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ON A TYPE OF DIFFERENTIAL CALCULUS IN THE FRAME OF GENERALIZED HILFER INTEGRO-DIFFERENTIAL EQUATION

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Abstract. In this paper, we investigate the existence and uniqueness of solutions to a new class of integro-differential equation boundary value problems (BVPs) with $\top$-Hilfer operator. Our problem is converted into an equivalent fixed-point problem by introducing an operator whose fixed points coincide with the solutions to the given problem. Using Banach's and Schauder's fixed point techniques, the uniqueness and existence result for the given problem are demonstrated. The stability results for solutions of the given problem are also discussed. In the end. One example is provided to demonstrate the obtained results

1. Introduction

Because of their numerous applications in mathematics, biology, physics, finance, engineering, dynamical systems and control theory, fractional differential equations (FDEs) are of great interest, see [2, 9, 11, 13, 15, 22] and the references therein. However, because of the complexities of their initial values,

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several physical interpretations of FDEs are still unknown, so the theory of FDEs is still in its infancy. Nonetheless, because of their numerous practical applications and theoretical significance, these equations have become the most popular topic of discussion among a number of examiners. There has also been shown a significant interest in the study of FDEs by many authors, for instance [5, 6, 12, 16, 18, 20].

Sousa and Oliveira [25] recently proposed a new type of fractional differential (FD) operator called a ψ-Hilfer fractional operator, which generalises the Hilfer fractional operator [9, 10]. It is important to note that the ψ-Hilfer fractional derivative is defined with respect to another function, and it unifies the various fractional derivative definitions found in the literature.

Thabet et al. [24] investigated various criteria for the existence of solutions for the following boundary value problem of Hilfer fractional integro-differential equations with boundary conditions using the measure of noncompactness technique and Mönch fixed point theorems (FPTs):

$$\begin{cases} \left(\mathcal{D}_{\mathfrak{a}^+}^{\sqsupset_1,\sqsupset_2}\right)\Xi(\upsilon) = \aleph(\upsilon,\Xi(\upsilon),\Re(\Xi)(\upsilon)),\ \upsilon\in\ [0,1],\ 0<\sqsupset_1<1, 0\le\sqsupset_2\le 1, \\ \mathcal{I}_{\mathfrak{a}^+}^{1-\zeta}\left(d_1\Xi(\mathfrak{a}^+)+d_2\Xi(\mathfrak{b}^-)\right)=d_3, \qquad \sqsupset_1\le\zeta=\sqsupset_1+\sqsupset_2(1-\sqsupset_1), \end{cases} \tag{1.1}$$

where $\mathcal{D}_{\mathfrak{a}^+}^{\sqsupset_1,\sqsupset_2}$ is the Hilfer FD of order $\sqsupset_1$ and type $\sqsupset_2$.

Significant attention was paid to the topic of fractional boundary value problems with a variety of boundary conditions. Many researchers, in particular, have established the theoretical aspects of such problems using the tools of fixed point theory. We refer the reader to the works for more information and examples [1, 17, 19]. Also, the authors in [4, 14] investigate some nonlocal fractional BVP problems with $\top$-Hilfer FDs, for exrea studies about Multipoint BVP (see [3]).

In this manuscript, we apply FPTs to examine the existence of solutions for a generalized Hilfer-type integro-differential equation with positive constant coefficient listed below:

$$\begin{cases} {}^H\mathcal{D}_{\mathfrak{a}^+}^{\sqsupset_1,\sqsupset_2;\top}\Xi(\upsilon)=\lambda\Xi(\upsilon)+\aleph(\upsilon,\Xi(\upsilon),\Re(\Xi)(\upsilon)),\ \upsilon\in ß=[\mathfrak{a},\mathfrak{b}], \\ \qquad\qquad 0<\sqsupset_1<1,\quad 0\le\sqsupset_2\le 1, \\ \mathcal{N}_1\Xi(\mathfrak{a}^+)+\mathcal{N}_2\Xi(\mathfrak{b}^-)=\mathcal{N}_3,\ \mathcal{N}_i\in\mathbb{R},\quad (i=1,2,3), \\ \qquad\qquad \sqsupset_1\le\zeta=\sqsupset_1+\sqsupset_2(1-\sqsupset_1),\quad \lambda>0, \end{cases} \tag{1.2}$$

where ${}^H\mathcal{D}_{\mathfrak{a}^+}^{\sqsupset_1,\sqsupset_2;\top}$ is the $\top$-Hilfer FD of order $\sqsupset_1$ and type $\sqsupset_2$. $\aleph : ß\times \times \mathbb{R}^2\to\mathbb{R}$ is a continuous function, and $\Re$ is a linear integral operator defined by $\Re(\Xi)(\upsilon)=\int_{\mathfrak{a}}^{\upsilon}K(\upsilon,\tau)\Xi(\tau)d\tau$ with $\Im=\max\{\int_{\mathfrak{a}}^{\upsilon}K(\upsilon,\tau)d\tau) : (\upsilon,\tau)\in ß\times ß\}$, $K\in\mathcal{C}(ß\times ß,\mathbb{R})$.

The goal of this study is to further develop the area of fractional boundary value problems by considering integro-differential equations with generalized fractional derivative operators. In general, our new results remain valid for different values of the function $\top$ and cover a wide range of corresponding problems, for example. Letting $\top(\upsilon) = \log \upsilon$, $\top(\upsilon) = \upsilon^{\beta}$, $\beta > 0$, $\top(\upsilon) = \upsilon$, and $\daleth_2 = 1$, $\top(\upsilon) = \upsilon$, and $\daleth_2 = 0$, the $\top$-Hilfer FD reduces to the Hilfer-Hadamard, Hilfer-Katugampola, Caputo-type, Riemann–Liouville (RL)-type, respectively.

