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QUALITATIVE ANALYSIS OF ABR-FRACTIONAL VOLTERRA-FREDHOLM SYSTEM

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Abstract. In this work, we explore the existence and uniqueness results for a class of boundary value issues for implicit Volterra-Fredholm nonlinear integro-differential equations (IDEs) with Atangana-Baleanu-Riemann fractional (ABR-fractional) that have non-instantaneous multi-point fractional boundary conditions. The findings are supported by Krasnoselskii's fixed point theorem, Gronwall-Bellman inequality, and the Banach contraction principle. Finally, a demonstrative example is provided to support our key findings.

1. Introduction

In recent years, research and development have made significant strides thanks to the fractional calculus (FC) theory, which deals with differential equations of fractional order. This is mostly due to the FC theory's ability to provide models with adequate solutions, particularly for real-world problems. Additionally, generalized differential equations are considered to be fractional differential equations. The IDE is an operator that contains both integer-order integrals and integer-order derivatives as special cases, which is why FC is becoming more popular and many applications arise from the term in the present. This field of mathematical physics deals with IDEs in which the

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integrals are of the convolution form and also have predominantly power law or logarithm type single kernels [1, 3, 5, 8, 12, 13, 14, 18, 20].

By employing the Mittag-Leffler function as its kernel, the nonsingular Riemann-Liouville and Caputo form of the fractional operator is presented by Baleanu and Baleanu in [6]. The AB-fractional derivative was used by Bonyah et al. [7] to create a mathematical model for the illnesses of cancer and hepatitis coinfection. Ahmad et al. [2] offered the AB-fractional derivative of the fractional-order vitamin-immune-tumor model and highlighted the model's existence, uniqueness, and Hyres-Ulam stability findings. Researchers employing the dynamical fractional immune-tumor model with AB-fractional derivative created a chaotic and comparative study of tumour and effector cells [25]. In [10], the fractional AB-derivative was used to investigate the numerical solution of the fractional immunogenetic tumour model. Several applications were cited in reference [4, 28] in a similar manner.

The AB-fractional derivative was used by Logeswari et al. [26] to study the mathematical model for the global spread of the COVID-19 virus. In order to predict the impact of the disease spreading throughout India, they also devised a framework for producing numerical results. A few additional significant papers that attempted to address the issue of different illnesses modeled as FDEs employing AB-fractional derivative are [21]. Tidke presented the following equation in [29]:

$$\begin{cases} \frac{d^r y}{dv^r} = M(v)y(v) + \Im(v, y(v)), \ v \in [0, a], \\ y(0) = y_0, \end{cases}$$

where $0 < r < 1$, the unknown $y(.)$ takes values in the Banach space $X; \Im \in C(J \times X, X)$, and $M(v)$ is a linear bounded operator on X and $y_0 \in X$.

The authors of [24] worked with the following fractional implicit nonlinear equations and looked at the uniqueness, existence, and qualities of continuous dependence as well as the interval of existence and existence of solution

$$\begin{cases} {}^c\Delta^\alpha y(v) = M\left(v, y(v)^c, \Delta^\alpha(v)\right), \\ y(0) = y_0 \in \mathbb{R}, v \in [0, b], \end{cases}$$

where ${}^c\Delta^\alpha(\alpha \in (0, 1))$ denotes the Caputo fractional derivative and $M \in [0, b] \times \mathbb{R} \times \mathbb{R}$ is a given continuous function.

Guo et al. [11] examined the boundary value issues for impulsive fractional differential equations with the following:

$$\begin{cases} {}^C\Delta_v^\alpha y(v) = M(v, y(v)), \ v \in J' \setminus J : v_1, v_2, \ldots, v_m, \ J = [0, \Omega], \\ \Delta y\left(v_k\right) = y\left(v_k^+ - v_k^-\right) = I_k\left(v_k^-\right), \ k = 0, 1, 2, \ldots, m, \\ e_1 y(0) + e_2 y(\Omega) = e_3, \end{cases}$$

where ${}^{C}\Delta_{v}^{\alpha}$, $0<\alpha<1$ denotes the fractional Caputo derivative, $M: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and v_k satisfy $0=v_0<v_1<\ldots<v_m<v_{m+1}=\Omega$, $y\left(v_k^{+}\right)=\lim_{\epsilon\to 0^{+}} y\left(v_k+\epsilon\right)$ and $y\left(v_k^{-}\right)=\lim_{\epsilon\to 0^{-}} y\left(v_k+\epsilon\right)$ represent the right and left limit of $y(v)$ at $v=v_k$. $I_k\in C(\mathbb{R},\mathbb{R})$ and e_1,e_2,e_3 are real constants with $e_1+e_2\neq 0$.

