_n, p) = 0,$$

and so,

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, assume that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Theorem 3.1, we have

$$d(x_{n+1}, p) \leq d(x_n, p),\ \forall\ p \in F(T),$$

so we have,

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)). \tag{3.15}$$

Thus, $d(x_n, F(T))$ is a decreasing sequence which is bounded below by zero as well, thus $\lim_{n\to\infty} d(x_n, F(T))$ exists. As, $\lim_{n\to\infty} inf d(x_n, F(T)) = 0$ so $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j\}$ in $F(T)$ such that $d(x_{n_j}, x_j) \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Theorem 3.1, we have

$$\begin{aligned}d(x_{n_{j+1}}, x_j) &\leq d(x_{n_j}, x_j)\\ &\leq \frac{1}{2^j}.\end{aligned}$$

Using triangle inequality, we get

$$\begin{aligned} d(x_{n_{j+1}}, x_j) &\leq d(x_{j+1}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_j) \\ &\leq \frac{1}{2^{j+1}} + \frac{1}{2^j} \\ &\leq \frac{1}{2^{j-1}} \\ &\to 0 \ \ as \quad j \to \infty. \end{aligned}$$

So, $\{x_j\}$ is a Cauchy sequence in $F(T)$. By Lemma 2.7, we have $F(T)$ is closed, so $\{x_j\}$ converges to some $x \in F(T)$. Again, owing to triangle inequality, we have

$$d(x_{n_j}, x) \leq d(x_{n_j}, x_j) + d(x_j, x).$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $x \in F(T)$.

Since $\liminf_{n\to\infty} d(x_n, x)$ exists by Theorem 3.1, $\{x_n\}$ converges to $x \in F(T)$. □

Eventually, we discuss the strong convergence for our scheme (1.5) by using the condition(I) given by Definition 2.4.

Theorem 3.5. *Let C be a nonempty closed convex subset of complete $CAT(0)$ space X and $T: C \to C$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by* (1.5) *and T satisfies the Condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From (3.15), $\lim_{n\to\infty} d(x_n, F(T))$ exists. Also, by Theorem 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the Condition (I) that

$$\begin{aligned} \lim_{n\to\infty} f(d(x_n, F(T))) &\leq \lim_{n\to\infty} d(x_n, Tx_n) \\ &= 0. \end{aligned}$$

So $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Theorem 3.4, the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. □

4. Numerical example

Let $C = [0, \infty)$ which is a closed convex subset of the $CAT(0)$ space $X = \mathbb{R}$, endowed with the usual metric. Define a mapping $T: C \to C$ by

$$Tx = \begin{cases} \frac{x}{2}, \ if \ x > 2, \\ 0, \ if \ x \in [0, 2], \end{cases}$$

for all $x \in C$. Then, we know that

(1) T does not satisfy condition (C);

(2) T is a generalized α-nonexpansive mapping with $\alpha = \frac{1}{3}$.
And, take $\alpha_n = \frac{3n}{8n+4}$, $\beta_n = \frac{1}{n+4}$ and $\gamma_n = \frac{n}{(2n+6)^2}$. Also the initial values $x_1 = 1000$ and $p = 0$ is a fixed point of T. Table 1 shows that behavior of iteration processes Abbas et al. [10], Thakur et al. [22] and Piri et al. [19] and new iteration (1.5) of T in 7-step.

Table 1. Convergence of new iteration (1.5) for fixed point 0.

No. of iteration	Abbas iteration	Thakur new iteration	Piri iteration	New iteration
0	1000	1000	1000	1000
1	338.73333	247.18750	198.59375	98.634895
2	114.74027	61.101660	39.439477	9.7288426
3	38.866354	15.103566	7.8324337	0
4	13.165329	3.7334128	1.2140272	0
5	4.4595260	0	0	0
6	1.5105930	0	0	0
7	0	0	0	0

In what follows, we numerically compare our new iteration process (1.5) with some existing iteration processes.