The remainder of the article is structured as follows. Section 2: contains some preliminary results. Section 3 and 4: demonstrate the main results. Section 5: provides an example of how the theoretical results can be applied in practice. Section 6: concludes with a summary of our findings.

2. Preliminary notions

In this portion, we introduce some notations and definitions of fractional calculus. Let $L(\text{ß}, \mathbb{R})$ and $\mathcal{C}(\text{ß}, \mathbb{R})$ are Lebesgue integrable functions and Banach space from $\text{ß} \to \mathbb{R}$ with the norms

$$\|\Xi\|_L = \int_{\mathfrak{a}}^{\mathfrak{b}} |\Xi(\upsilon)|\, d\upsilon$$

and

$$\|\Xi\|_\infty = \sup\{|\Xi| : \upsilon \in \text{ß}\},$$

respectively.

Definition 2.1. ([13]) Let $\daleth_1 > 0$ and $\Xi \in L^1(\text{ß}, \mathbb{R})$. The left sided $\top-$RL fractional integral of order $\daleth_1$ defined by

$$\mathcal{I}^{\daleth_1;\top}\Xi(\upsilon) = \frac{1}{\Gamma(\daleth_1)} \int_{\mathfrak{a}}^{\upsilon} \top'(\iota)(\top(\upsilon) - \top(\iota))^{\daleth_1 - 1}\Xi(\iota)d\iota.$$

Definition 2.2. ([25]) Let $n-1 < \daleth_1 < n$, $0 \le \daleth_2 \le 1$. The $\top-$Hilfer FD of order $\daleth_1$ and type $\daleth_2$ is given by

$${}^{H}\mathcal{D}^{\daleth_1,\daleth_2;\top}\Xi(\upsilon) = \mathcal{I}^{\daleth_2(n-\daleth_1);\top}\Big(\frac{1}{\top'(\upsilon)}\frac{d}{d\upsilon}\Big)^n\, \mathcal{I}^{(1-\daleth_2)(n-\daleth_1);\top}\Xi(\upsilon),$$

where $\upsilon > \mathfrak{a}$.

Lemma 2.3. ([13, 25]) *Let* $\daleth_1, \eta$, *and* $\alpha > 0$. *Then*

(1) $\mathcal{I}^{\daleth_1;\top}\mathcal{I}^{\eta;\top}\Xi(\upsilon) = \mathcal{I}^{\daleth_1+\eta;\top}\Xi(\upsilon)$.
(2) $\mathcal{I}^{\daleth_1;\top}(\top(\upsilon) - \top(\mathfrak{a}))^{\alpha-1} = \frac{\Gamma(\alpha)}{\Gamma(\daleth_1+\alpha)}(\top(\upsilon) - \top(\mathfrak{a}))^{\daleth_1+\alpha-1}$.

We also note that ${}^{H}\mathcal{D}^{\daleth_1,\daleth_2;\top}(\top(\upsilon) - \top(\mathfrak{a}))^{\zeta-1} = 0$.

Lemma 2.4. ([25]) *Let* $\Xi \in L(\mathfrak{a},\mathfrak{b})$, $\beth_1 \in (n-1,n]$ $(n\in\mathbb{N})$, $\beth_2\in[0,1]$. *Then*

$$\left(\mathcal{I}^{\beth_1;\top H}\mathcal{D}^{\beth_1,\beth_2;\top}\Xi\right)(\upsilon)=\Xi(\upsilon)-\sum_{k=1}^{n-1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-k}}{\Gamma(\zeta-k+1)}\Xi_{\top}^{[n-k]}\mathcal{I}^{(1-\beth_2)(n-\beth_1);\top}\Xi(\mathfrak{a}),$$

where $\Xi_{\top}^{[n-k]}=(\frac{1}{\top'(\upsilon)}\frac{d}{d\upsilon})^{[n-k]}\Xi(\upsilon)$.

Lemma 2.5. *Let* $\zeta=\beth_1+\beth_2(1-\beth_1)$, *where* $0<\beth_1<1$, $0\le\beth_2\le 1$, *and* $\Xi\in\mathcal{C}(\beta,\mathbb{R})$. *Then, the solution of the following* $\top-$*Hilfer type of BVP*

$$\begin{cases}{}^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\Xi(\upsilon)=\lambda\Xi(\upsilon)+\mathcal{M}(\upsilon),\ \upsilon\in\beta,\\ \mathcal{N}_1\Xi(\mathfrak{a}^+)+\mathcal{N}_2\Xi(\mathfrak{b}^-)=\mathcal{N}_3,\ \mathcal{N}_i\in\mathbb{R},\quad(i=1,2,3)\end{cases}\tag{2.1}$$

is given by

$$\Xi(\upsilon)=\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\mathfrak{b})+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\upsilon),\tag{2.2}$$

where

$$\hbar=\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1\Gamma(\zeta)}(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}\right)\neq 0.\tag{2.3}$$

Proof. Let Ξ be a solution of the first equation of (2.1). Applying $\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}$ on the first equation (2.1) with Lemma 2.4, and setting $\mathcal{I}_{\mathfrak{a}^+}^{1-\zeta;\top}\Xi(\mathfrak{a})=\Xi_{\mathfrak{a}^+}$, we obtain

$$\Xi(\upsilon)=\frac{\Xi(\mathfrak{a}^+)}{\Gamma(\zeta)}(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}+\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\upsilon)+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\upsilon).\tag{2.4}$$