In [30], Yukunthorn et al. investigated the issues of boundary value for the fractional Hadamard impulsive equations:

$$\begin{cases} {}^{C}\Delta_{v_k}^{p_k} y(v)=M(v,y(v)),\ v\in J_k\subset\left[v_0,\Omega\right], v=v_k,\\ \Delta y\left(v_k\right)=\varphi_k\left(y\left(v_k\right)\right),\ k=1,2,\ldots,m,\\ e_1 y\left(v_0\right)+e_2 y(\Omega)=\sum_{i=0}^{m}\gamma_i J_{v_i}^{q_i} y\left(v_{i+1}\right), \end{cases}$$

where ${}^{C}\Delta_{v_k}^{p_k}$ is the Hadamard fractional derivative of order $p_k\in(0,1]$ on intervals $J_k=\left(v_k,v_{k+1}\right], k=1,2,\ldots,m$ with $J_0=\left[v_0,v_1\right], 0<v_1<v_2<v_3<\ldots<v_k<\ldots<v_m<v_{m+1}=\Omega$ which are the impulse points, $J:=\left[\chi_0,\Omega\right], M: J\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous and $\varphi_k\in C(\mathbb{R},\mathbb{R})$, $J_{v_i}^{q_i}, q_i>0, i=1,2,\ldots,m$ is the fractional Hadamard integral. The jump conditions are defined by

$$D y\left(v_k\right)=y\left(v_k^{+}\right)-y\left(v_k\right),$$

where $y\left(v_k^{+}\right)=\lim_{\epsilon\to 0^{+}} y\left(v_k+\epsilon\right), k=1,2,3,\ldots,m$.

We explore AB-fractional derivatives of the following form in multi-derivative nonlinear impulsive FDEs, which are inspired by the aforementioned works:

$$\begin{cases} {}_{0}^{*}D_{\Xi}^{\alpha}(\Xi)=\Im(\Xi,\Lambda(\Xi),\Psi\Lambda(\Xi),\Upsilon\Lambda(\Xi)), \quad \Xi\in J,\\ \Lambda\left(\chi_k^{+}\right)=\left(\chi_k^{-}\right)+y_k,\ y_k\in\mathbb{R},\\ \Lambda(0)=\int_0^{\Omega}\frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\vartheta(\sigma,\Lambda(\sigma))d\sigma, \end{cases} \tag{1.1}$$

where $J=[0,\Omega]$, $\Omega>0$, $0<\alpha<1$, ${}^{*}D_{\Xi}^{\alpha}$ is the ABR-fractional differential operator and $\Im\in C(J\times\mathbb{R}^3\longrightarrow\mathbb{R})$, $\vartheta\in C(J\times\mathbb{R}\longrightarrow\mathbb{R})$ are nonlinear functions, and $\Psi\Lambda(\Xi)=\int_0^{\Xi} k(\Xi,s,\Lambda(s))\mathrm{d}s$, $\Upsilon\Lambda(\Xi)=\int_0^{\Omega} h(\Xi,s,\Lambda(s))\mathrm{d}s$ and $k,h:\Delta\times[0,\Omega]\longrightarrow\mathbb{R}, \Delta=\{(\Xi,s): 0\leq s\leq\Xi\leq\Omega\}$.

$0=\Xi_0<\Xi_1<\Xi_2<\ldots<\Xi_m=1$, $\Delta\Lambda|_{\Xi=\Xi_k}=\Lambda\left(\Xi_k^{+}\right)-\Lambda\left(\Xi_k^{-}\right)$, and $\Lambda\left(\Xi_k^{+}\right)=\lim_{h\longrightarrow 0^{+}}\Lambda\left(\Xi_k+h\right)$ and $\Lambda\left(\Xi_k^{-}\right)=\lim_{h\to 0^{-}}\Lambda\left(\Xi_k+h\right)$ indicate the boundaries of the right and left hands of $\Xi(t)$ at $\Xi=\Xi_k$.

2. Preliminaries

Now, a few fractional calculus notations, definitions, and known outcomes are recalled [6, 15, 16, 17, 19, 22, 23, 27].

Definition 2.1. ([22]) For $p \in [1,\infty)$ and Λ form an open subset of $\mathbb{R}$, then the Sobolev space $\Upsilon^p(\Lambda)$ can be defined as follows:

$$\Upsilon^p(\Lambda) = \left\{ \Im \in L^2(\Lambda) : D^\beta \Im \in L^2(\Lambda) \text{ for all } \quad |\beta| \leq \Lambda \right\}.$$

Definition 2.2. ([6]) We assume $\Lambda \in \Upsilon^1(0,1)$. Next, the left AB fractional derivative in Riemann-Liouville viewpoint of Λ of order $0 < \alpha < 1$ (ABR derivative) characterized by

$$D_\Xi^\alpha = \frac{B(\alpha)}{(1-\alpha)} \frac{d}{d\sigma} \int_0^\Xi \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\Xi - \sigma)^\alpha \right) \Lambda(\sigma) d\sigma,$$

where $B(\alpha) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$ and $\mathbb{E}$ is one parameter Mittag-Leffler function.