Case I: Taking, $\alpha_n = \frac{3n}{8n+4}$, $\beta_n = \frac{1}{n+4}$, $\gamma_n = \frac{n}{(2n+6)^2}$ and $x_1 = 1000$.
Case II: Taking, $\alpha_n = \frac{n}{n+1}$, $\beta_n = \frac{1}{(n+7)^2}$, $\gamma_n = \frac{2n}{(5n+2)}$ and $x_1 = 1000$.
Case III: Taking, $\alpha_n = 1 - \frac{1}{(2n+8)}$, $\beta_n = \frac{n}{16n+1}$, $\gamma_n = \frac{n}{(n+5)}$ and $x_1 = 1000$.
Case IV: Taking, $\alpha_n = \frac{2n}{3n+2}$, $\beta_n = \frac{n}{\sqrt{n+2}}$, $\gamma_n = \sqrt{\frac{2n}{(3n+5)}}$ and $x_1 = 1000$.

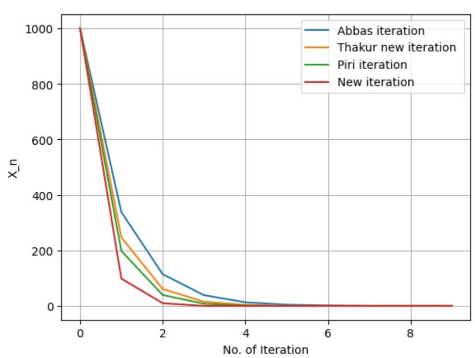

Figure 1. Case (I)

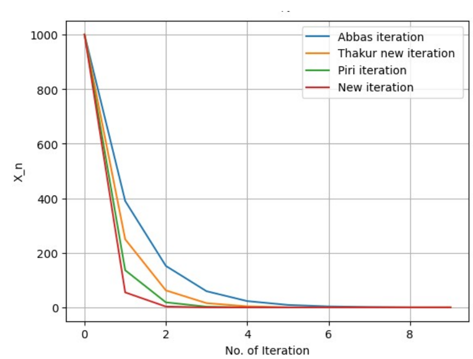

Figure 2. Case (II)

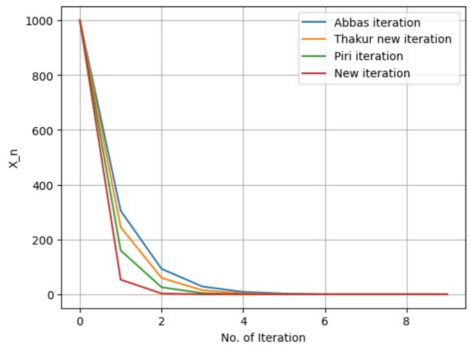

Figure 3. Case (III)

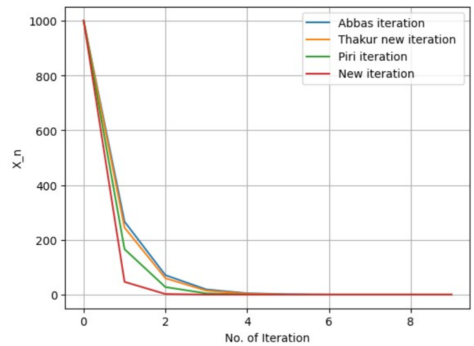

Figure 4. Case (IV)

We see from Figure 1, 2, 3 and 4 that Abbas et al. [10], Thakur et al. [22], Piri et al. [19] and new iteration (1.5) processes are stable with respect to the choice of different parameters.

5. Conclusion

In this paper, we have presented a new type of iteration procedure called SRJ-iteration for generalized α-nonexpansive mapping in $CAT(0)$ spaces. Our result generalizes results of Abbas et al. [10], Thakur et al. [22] and Piri et al. [19] in the sense of faster iteration process.

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