Now, taking the limit $\upsilon\to\mathfrak{b}^-$ in (2.4), we get

$$\Xi(\mathfrak{b}^-)=\frac{\Xi(\mathfrak{a}^+)}{\Gamma(\zeta)}(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}+\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\mathfrak{b}^-)+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\mathfrak{b}^-).\tag{2.5}$$

By use the boundary value condition $\mathcal{N}_1\Xi(\mathfrak{a}^+)+\mathcal{N}_2\Xi(\mathfrak{b}^-)=\mathcal{N}_3$, we have

$$\Xi(\mathfrak{a}^+)=\frac{\mathcal{N}_3}{\mathcal{N}_1}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\Xi(\mathfrak{b}^-),\ \text{where }\mathcal{N}_1\neq 0.\tag{2.6}$$

Submitting (2.5) into (2.6), we obtain

$$\begin{aligned}\Xi(\mathfrak{a}^+)&=\frac{\mathcal{N}_3}{\mathcal{N}_1}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{\Xi(\mathfrak{a}^+)}{\Gamma(\zeta)}(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}\\&\quad-\frac{\mathcal{N}_2}{\mathcal{N}_1}\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\mathfrak{b})+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\mathfrak{b})\right],\end{aligned}$$

$$\Xi(\mathfrak{a}^+)(1+\frac{\mathcal{N}_2}{\mathcal{N}_1\Gamma(\zeta)}(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1})$$
$$=\frac{\mathcal{N}_3}{\mathcal{N}_1}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\mathfrak{b})+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\mathfrak{b})\right],$$

which gives

$$\Xi(\mathfrak{a}^+)=\frac{1}{\hbar}\left(\frac{\mathcal{N}_3}{\mathcal{N}_1}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\mathfrak{b})+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\mathcal{M}(\mathfrak{b})\right]\right), \tag{2.7}$$

where $\hbar$ is defined in (2.3). Submitting (2.7) into (2.4), we obtain (2.2). The proof is completed. □

We are ready to present our main findings. Our theorems are concerned with the uniqueness and existence solutions of problem (1.2) and relies on Banach's FPT [8] and Schauder's FPT [26].

Now, according to Lemma 2.5, we define the operator $\wp:\mathcal{C}(ß,\mathbb{R})\to\mathcal{C}(ß,\mathbb{R})$ by

$$\begin{aligned}(\wp\Xi)(\upsilon) \quad = \quad & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}-\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\\ & \times\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\mathfrak{b})\ +\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\aleph(\mathfrak{b},\Xi(\mathfrak{b}),\Re(\Xi)(\mathfrak{b}))\right]\\ & +\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Xi(\upsilon)+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\aleph(\upsilon,\Xi(\upsilon),\Re(\Xi)(\upsilon)).\end{aligned} \tag{2.8}$$

It should be noted that the integro-differential type problem (1.2) has a solution if and only if $\wp$ has a fixed point. For the sake of consistency, we have set the constants:

$$\Lambda:=\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}+\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right) \tag{2.9}$$

and

$$\Omega=(\lambda+\ell\Im)\,\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right). \tag{2.10}$$

3. Integro-differential type problem (1.2)

We introduce the hypotheses required in the sequel before proceeding with the main results.

($\mathcal{H}_1$) There exists $\emptyset_1,\emptyset_2\in(0,1)$ such that

$$|\aleph(\upsilon,\Xi_1,\Xi_1^*)-\aleph(\upsilon,\Xi_2,\Xi_2^*)|\le\emptyset_1\,|\Xi_1-\Xi_2|+\emptyset_2\,|\Xi_1^*-\Xi_2^*|$$

for any $\Xi_1,\Xi_1^*,\Xi_2,\Xi_2^*\in\mathbb{R}$ and $\emptyset_1,\emptyset_2\inß$.

($\mathcal{H}_2$) Let $\aleph \in \mathcal{C}(ß\times\mathbb{R}^2, \mathbb{R})$ be a function such that $\aleph(., \Xi(.), \Re(\Xi)(.)) \in \mathcal{C}(ß\times\mathbb{R}^2)$ for any $\Xi \in \mathcal{C}(ß, \mathbb{R}^+)$ and there exists a constant $\ell, \Im > 0$ such that

$$|\aleph(\upsilon, \Xi, \Re(\Xi))| \leq \ell(1 + \Im\,|\Xi|), \quad \forall(\upsilon, \Xi, \Re(\Xi)) \in ß \times \mathbb{R}^2.$$

($\mathcal{H}_3$) Let $\aleph \in \mathcal{C}(ß\times\mathbb{R}^2, \mathbb{R})$ be a function such that $\aleph(., \Xi(.), \Re\Xi(.)) \in \mathcal{C}(ß\times\mathbb{R}^2)$ and $\Phi \in \mathcal{C}(ß, \mathbb{R}^+)$ such that

$$|\aleph(\upsilon, \Xi, \Re\Xi)| \leq \Phi(\upsilon), \quad \forall(\upsilon, \Xi, \Re\Xi) \in ß \times \mathbb{R}^2.$$

Theorem 3.1. *Assume that* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold. If*

$$\Omega_1 = (\lambda + \emptyset_1 + \emptyset_2\Im)\left(\frac{(\top(\mathfrak{b}) - \top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1 + 1)}\right)\left(1 + \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b}) - \top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right) < 1, \tag{3.1}$$

then the integro-differential type problem (1.2) *has a unique solution on ß.*

Proof. We convert (1.2) into a fixed point problem, that is, $\Xi = \wp\Xi$ such that $\wp : \mathcal{C}(ß, \mathbb{R}) \to \mathcal{C}(ß, \mathbb{R})$ defined by (2.8).

Note that the fixed points of $\wp$ are solutions of (1.2). we will prove that $\wp$ has a unique fixed point by applying Banach theorem [8]. Indeed, we choose

$$\gamma \geq \frac{\Lambda}{1 - \Omega},$$

where Λ and Ω are define in (2.9) and (2.10), respectively.