Definition 2.3. ([6]) Let $\Lambda \in \Upsilon^1(0,1)$. Then, in Caputo meaning, the left AB-fractional derivative of x of order $0 < \vartheta < 1$ characterized by

$$D_t^\vartheta = \frac{B(\vartheta)}{(1-\vartheta)} \int_0^t \mathbb{E}_\vartheta \left(\frac{-\vartheta}{(1-\vartheta)} (\Xi - \sigma)^\vartheta \right) \Lambda'(\sigma) d\sigma.$$

Definition 2.4. ([23]) A Mittag-Leffler generalized function $[\gamma]_{\vartheta,\Delta}(\delta)$ for the complex ϑ, Δ with $\mathrm{Re}(\vartheta) > 0$ can be defined as follows:

$$\mathbb{E}^\gamma_{\vartheta,\Delta}(\delta) = \sum_{k=0}^\infty \frac{\gamma_k}{\Gamma(\vartheta k + \Delta)} \frac{\delta^k}{k!},$$

where γ_k is the Pochhammer symbol given by

$$\gamma_0 = 1, \quad \gamma_k = \gamma(\gamma + 1 \ldots (\gamma + k - 1)), \quad k \in \mathbb{N}.$$

We note that

$$\mathbb{E}^1_{\vartheta,\Delta}(z) = \mathbb{E}_{\vartheta,\Delta}(z), \quad \mathbb{E}^1_{\vartheta,1}(z) = \mathbb{E}_\vartheta(z).$$

Lemma 2.5. ([6]) *Suppose* $L\{\Im(\Xi); p\} = \bar{F}(p)$. *Then*

$$D_t^\vartheta\{\Im(\Xi); p\} = B(\vartheta)/1 - \vartheta p^\vartheta \bar{F}(p)/p^\vartheta + \vartheta/1 - \vartheta.$$

Lemma 2.6. ([23]) $L\left\{ t^{k\vartheta + \Delta - 1} \mathbb{E}^{(k)}_{\vartheta,\Delta} \left(\pm a t^\vartheta \right); p \right\} = k! p^{\vartheta - \Delta /} \left(p^\vartheta \pm a \right)^{k+1}$ *and* $\mathbb{E}^{(k)} t = d^k / dt^k t$.

Definition 2.7. ([23]) Let $\rho, \mu, \Lambda, \gamma \in \mathbb{C}(\mathrm{Re}(\rho), \mathrm{Re}(\mu) > 0)$, $b > a$. On a class $L(a,b)$, the fractional integral operator $E^\gamma_{\rho,\mu,\Lambda;a+}$ is defined by

$$\left(E^\gamma_{\rho,\mu,\Lambda;a+} \phi \right) \Xi = \int_a^t (\Xi - \sigma)^{\gamma - 1} \mathbb{E}^\gamma_{\rho,\mu} \left[\Lambda (\Xi - \sigma)^\rho \right] \phi(\sigma) d\sigma, \quad \Xi \in [a,b].$$

Lemma 2.8. ([27]) *Let $\rho,\mu,\Lambda,\gamma \in \mathbb{C}(\mathrm{Re}(\rho),\mathrm{Re}(\mu) > 0)$, $b > a$. Then the operator $E^{\gamma}_{\rho,\mu,\Lambda;a+}$ is bounded on $C[a,b]$ such that*

$$\left\| \left(E^{\gamma}_{\rho,\mu,\Lambda;a+}\phi \right)(\Xi) \right\| \leq \mathbb{Q}\|\phi\|,$$

where

$$\mathbb{Q} = (b-a)^{Re(u)} \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{|\Gamma(\rho k+\mu)|[(\mathrm{Re}(\rho)k+\rho(\mu))]} \frac{\left|\Lambda(b-a)^{Re(\rho)}\right|^k}{k!}.$$

Lemma 2.9. ([23]) *Let $\rho,\mu,\Lambda,\gamma \in \mathbb{C}(\mathrm{Re}(\rho),\mathrm{Re}(\mu) > 0)$. Then the $E^{\gamma}_{\rho,\mu,\Lambda;a+}$ is invertible operator in $L(a,b)$ and for all $\Im \in L(a,b)$, is an inversion to the left*

$$\left(\left[E^{\gamma}_{\rho,\mu,\Lambda;a+} \right]^{-1} \Im \right) \Xi = \left(D^{\mu+\nu}_{a+} E^{-\gamma}_{\rho,\mu,\Lambda;a+} \Im \right) \Xi, \quad a < \Xi \leq b,$$

where $v \in C$, $(\mathrm{Re}(v) > 0)$ and $D^{\mu+\nu}_{a+}$ denotes the RL-fractional operator of order $\mu + v$ with a.

Lemma 2.10. ([27]) *Let $n,u,\Lambda,\gamma \in \mathbb{C}(\mathrm{Re}(n),\ \mathrm{Re}(u) > 0)$. Assume $\int_a^{\omega}(\omega - \sigma)^{\gamma-1}\mathbb{E}^{\gamma}_{n,u}\left[x(\omega-\sigma)^n\right]\phi(\sigma)d\sigma = \Im(\omega)$, $c < \omega \leq d$, is solvable in $L(c,d)$. Then*

$$\phi(\Xi) = \left(D^{u+v}_{c+} E^{-\gamma}_{n,u,\Lambda;c+} \Im \right) \Xi, \quad c < \Xi \leq d,$$

where $v \in C$, $(\mathrm{Re}(v) > 0)$ and $D^{u+\nu}_{c+}$ is the fractional Riemann-Liouville operator of order $u + v$ with a, which is a unique solution.

Lemma 2.11. ([1]) (Krasnoselskii's Theorem) *Suppose Λ is a Banach space. Suppose $\mathcal{S}$ is a bounded closed convex subset of Λ, and Suppose F_1, F_2 are operators of $\mathcal{S} \longrightarrow \Lambda$, when $F_1\Lambda +\, F_2 v \in \mathcal{S}$ for all $\Lambda, v \in \mathcal{S}$. If F_1 is contraction and F_2 is completely continuous, then*

$$F_1\Lambda + F_2 v = \Lambda$$

has a solution on $\mathcal{S}$.

Lemma 2.12. ([23]) (Gronwall-Bellman inequality) *Suppose y and $\Im$ are non-negative and continuous functions defined on $J = [0,\Omega]$ and $c > 0$. Then*

$$y(\Xi) \leq c + \int_0^{\Xi} \Im(\sigma)y(\sigma)d(\sigma), \quad \Xi \in J$$

implies that

$$y(\Xi) \leq c\ \exp\left(\int_0^{\Xi} \Im(\sigma)d(\sigma) \right), \quad \Xi \in J.$$

Lemma 2.13. *The function $\Lambda \in C(J)$ is a solution of the given system, for any function $h \in C(J)$,*

$$\begin{cases} {}^{*}_{0}D^{\alpha}_{\Xi}\Lambda(\Xi) = h(\Xi), \quad \Xi \in J, \\ \Lambda\left(\Xi_k^{+}\right) = \Lambda\left(\Xi_k^{-}\right) + y_k, \quad y_k \in \mathbb{R}, \end{cases} \tag{2.1}$$

with condition of the form

$$\Lambda(0) = \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \tag{2.2}$$

if and only if Λ is a solution of equation

$$\Lambda(\Xi) = \begin{cases} \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ \quad + \frac{B(\alpha)}{1-\alpha} \int_0^{\Xi_1} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma \\ \quad + \int_0^{\Xi} h(\sigma) d\sigma, \quad for \quad \Xi \in [0, \Xi_1), \\ y_1 + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ \quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_1}^{\Xi_2} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma \\ \quad + \int_0^{\Xi} h(\sigma) d\sigma, \quad for \quad \Xi \in (\Xi_1, \Xi_2), \\ y_1 + y_2 + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ \quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_2}^{\Xi_3} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma \\ \quad + \int_0^{\Xi} h(\sigma) d\sigma, \quad for \quad \Xi \in (\Xi_2, \Xi_3), \\ \quad \vdots \\ \sum_{i=1}^{m} y_i + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ \quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_m}^{\Omega} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma \\ \quad + \int_0^{\Xi} h(\sigma) d\sigma, \quad for \quad \Xi \in (\Xi_m, \Omega]. \end{cases}$$

Proof. Let Λ satisfies (2.1) and (2.2). If $\Xi \in [0, \Xi_1]$, then

$${}^{*}_{0}D^{\alpha}_{\Xi}\Lambda(\Xi) = h(\Xi),$$

$$\Lambda(0) = \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma$$

and

$$\begin{aligned}\Lambda(\Xi) &= \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_0^{\Xi_1} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma.\end{aligned}$$

If $\Xi \in (\Xi_1, \Xi_2)$, then we have

$$\begin{cases} {}_0^{*}D_{\Xi}^{\alpha} \Lambda(\Xi) = h(\Xi), \\ \Lambda\left(\Xi_k^{+}\right) = \Lambda\left(\Xi_k^{-}\right) + y_k, \quad y_k \in \mathbb{R}, \end{cases} \tag{2.3}$$

and so

$$\begin{aligned}\Lambda(\Xi) &= \Lambda\left(\Xi_1^{+}\right) - \int_0^{\Xi_1} h(\sigma) d\sigma + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_1}^{\Xi_2} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma \\ &= \Lambda\left(\Xi_1^{+}\right) + y_1 - \int_0^{\Xi_1} h(\sigma) d\sigma + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_1}^{\Xi_2} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma \\ &= y_1 + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_1}^{\Xi_2} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma.\end{aligned}$$

If $\Xi \in (\Xi_2, \Xi_3)$, then we find

$$\begin{aligned}\Lambda(\Xi) &= \Lambda\left(\Xi_2^{+}\right) - \int_0^{\Xi_2} h(\sigma) d\sigma + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_2}^{\Xi_3} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma \\ &= \Lambda\left(\Xi_2^{+}\right) + y_2 - \int_0^{\Xi_1} h(\sigma) d\sigma + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_2}^{\Xi_3} \mathbb{E}_{\alpha}\left(\frac{-\alpha}{(1-\alpha)}(\Xi-\sigma)^{\alpha}\right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma\end{aligned}$$

$$= y_1 + y_2 + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma$$

$$+ \frac{B(\alpha)}{1-\alpha} \int_{\Xi_2}^{\Xi_3} \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\Xi-\sigma)^\alpha \right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma.$$

Assume the case $\Xi \in (\Xi_m, \Omega]$. Then

$$\begin{aligned} \Lambda(\Xi) &= \sum_{i=1}^{m} y_i + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_{\Xi_m}^{\Omega} \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\Xi-\sigma)^\alpha \right) \Lambda'(\sigma) d\sigma + \int_0^{\Xi} h(\sigma) d\sigma. \end{aligned}$$