First, we show that $\wp\mathfrak{Z}_\gamma \subset \mathfrak{Z}_\gamma$, where

$$\mathfrak{Z}_\gamma = \{\Xi \in \mathcal{C}(ß, \mathbb{R}) : \|\Xi\| \leq \gamma\}. \tag{3.2}$$

For any $\Xi \in \mathfrak{Z}_\gamma$, we get

$$\begin{aligned}
|(\wp\Xi)(\upsilon)| \leq\ & \sup_{\upsilon\in ß}\Bigg\{\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon) - \top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \left(\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon) - \top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right) \\
& \times\left(\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\Xi(\mathfrak{b})| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\aleph(\iota, \Xi(\iota), \Re(\Xi)(\iota)|(\mathfrak{b})\right) \\
& + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\Xi(\upsilon)| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\aleph(\iota, \Xi(\iota), \Re(\Xi)(\iota)|(\upsilon)\Bigg\} \\
\leq\ & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon) - \top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon) - \top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \\
& \times\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\Xi(\mathfrak{b})| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\ell(1 + \Im\,|\Xi|)(\mathfrak{b})\right] \\
& + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\Xi(\upsilon)| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\ell(1 + \Im\,|\Xi|)(\upsilon)
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}+\left(\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right)\\
&\times\Big[\lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi\|+\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\\
&+\ell\Im\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi\|\Big]+\lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi\|\\
&+\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}+\ell\Im\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi\|\\
\leq\ & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}+\frac{\mathcal{N}_2}{\mathcal{N}_1}\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta+\beth_1-1}}{|\hbar|\,\Gamma(\zeta)\Gamma(\beth_1+1)}\\
&+\frac{\mathcal{N}_2}{\mathcal{N}_1}(\lambda+\ell\Im)\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta+\beth_1-1}}{|\hbar|\,\Gamma(\zeta)\Gamma(\beth_1+1)}\|\Xi\|\\
&+\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}+(\lambda+\ell\Im)\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi\|\\
=\ & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\\
&+\ell\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right)\\
&+(\lambda+\ell\Im)\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right)(\gamma)\\
\leq\ & \Lambda+\Omega\gamma\\
\leq\ & \gamma.
\end{aligned}
$$

This means that $\wp\mathfrak{Z}_\gamma\in\mathfrak{Z}_\gamma$. that is, $\wp\mathfrak{Z}_\gamma\subset\mathfrak{Z}_\gamma$.

Next, for each $\Xi,\Xi^\star\in\mathcal{C}(ß,\mathbb{R})$ and $\upsilon\in ß$, we have

$$
\begin{aligned}
|(\wp\Xi)(\upsilon)-(\wp\Xi^\star)(\upsilon)|\leq\ & \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\Big[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|(\Xi(\iota)-\Xi^\star(\iota))|(\mathfrak{b})\\
&+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\aleph(\iota,\Xi(\iota),\Re(\Xi)(\iota))-\aleph(\iota,\Xi^\star(\iota),\Re\Xi^\star(\iota))|(\mathfrak{b})\Big]\\
&+\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\Xi(\iota)-\Xi^\star(\iota)|(\upsilon)\\
&+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}(|\aleph(\iota,\Xi(\iota),\Re\Xi(\iota))|-|\aleph(\iota,\Xi^\star(\iota),\Re\Xi^\star(\iota))|)(\upsilon)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\|\Xi-\Xi^\star\|+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}(\emptyset_1+\emptyset_2\Im)\|\Xi-\Xi^\star\|\right]\\
&\quad+\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\|\Xi-\Xi^\star\|+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\;(\emptyset_1+\emptyset_2\Im)\|\Xi-\Xi^\star\|\\
&\leq \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\Big[\lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\\
&\quad+(\emptyset_1+\emptyset_2\Im)\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\Big]\|\Xi-\Xi^\star\|\\
&\quad+\lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi-\Xi^\star\|+(\emptyset_1+\emptyset_2\Im)\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\|\Xi-\Xi^\star\|\\
&\leq (\lambda+\emptyset_1+\emptyset_2\Im)\left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right)\|\Xi-\Xi^\star\|\\
&\leq \Omega_1\|\Xi-\Xi^\star\|,
\end{aligned}$$

which implies that

$$\|\wp\Xi-\wp\Xi^\star\|\leq\Omega_1\|\Xi-\Xi^\star\|.$$

Then, based on the conditions (3.1), we can conclude that $\wp$ is a contraction operator. Thus, according to Banach's FPT, has a unique fixed point. As a result, problem (1.2) has a single solution on ß. This completes the proof. □

The following result provides existence criteria for problem solutions (1.2), and its proof is based on Schauder's fixed point technique [26].

Theorem 3.2. *Assume that the assumption ($\mathcal{H}_1$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$) hold. If*

$$\Omega_2=\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left[1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right]<1, \tag{3.3}$$

then, the problem (1.2) *has at least one solution in the space* $\mathcal{C}(ß,\mathbb{R}^+)$.