Conversely, let Λ satisfies Eq. (2.3). The equation (2.3) to the system (2.1) and (2.2) is $E^{\gamma}_{\rho,\mu,\Lambda;a+}$ as given below:

$$\Lambda(\Xi) = \sum_{i=1}^{m} y_i + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma$$

$$+ \frac{B(\alpha)}{1-\alpha} \left(\mathbb{E}^{1}_{\alpha,1,-\alpha/1-\alpha;0+} \Lambda \right)(\Xi) + \int_0^{\Xi} h(\sigma) d\sigma.$$

□

Theorem 2.14. *The function $\Lambda \in C(J)$ is a solution of ABR-FDEs* (1.1), *for any $f \in C(J \times R \times R \times R, R)$ if and only if Λ is a solution of equation*

$$\begin{aligned} \Lambda(\Xi) &= \sum_{i=1}^{m} y_i + \int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \frac{B(\alpha)}{1-\alpha} \int_0^{\Xi} \mathbb{E}_\alpha \left(\frac{-\alpha}{(1-\alpha)} (\Xi-\sigma)^\alpha \right) \Lambda(\sigma) d\sigma \\ &\quad + \int_0^{t} \Im(\sigma, \Lambda(\sigma), \Psi\Lambda(\sigma), \Upsilon\Lambda(\sigma)) d\sigma, \quad t \in J. \end{aligned} \tag{2.4}$$

Proof. Proof is provided by using Lemma 2.13,

$$h(\Xi) = \Im(\Xi, \Lambda(\Xi), \Psi\Lambda(\Xi), \Upsilon\Lambda(\Xi)), \quad \Xi \in J.$$

□

The theorem in following is proven using the characteristics of the fractional integral operator $E^{\gamma}_{\rho,\mu,\Lambda;a+}$.

Theorem 2.15. *Let $0 < \alpha < 1$. A function F in $C(J)$ according to*

$$(F\Lambda)(\Xi) = \frac{B(\alpha)}{1-\alpha}\left(E^{1}_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda\right)(\Xi), \quad \Lambda \in C(J), \quad \Xi \in J.$$

Then we have the following statements:

(1) *F is a bounded linear operator in $C(J)$;*
(2) *F is satisfied the Lipschitz condition;*
(3) *Let S be a bounded subset in $C(J)$. Then $F(S)$ is equicontinuous;*
(4) *The operator $F\Lambda = \Im$ has a unique solution in $C(J)$ for any $\Im \in C(J)$, and F is invertible.*

Proof. (1) As a result of the integral operator $E^{1}_{\alpha,1,-\alpha/1-\alpha;0+\Lambda}$ is a bounded and linear in $C(J)$, as a result, the equation

$$\|E^{1}_{\alpha,1,-\alpha/1-\alpha;0+\Lambda}\| \leq \mathbb{Q}\|\Lambda\|, \quad \Xi \in \Omega,$$

where we locate

$$\begin{aligned}\mathbb{Q} &= \sum_{k=0}^{\infty} \frac{(1)_k}{\|\Gamma(\alpha k+1)(\alpha k+1)\|} \frac{|-\alpha/1-\alpha|\Omega^{\alpha^k}}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\alpha/1-\alpha\Omega^{\alpha^k}}{\Gamma(\alpha k+2)} \\ &= \mathbb{E}_{\alpha,2}\left(\frac{\alpha}{1-\alpha}\Omega^{\alpha+1}\right).\end{aligned}$$

Then, we have

$$\begin{aligned}F\Lambda &= \left|\pi\frac{B(\alpha)}{1-\alpha}\right| \left\|E^{1}_{\alpha,1,-\alpha/1-\alpha;0+\Lambda}\right\| \\ &\leq \mathbb{Q}\frac{B(\alpha)}{1-\alpha}\|\Lambda\|, \quad \forall \Lambda \in C(J).\end{aligned}$$

Thus, the operator F is bounded on $C(J)$.

(2) For any $\Lambda, \sigma \in C(J)$. Thus, for any $\Xi \in J$, using the linearity of F and the boundedness of the operator $E^{1}_{\alpha,1,-\alpha/1-\alpha;0+}$, we discover that we have:

$$\begin{aligned}|F\Lambda(\Xi) - F\sigma(\Xi)| &= |(F\Lambda - F\sigma)(\Xi)| \\ &= \frac{B(\alpha)}{1-\alpha}\left|\left(E^{1}_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda - \sigma\right)(\Xi)\right| \\ &\leq \frac{B(\alpha)}{1-\alpha}\left\|E^{1}_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda - \sigma\right\| \\ &\leq \mathfrak{Q}\frac{B(\alpha)}{1-\alpha}\|\Lambda - \sigma\|. \qquad (2.5)\end{aligned}$$

This gives

$$\|F\Lambda - F\sigma\| \leq Q\frac{B(\alpha)}{1-\alpha}\|\Lambda - \sigma\|, \quad \Lambda, \sigma \in C(J).$$

With constant of Lipschitz $\Omega B(\alpha)/1-\alpha \mathbb{E}_{\alpha,2}(\alpha/1-\alpha\ \Omega^{\alpha}\ )$, the operator F satisfies the Lipschitz condition.