Proof. Consider the ball

$$\mathfrak{Z}_\rho=\{\Xi\in\mathcal{C}(ß,\mathbb{R}):\|\Xi\|\leq\rho\},$$

where $\rho>0$ with

$$\rho\geq\frac{1}{1-\lambda\Omega_2}\left[\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}+\Omega_2\|\Phi\|\right],$$

where $\|\Phi\|=\sup_{\upsilon\in ß}|\Phi(\upsilon)|$. We build the operators $\wp_1,\wp_2$ on $\mathfrak{Z}_\rho$ where $\mathfrak{Z}_\rho$ by

$$(\wp_1\Xi)(\upsilon)=\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\aleph(\iota,\Xi(\iota),\Re(\Xi)(\iota)|(\upsilon),$$

and

$$
\begin{aligned}
(\wp_2\Xi)(\upsilon) \quad = \quad & \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\Xi(\upsilon)| \\
& + \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \\
& \times\Big[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\Xi(\mathfrak{b})| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\aleph(\iota,\Xi(\iota),\Re(\Xi)(\iota)|\,(\mathfrak{b})\Big].
\end{aligned}
$$

For any Ξ, $\Xi^* \in \mathfrak{Z}_\rho$, we have

$$
\begin{aligned}
&|(\wp_1\Xi)(\upsilon) + (\wp_2\Xi^*)(\upsilon)| \\
&\quad\le \sup_{\upsilon\in\beta}\Bigg\{\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \left(\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right) \\
&\quad\quad \times\left(\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\Xi^*(\mathfrak{b})| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\aleph(\iota,\Xi^*(\iota),\Re(\Xi^*)(\iota)|\,(\mathfrak{b})\right) \\
&\quad\quad + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\Xi^*(\upsilon)| + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\,|\aleph(\iota,\Xi(\iota),\Re(\Xi)(\iota)|\,(\upsilon)\Bigg\} \\
&\quad\le \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \\
&\quad\quad + \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\Big[\lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\,\|\Xi^*\| + \frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\,\|\Phi\|\Big] \\
&\quad\quad + \lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\,\|\Xi^*\| + \frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\,\|\Phi\| \\
&\quad\le \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \lambda\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\Big[1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\Big]\|\Xi^*\| \\
&\quad\quad + \frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\left[1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)}\right]\|\Phi\| \\
&\quad\le \frac{1}{1-\lambda\Omega_2}\left[\frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} + \Omega_2\,\|\Phi\|\right] \\
&\quad\le \rho.
\end{aligned}
$$

This proves that $\wp_1\Xi + \wp_2\Xi^* \in \mathfrak{Z}_\rho$.

Now, we have to show $\wp_1$ is continuous, due to $\Re(\Xi)(\cdot) =: \aleph(\cdot,\Xi(\cdot),\Re(\Xi)(\cdot)) \in \mathcal{C}(\cdot,\mathbb{R}^2,\mathbb{R})$. Also, $\wp_1$ is uniformly bounded on $\mathfrak{Z}_\rho$ because we have from (H_3) that

$$
\|(\wp_1\Xi)\| \le \frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\,\|\Phi\|\,.
$$

In addition, we prove the compactness of $\wp_1$ as follows. Let $\upsilon_1,\upsilon_2 \in ß$ such that $\upsilon_1 < \upsilon_2$. Then

$$\begin{aligned}
&|\wp_1\Xi(\upsilon_2)-\wp_1\Xi(\upsilon_1)|\\
&\leq \frac{1}{\Gamma(\beth_1)}\int_{\mathfrak{a}}^{\upsilon_1} \top'(\iota)\left[(\top(\upsilon_2)-\top(\iota))^{\beth_1-1}-(\top(\upsilon_1)-\top(\iota))^{\beth_1-1}\right]|\Re(\Xi)|(\iota)d\iota\\
&\quad+\frac{1}{\Gamma(\beth_1)}\int_{\upsilon_1}^{\upsilon_2} \top'(\iota)(\top(\upsilon_2)-\top(\iota))^{\beth_1-1}|\Re(\Xi)|(\iota)d\iota\\
&\leq \frac{\|\Phi\|}{\Gamma(\beth_1+1)}\Big[2(\top(\upsilon_2)-\top(\upsilon_2))^{\beth_1}+\Big|(\top(\upsilon_2)-\top(a))^{\beth_1}-(\top(\upsilon_1)-\top(a))^{\beth_1}\Big|\Big].
\end{aligned}$$

The last inequality with $\upsilon_2-\upsilon_1 \to 0$ gives

$$|\wp_1\Xi(\upsilon_2)-\wp_1\Xi(\upsilon_1)| \to 0, \text{ for all } |\upsilon_2-\upsilon_1| \to 0, \ \Xi\in \mathfrak{Z}_\rho.$$

Then, $\wp_1$ is relatively compact on $\mathfrak{Z}_\rho$. An application of the Arzel-Ascoli theorem, $\wp_1$ is compact on $\mathfrak{Z}_\rho$. Hence, all the assumptions of Krasnoselskii's fixed point theorem are satisfied. So, we infer that (1.2) has at least one solution on ß. This completes the proof. □

4. U-H and G-U-H stability analysis

In this part, we discuss the U-H and G-U-H stability of the problem (1.2). The following observations are taken from [7, 21].

Definition 4.1. The problem (1.2) is said to be U-H stable if there exists a constant $K_\aleph > 0$ such that for each $\epsilon > 0$ and every solution $\widetilde{\Xi} \in \mathcal{C}(ß,\mathbb{R})$ of the inequalities

$$\left|{}^H\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\widetilde{\Xi}(\upsilon)-\left[\lambda\Xi(\upsilon)+\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))\right]\right| \leq \epsilon, \quad \forall \upsilon \in ß, \tag{4.1}$$

there exists a solution $\Xi \in \mathcal{C}(ß,\mathbb{R})$ of the problem (1.2) that satisfies

$$\left|\widetilde{\Xi}(\upsilon)-\Xi(\upsilon)\right| \leq K_\aleph\epsilon. \tag{4.2}$$