(3) Let $\mathcal{S} = \{\Lambda \in C(J) : \|\Lambda\| \leq R\}$ be a bounded closed subset of $C(J)$. Then, for every $\Lambda \in \mathcal{S}$ and $\Xi_1, \Xi_2 \in J$ with $\Xi_1 < \Xi_2$, we get

$$\begin{aligned}
&\Big|F\Lambda\left(\Xi_1\right) - F\sigma\left(\Xi_2\right)\Big| \\
&= \Big|\frac{B(\alpha)}{1-\alpha}\left(E^1_{\alpha,1,(-\alpha/1-\alpha);0+}\Lambda\right)\left(\Xi_1\right) - \frac{B(\alpha)}{1-\alpha}\left(E^1_{\alpha,1,(-\alpha/1-\alpha);0+}\Lambda\right)\left(\Xi_2\right)\Big| \\
&= \frac{B(\alpha)}{1-\alpha}\Big|\int_0^{\Xi}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}\left(\Xi_1-\sigma\right)^\alpha\right)\Lambda(\sigma)d\sigma \\
&\quad - \int_0^{\Xi}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}\left(\Xi_2-\sigma\right)^\alpha\right)\Lambda(\sigma)d\sigma\Big| \\
&\leq \frac{B(\alpha)}{1-\alpha}\Big|\int_0^{\Xi}\Big\{\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}\left(\Xi_1-\sigma\right)^\alpha\right) \\
&\quad - \int_0^{\Xi}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}\left(\Xi_2-\sigma\right)^\alpha\right)\Lambda(\sigma)d\sigma\Big\}\Big| \\
&\quad + \frac{B(\alpha)}{1-\alpha}\Big|\int_{\Xi_1}^{\Xi_2}\mathbb{E}_\alpha\left(\frac{-\alpha}{(1-\alpha)}\left(\Xi_2-\sigma\right)^\alpha\right)\Lambda(\sigma)d\sigma\Big| \\
&\leq \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\Big|\left(\frac{-\alpha}{1-\alpha}\right)^k\Big|\frac{1}{\Gamma(\alpha k+1)}\int_0^{\Xi_1}\Big|\left(\Xi_1-\alpha\right)^{k\alpha} - \left(\Xi_2-\alpha\right)^{k\alpha}\Lambda(\sigma)d\sigma\Big| \\
&\quad + \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\Big|\left(\frac{-\alpha}{1-\alpha}\right)^k\Big|\frac{1}{\Gamma(\alpha k+1)}\int_{\Xi_1}^{\Xi_2}\Big|\left(\Xi_2-\alpha\right)^{k\alpha}\Big|\Lambda(\sigma)d\sigma \\
&\leq \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha}{1-\alpha}\right)^k\frac{1}{\Gamma(\alpha k+1)}\int_0^{\Xi_1}\Big\{\left(\Xi_2-\alpha\right)^{k\alpha} - \left(\Xi_1-\alpha\right)^{k\alpha}\Big\}\Lambda(\sigma)d\sigma \\
&\quad + \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha}{1-\alpha}\right)^k\frac{1}{\Gamma(\alpha k+1)}\int_{\Xi_1}^{\Xi_2}\left(\Xi_2-\alpha\right)^{k\alpha}\Lambda(\sigma)d\sigma
\end{aligned}$$

$$\leq \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+2)}$$

$$\times\left\{-\left(\Xi_2-\Xi_1\right)^{k\alpha+1}+\left(\Xi_2\right)^{k\alpha+1}-\left(\Xi_1\right)^{k\alpha+1}+\left(\Xi_2-\Xi_1\right)^{k\alpha+1}\right\}$$

$$\leq \frac{RB(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(\alpha k+2)}\left\{\left(\Xi_2\right)^{k\alpha+1}-\left(\Xi_1\right)^{k\alpha+1}\right\}. \tag{2.6}$$

From above inequality, it follows that, if $\mid \Xi_1 - \Xi_2 \mid \to 0$, then

$$\left|F\Lambda\left(\Xi_1\right)-F\sigma\left(\Xi_2\right)\right| \to 0.$$

This establishes the equicontinuity of $F(S)$ on J.

(4) For any $\Im \in C(J)$, we obtain

$$\left(E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Im\right)^{-1}(\Xi)=\left(D^{1+\beta}_{0+}E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Im\right)^{-1}(\Xi), \Xi \in (a,b), \tag{2.7}$$

where $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta)>0$. Thus, by definition of F and (2.7), we get

$$\begin{aligned}\left(F^{-1}\Im\right)(\Xi) &= \left(\frac{B\alpha}{1-\alpha}E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Im\right)^{-1}(\Xi)\\ &= \frac{1-\alpha}{B\alpha}\left(D^{1+\beta}_{0+}E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Im\right)^{-1}(\Xi), \quad \Xi \in (a,b).\end{aligned}$$

As a result, the operator F is shown to be invertible on $C(J)$.

$$(F\Lambda)(\Xi)=\Im(\Xi), \quad \Xi \in J$$

has the unique solution

$$\Lambda(\Xi)=\frac{1-\alpha}{B\alpha}\left(D^{1+\beta}_{0+}E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Im\right)(\Xi), \ \Xi \in (a,b).$$

This completes the proof. □

Therefore we have the following theorem from the above results.