Definition 4.2. The problem (1.2) is said to be G-U-H stable if there exists $\Psi \in \mathbb{C}([0,\infty),[0,\infty))$, $\Psi(0) = 0$ such that for each solution $\widetilde{\Xi} \in \mathcal{C}(ß,\mathbb{R})$ of the inequality

$$\left|{}^H\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\widetilde{\Xi}(\upsilon)-\left[\lambda\Xi(\upsilon)+\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))\right]\right| \leq \epsilon, \quad \upsilon \in ß, \tag{4.3}$$

there exists a solution $\Xi \in \mathcal{C}(ß,\mathbb{R})$ for the problem (1.2) such that

$$\left|\widetilde{\Xi}(\upsilon)-\Xi(\upsilon)\right| \leq \Psi(\epsilon), \ \upsilon \in ß.$$

Remark 4.3. A function $\widetilde{\Xi} \in \mathcal{C}(ß, \mathbb{R})$ satisfies the inequality (4.1) if and only if there exists a function $\omega \in \mathcal{C}(ß, \mathbb{R})$ with

(1) $|\omega(\upsilon)| \le \epsilon$, $\upsilon \in$ß,
(2) for all $\upsilon \in$ß,

$$^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\widetilde{\Xi}(\upsilon) = \aleph(\upsilon, \widetilde{\Xi}(\upsilon), \Re\widetilde{\Xi}(\upsilon)) + |\omega(\upsilon)|.$$

Lemma 4.4. *If $\widetilde{\Xi} \in \mathcal{C}(ß, \mathbb{R})$ is a solution to inequality* (4.1)*, then $\widetilde{\Xi}$ is a solution of the following integral inequality:*

$$\left|\widetilde{\Xi}(\upsilon) - Z_{\widetilde{\Xi}} - \frac{1}{\Gamma(\beth_1)}\int_a^\upsilon \top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1}\aleph(\upsilon, \widetilde{\Xi}(\upsilon), \Re\widetilde{\Xi}(\upsilon))(\iota)d\iota\right|$$
$$\le \left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\right)\epsilon,$$

where

$$Z_{\widetilde{\Xi}} = \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)} - \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}$$
$$\times\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\widetilde{\Xi}(\mathfrak{b}) + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\aleph(\mathfrak{b}, \widetilde{\Xi}(\mathfrak{b}), \Re(\widetilde{\Xi})(\mathfrak{b}))\right] + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\widetilde{\Xi}(\upsilon).$$

Proof. In view of Remark 4.3 and Theorem 3.1, we have

$$\widetilde{\Xi}(\upsilon) = \frac{\mathcal{N}_3}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)} - \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}$$
$$\times\left[\lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\widetilde{\Xi}(\mathfrak{b}) + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left(\aleph(\mathfrak{b}, \widetilde{\Xi}(\mathfrak{b}), \Re(\widetilde{\Xi})(\mathfrak{b})) + \omega(\upsilon)\right)\right] + \lambda\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\widetilde{\Xi}(\upsilon)$$
$$+ \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left(\aleph(\upsilon, \widetilde{\Xi}(\upsilon), \Re(\widetilde{\Xi})(\upsilon)) + \omega(\upsilon)\right),$$

it follows that

$$\left|\widetilde{\Xi}(\upsilon) - Z_{\widetilde{\Xi}} - \frac{1}{\Gamma(\beth_1)}\int_a^\upsilon \top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1}\aleph(\upsilon, \widetilde{\Xi}(\upsilon), \Re\widetilde{\Xi}(\upsilon))(\iota)d\iota\right|$$
$$\le \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\omega(\upsilon)|(\mathfrak{b}) + \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}|\omega(\upsilon)|(\iota)$$
$$\le \frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\epsilon + \left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\epsilon$$
$$\le \left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\right)\epsilon.$$

□

Now we state the following generalization of Gronwall's lemma.

Lemma 4.5. ([23]) *Let $v, \aleph$ be two integrable functions and z a continuous function with domain ß. Assume that*

(1) *v and $\aleph$ are nonnegative,*
(2) *z is nonnegative and nondecreasing.*

If

$$v(\upsilon) \leq \aleph(\upsilon) + z(\upsilon) \int_a^{\upsilon} \top'(\tau)(\top(\upsilon) - \top(\tau))^{\beth_1 - 1} v(\tau) d\tau, \quad \upsilon \in ß,$$

then

$$v(\upsilon) \leq \aleph(\upsilon) + \int_a^{\upsilon} \sum_{k=1}^{\infty} \frac{(z(\upsilon)\Gamma(\beth_1))^k}{\Gamma(k\beth_1)} \top'(\tau)(\top(\upsilon) - \top(\tau))^{k\beth_1 - 1} \aleph(\tau) d\tau, \quad \upsilon \in ß.$$

Remark 4.6. ([23]) In particular, if $\aleph(\upsilon)$ be a nondecreasing function on ß. Then we have

$$v(\upsilon) \leq \aleph(\upsilon) E_{\beth_1} \left[(z(\upsilon)\Gamma(\beth_1)(\top(\upsilon) - \top(\tau))^{\beth_1 - 1} \right], \ \upsilon \in ß,$$

where $E_{\beth_1}$ is the Mittage-Leffler function defined by

$$E_{\beth_1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beth_1 k + 1)}, \quad x \in ß.$$

Theorem 4.7. *Suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and condition (3.1) are satisfied. Under the Lemma 4.4, the following equation*

$$^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1, \beth_2; \top} \Xi(\upsilon) = \lambda \Xi(\upsilon) + \aleph(\upsilon, \Xi(\upsilon), \Re\Xi(\upsilon)), \ \upsilon \in ß \tag{4.4}$$

is U-H stable.