Theorem 2.16. *Suppose* $\Im \in C(J\times R, R)$. *Then the system* $D^{\alpha}_{\Xi}=\Im(\Xi,\Lambda(\Xi))$, $\Xi \in J$ *can be solvable on* $C(J)$ *and has a solution given by*

$$\Lambda(\Xi)=\frac{1-\alpha}{B\alpha}\left(D^{1+\beta}_{0+}E^1_{\alpha,1,-\alpha/1-\alpha;0+}\bar{\Im}\right)(\Xi), \quad \Xi \in J,$$

where $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta)>0$ *and* $\int_0^{\Xi}\Im(\sigma,\Lambda(\sigma))d(\sigma), \quad \Xi \in J$.

3. Main results

Theorem 3.1. *Suppose* $\Im \in C(J \times R^3, R)$ *and the Lipschitz condition of* $\vartheta \in C(J \times R)$ *is satisfied.*

(1) $|\Im(\Xi, \Lambda, \kappa_1, h_1) - \Im(\Xi, v, \kappa_2, h_2)| \leq p(\Xi)\left[|\Lambda - v| + |\kappa_1 - \kappa_2| + |h_1 - h_2|\right]$, $\Lambda, v, \kappa_i, h_i \in C(J),\ \ i = 1, 2$,

(2) $|\vartheta(\Xi, \Lambda) - \vartheta(\Xi, v)| \leq L_\vartheta |\Lambda - v|,\ \ \Lambda, v \in C(J)$,

(3) $|\vartheta(\Xi, \Lambda)| \leq C_\vartheta |\Lambda| + M_\vartheta,\ \ \Lambda \in C(J)$,

where $p : J \longrightarrow \mathbb{R}^+$ *with* $L = \sup(\Xi)$ *and for the real constants* $L_\vartheta, M_\vartheta, C_X > 0$. *If* $0 < L < \min\{1, 1/2\Omega\}$, *then the system* (1.1) *has a solution in* $C(J)$ *provided*

$$\left(\frac{\Omega^\alpha}{\Gamma(\alpha+1)} C_\vartheta + \frac{B(\alpha)\mathbb{E}_{\alpha,2}(\alpha/1-\alpha)\Omega^{\alpha+1}}{1-\alpha}\right) < 1. \tag{3.1}$$

Proof. Assume that

$$R = \frac{\Omega^\alpha/\Gamma(\alpha+1) M_C + M_\Im \Omega + M^*}{1 - L\Omega - \Omega^\alpha/\Gamma(\alpha+1) C_C - B(\alpha)\mathbb{E}_{\alpha,2}(\alpha/1-\alpha)\Omega^{\alpha+1}/1-\alpha}, \tag{3.2}$$

where $M_\Im = \sup |\Im(\Xi, 0, 0, 0)|$ and $M^* > 0$ is a constant such that $\sum_{i=1}^m |y_i| \leq M^*$. Then we have $R > 0$ from the choice of L and condition (3.1).

Assume that

$$\mathcal{S} = \{\Lambda \in C(J) : \|\Lambda\| \leq R\}. \tag{3.3}$$

Then the existence of $\mathcal{S}$ as a closed, convex and bounded subset of Λ may be demonstrated. Let $F_1 : S \to \Lambda$ and $F_2 : S \to \Lambda$, both of which are defined by

$$\begin{aligned} (F_1\Lambda)(\Xi) &= \sum_{i=1}^m y_i + \int_0^\Omega \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \vartheta(\sigma, \Lambda(\sigma)) d\sigma \\ &\quad + \int_0^\Xi \Im(\sigma, \Lambda(\sigma), \Psi\Lambda(\sigma), \Upsilon\Lambda(\sigma)) d\sigma, \\ (F_2\Lambda)(\Xi) &= \frac{B(\alpha)}{1-\alpha}\left(E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda\right)(\Xi),\ \ \Xi \in J. \end{aligned} \tag{3.4}$$

The operator equivalent (3.4) to the system (1.1) is as follows:

$$\Lambda = F_1\Lambda + F_2\Lambda,\ \ \Lambda \in PC(J). \tag{3.5}$$

The same has been shown using the steps below.

Step 1. F_1 is contraction.

For every $\Lambda, v \in PC(J)$ and $\Xi \in J$, we have using the Lipschitz on $\Im$,

$$|\Im(\Xi, \Lambda(\Xi), \Psi\Lambda(\sigma), \Upsilon\Lambda(\sigma)) - \Im(\Xi, v(\Xi), \Phi v(\Xi), \Upsilon v(\Xi))| \leq p(\Xi)|\Lambda - v|. \tag{3.6}$$

This gives

$$\|F_1\Lambda - F_1 v\| \le (L_\vartheta + L\Omega)\,\|\Lambda - v\|, \quad \Lambda, v \in PC(J). \tag{3.7}$$

Step 2. Next, we demonstrate that F_2 is a fully continuous variable. Using the Ascoli-Arzela Theorem and Theorem 2.15, it is simple to show that the operator F_2 is entirely continuous.

Step 3. We have to prove that $F_1\Lambda + F_2 v \in \mathcal{S}$ for any $\Lambda, v \in \mathcal{S}$.