Proof. Let $\overline{\Xi} \in \mathcal{C}(ß, \mathbb{R})$ be a function satisfies (4.1) and $\epsilon > 0$, let $\Xi \in \mathcal{C}(ß, \mathbb{R})$ be a unique solution of the following problem

$$\begin{cases} ^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1, \beth_2; \top} \Xi(\upsilon) = \lambda \Xi(\upsilon) + \aleph(\upsilon, \Xi(\upsilon), \Re\Xi(\upsilon)) \quad \upsilon \in ß, \\ \mathcal{N}_1 \Xi(\mathfrak{a}^+) = \mathcal{N}_1 \widetilde{\Xi}(\mathfrak{a}^+), \ \mathcal{N}_2 \Xi(\mathfrak{b}^-) = \mathcal{N}_2 \widetilde{\Xi}(\mathfrak{b}^-), \end{cases}$$

where $\mathcal{N}_i \in \mathbb{R}$, $(\ i = 1, 2, 3)$, $0 \leq \beth_2 \leq 1$. Then, by Lemma 2.5, we get

$$\Xi(\upsilon) = Z_{\Xi} + \frac{1}{\Gamma(\beth_1)} \int_a^{\upsilon} \top'(\iota)(\top(\upsilon) - \top(\iota))^{\beth_1 - 1} \aleph(\upsilon, \widetilde{\Xi}(\upsilon), \Re\widetilde{\Xi}(\upsilon))(\iota) d\iota, \quad \upsilon \in ß.$$

On the other hand, $\mathcal{N}_1 \Xi(\mathfrak{a}^+) = \mathcal{N}_1 \widetilde{\Xi}(\mathfrak{a}^+)$, $\mathcal{N}_2 \Xi(\mathfrak{b}^-) = \mathcal{N}_2 \widetilde{\Xi}(\mathfrak{b}^-)$, then $Z_{\Xi} = Z_{\widetilde{\Xi}}$. Indeed

$$\begin{aligned}
\left|Z_{\Xi}-Z_{\widetilde{\Xi}}\right| \le{}& \left|\frac{\mathcal{N}_2}{\mathcal{N}_1}\right| \frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \Big[\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\mathfrak{b})-\widetilde{\Xi}(\mathfrak{b})\right| \\
&+\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\aleph(\mathfrak{b},\Xi(\mathfrak{b}),\Re(\Xi)(\mathfrak{b}))-\aleph(\mathfrak{b},\widetilde{\Xi}(\mathfrak{b}),\Re(\widetilde{\Xi})(\mathfrak{b}))\right|\Big] \\
&+\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\upsilon)-\widetilde{\Xi}(\upsilon)\right| \\
\le{}& \left|\frac{\mathcal{N}_2}{\mathcal{N}_1}\right| \frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \Big[\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\mathfrak{b})-\widetilde{\Xi}(\mathfrak{b})\right| \\
&+(\emptyset_1+\Im\emptyset_2)\,\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\mathfrak{b})-\widetilde{\Xi}(\mathfrak{b})\right|\Big]+\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\upsilon)-\widetilde{\Xi}(\upsilon)\right| \\
\le{}& \left|\frac{\mathcal{N}_2}{\mathcal{N}_1}\right| \frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{|\hbar|\,\Gamma(\zeta)} \\
&\times\Big[\Big(\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}+(\emptyset_1+\Im\emptyset_2)\,\mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\Big)\left|\Xi(\mathfrak{b})-\widetilde{\Xi}(\mathfrak{b})\right|\Big] \\
&+\lambda \mathcal{I}_{\mathfrak{a}^+}^{\beth_1;\top}\left|\Xi(\mathfrak{b})-\widetilde{\Xi}(\mathfrak{b})\right| \\
={}& 0.
\end{aligned}$$

Therefore, $Z_{\Xi}=Z_{\widetilde{\Xi}}$. We have

$$\Xi(\upsilon)=Z_{\widetilde{\Xi}}+\frac{1}{\Gamma(\beth_1)}\int_a^{\upsilon}\top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1}\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))(\iota)d\iota.$$

It follows from Lemma 4.4 that, for any $\upsilon \in ß$,

$$\begin{aligned}
&\left|\widetilde{\Xi}(\upsilon)-\Xi(\upsilon)\right| \\
&=\left|\widetilde{\Xi}(\upsilon)-Z_{\widetilde{\Xi}}-\frac{1}{\Gamma(\beth_1)}\int_a^{\upsilon}\top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1}\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))(\iota)d\iota\right| \\
&\quad+\frac{1}{\Gamma(\beth_1)}\int_a^{\upsilon}\top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1} \\
&\quad\times\left|\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))(\iota)-\aleph(\upsilon,\Xi(\upsilon),\Re\Xi(\upsilon))(\iota)\right|d\iota \\
&\le\left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\right)\epsilon \\
&\quad+(\emptyset_1+\Im\emptyset_2)\frac{1}{\Gamma(\beth_1)}\int_a^{\upsilon}\top'(\iota)(\top(\upsilon)-\top(\iota))^{\beth_1-1}\left|\widetilde{\Xi}(\iota)-\Xi(\iota)\right|d\iota.
\end{aligned}$$

Using generalized Gronwall inequality (Lemma 4.5), we get

$$\begin{aligned}
\left|\widetilde{\Xi}(\upsilon)-\Xi(\upsilon)\right| &\le \epsilon\mathcal{Q}E_{\beth_1}\left[(\emptyset_1+\Im\emptyset_2)(\top(\upsilon)-\top(\mathfrak{a}))^{\beth_1}\right] \\
&\le \epsilon K_{\aleph},
\end{aligned}$$

where
$$\mathcal{Q}=\left(\frac{(\top(\mathfrak{b})-\top(\mathfrak{a}))^{\beth_1}}{\Gamma(\beth_1+1)}\right)\left(1+\frac{\mathcal{N}_2}{\mathcal{N}_1}\frac{(\top(\upsilon)-\top(\mathfrak{a}))^{\zeta-1}}{\hbar\Gamma(\zeta)}\right)$$
and
$$K_{\aleph}=\mathcal{Q}E_{\beth_1}\left[(\emptyset_1+\Im\emptyset_2)\,(\top(\upsilon)-\top(\mathfrak{a}))^{\beth_1}\right],$$
which emplised that the problem (4.4) is U-H stable. □

Corollary 4.8. *Under the hypotheses of Theorem* 4.7, *if there exists* $\Psi\in\mathbb{C}([0,\infty),[0,\infty))$, $\Psi(0)=0$, *then the problem* (1.2) *is G-U-H stable.*

Proof. In a manner similar to above Theorem 4.7 with selecting $\Psi(\epsilon)=K_{\aleph}\epsilon$ such that $\Psi(0)=0$, we obtain
$$\left|\widetilde{\Xi}(\upsilon)-\Xi(\upsilon)\right|\le\Psi(\epsilon),\quad \upsilon\in\text{ß},$$
then the problem (1.2) is G-U-H stable. □

5. An example

Example 5.1. Consider the following problem of BVP for a generalized Hilfer-type integro-differential equation:
$$\begin{cases}{}^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\Xi(\upsilon)=\frac{1}{6}\Xi(\upsilon)+\aleph(\upsilon,\Xi(\upsilon),\Re(\Xi)(\upsilon)),\ \upsilon\in\text{ß}=[0,1],\ 0<\beth_1<1,\\ \frac{1}{16}\Xi(\mathfrak{a}^+)+\frac{1}{18}\Xi(\mathfrak{b}^-)=\frac{1}{19},\quad \alpha\le\zeta=\beth_1+\beth_2(1-\beth_1),\quad 0\le\beth_2\le1,\end{cases}\tag{5.1}$$
where $\mathcal{N}_1=\frac{1}{16}$, $\mathcal{N}_2=\frac{1}{18}$, $\mathcal{N}_3=\frac{1}{19}$, $\top:\text{ß}\to\mathbb{R}$ is a given function, and $\aleph\in\mathcal{C}(\text{ß}\times\mathbb{R}^2,\mathbb{R})$ is defined by
$$\aleph(\upsilon,\Xi(\upsilon),\Re(\Xi)(\upsilon))=\Xi(\upsilon)^{-\frac{1}{7}}+\frac{1}{99}(\upsilon^{\frac{5}{6}}\sin\Xi(\upsilon)+\int_0^{\upsilon}e^{(\upsilon-\iota)}\Xi(\iota)d\iota)$$
for $\upsilon\in[0,1]$, $\Xi_1,\Xi_2\in\mathbb{R}^+$ with $\beth_1=\frac{1}{2}$, $\beth_2=\frac{1}{2}$, $\zeta=\frac{3}{4}$ and $\top=\frac{1}{2}$. Clearly, the function $\aleph\in\mathcal{C}([0,1])$. For each $\Xi,\Xi^*\in\mathbb{R}^+$ and $\upsilon\in[0,1]$,
$$|\aleph(\upsilon,\Xi,\Re(\Xi))-\aleph(\upsilon,\Xi^*,\Re(\Xi^*))|\le\frac{1}{99}\left(|\Xi-\Xi^*|+|\Re(\Xi)-\Re(\Xi^*)|\right).$$
By some calculations, we get $\Im=\frac{e-1}{99}$. Hence, the condition $(\mathcal{H}_1)$ is satisfied with $\emptyset_1=\emptyset_2=\frac{1}{99}$. It is easy to verify that $\Omega_1=0.283\,60<1$. Since all the assumptions of Theorem 3.1 are fulfilled, therefore problem (5.1) has a unique solution.

On the other hand, as shown in Theorem 4.7, for every $\epsilon>0$, if $\widetilde{\Xi}\in\mathcal{C}(\text{ß}\times\mathbb{R})$ is a solution of the inequality
$$\left|{}^{H}\mathcal{D}_{\mathfrak{a}^+}^{\beth_1,\beth_2;\top}\widetilde{\Xi}(\upsilon)-\aleph(\upsilon,\widetilde{\Xi}(\upsilon),\Re\widetilde{\Xi}(\upsilon))\right|\le\epsilon,\ \upsilon\in(0,1),$$

then there exists a unique solution $\Xi \in \mathcal{C}(ß\times\mathbb{R})$ of the $\top$-Hilfer problem (5.1) such that

$$\left\|\Xi - \widetilde{\Xi}\right\| \leq K_{\aleph}\epsilon,$$

where

$$K_{\aleph} = \mathcal{Q}E_{\beth_1}\left[\left(\emptyset_1 + \Im\emptyset_2\right)\left(\top(\upsilon) - \top(\mathfrak{a})\right)^{\beth_1}\right] = 1.\,621\,5 > 0.$$

Hence the problem (5.1) is U-H stable. Moreover, if $\Psi(\epsilon) = K_{\aleph}\epsilon = (1.\,621\,5)\,\epsilon$ such that $\Psi(0) = 0$, then the problem (5.1) is G-U-H stable.

6. Conclusions

We introduced a new boundary value problem in this paper by considering a nonlinear fractional integro-differential equation with $\top$-Hilfer fractional derivative operator. Our approach to studying the given problem is based on modern functional analysis tools. We used the FPTs of Schauder and Banach to prove the existence and uniqueness of solutions to the problem at hand. Other qualitative analyses of the solution, such as stability results, can be discussed using the results of these investigations. Finally, we provided some examples.

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