For any $\Lambda, v \in \mathcal{S}$, from Theorem 2.15, we have

$$\begin{aligned}
&|(F_1\Lambda + F_2 v)(\Xi)| \\
&\quad \le |(F_1\Lambda)(\Xi)| + |(F_2 v)(\Xi)| \\
&\quad \le \left|\int_0^\Omega \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(v)}\vartheta(\sigma,\Lambda(\sigma))d\sigma\right| + \sum_{i=1}^m y_i \\
&\qquad + \int_0^\Xi |\Im(\sigma,\Lambda(\sigma),\Psi\Lambda(\sigma),\Upsilon\Lambda(\sigma))|d\sigma + \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}\Omega^{\alpha+1}\right]\|v\| \\
&\quad \le \int_0^\Omega \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}|\vartheta(\sigma,\Lambda(\sigma))|d\sigma + M^* \\
&\qquad + \int_0^\Xi |\Im(\sigma,\Lambda(\sigma),\Psi\Lambda(\sigma),\Upsilon\Lambda(\sigma))| - \Im(\sigma,0,0,0)\mid d\sigma \\
&\qquad + \int_0^\Xi |\Im(\sigma,0,0,0)|d\sigma + \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}\Omega^{\alpha+1}\right]R \\
&\quad \le \frac{\Omega^\alpha}{\Gamma(\alpha+1)}(C_\vartheta\|\Lambda\| + M_C) + M^* + L\int_0^\Xi |\Im(\sigma,\Lambda(\sigma),\Psi\Lambda(\sigma),\Upsilon\Lambda(\sigma))| \\
&\qquad + M_\Im\int_0^\Xi d\sigma + \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}\Omega^{\alpha+1}\right]R \\
&\quad \le \frac{\Omega^\alpha}{\Gamma(\alpha+1)}(C_\vartheta R + M_\vartheta) + M^* + LR\Xi \\
&\qquad + M_\Im\Xi + \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}\Omega^{\alpha+1}\right]R \\
&\quad \le \frac{\Omega^\alpha}{\Gamma(\alpha+1)}(C_\vartheta R + M_\vartheta) + M^* + LR\Omega \\
&\qquad + M_\Im\Omega + \frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha,2}\left[\frac{\alpha}{(1-\alpha)}\Omega^{\alpha+1}\right]R,
\end{aligned} \tag{3.8}$$

that is, by condition (3.1) with (3.2), we have

$$\frac{\Omega^{\alpha}}{\Gamma(\alpha+1)}(M_C)+M_{\Im}\Omega+M^* = R1-L\Omega-\frac{\Omega^{\alpha}}{\Gamma(\alpha+1)}C_l -\frac{B(\alpha)\mathbb{E}_{\alpha,2}(\alpha/1-\alpha)\Omega^{\alpha+1}}{1-\alpha}. \quad (3.9)$$

From (3.8) and (3.9), we get

$$|(F_1\Lambda+F_2v)(\Xi)|\leq R,\ \ \Xi\in J.$$

This gives

$$\|(F_1\Lambda+F_2v)\|\leq R,\ \ \forall\Lambda,v\in\mathcal{S}.$$

This shows that $F_1\Lambda+F_2v\in\mathcal{S}$ for $\Lambda,v\in\mathcal{S}$. Thus,

$$\Lambda=F_1\Lambda+F_2\Lambda,$$

which is the solution of the system (1.1) has a fixed point in S. The evidence is now complete. □

The following theorem provides two ways to show the uniqueness of solutions to the system (1.1). Using the Gronwall-Bellman inequality and the characteristics of the fractional integral operator $E^1_{\alpha,1,-\alpha/1-\alpha;0+}$, we first demonstrate the conclusion.

Theorem 3.2. *The system* (1.1) *has a unique solution in* $C(J)$ *under the conditions of Theorem* 3.1.

Proof. The equivalent fractional integral equation to the ABR-FDEs (1.1) in operator equation form is

$$\left(E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda\right)(\Xi)=\tilde{\Im}(\Xi),\ \ \Xi\in J,$$

where

$$\tilde{\Im}(\Xi)=\frac{1-\alpha}{B(\alpha)}\left(\int_0^{\Omega}\frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\vartheta(\sigma,\Lambda(\sigma))d\sigma-\Lambda_{\Xi} +\int_0^{\Xi}\Im(\sigma,\Lambda(\sigma),\Psi\Lambda(\sigma),\Upsilon\Lambda(\sigma))d\sigma\right)+\sum_{i=1}^{m}y_i,\ \ \Xi\in J.$$

Theorem 3.1 states that the operator (1.1) can be solved in $C(J)$. Applying Lemma 2.10 yields, the operator (1.1) has a unique solution in $C(J)$, which is the system (1.1).

Let Λ and v be two ABR-FDEs solutions that is (1.1). Using the linearity of the fractional integral operator, we get for every $\Xi\in J$,

$$
\begin{aligned}
&|\Lambda(\Xi)-v(\Xi)| \\
&= \Big(\int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\vartheta(\sigma,\Lambda(\sigma))d\sigma - \frac{B(\alpha)}{1-\alpha}\left(E^1_{\alpha,1,-\alpha/1-\alpha;0+}\Lambda\right)(\Xi) \\
&\quad + \int_0^{\Xi} \Im(\sigma,\Lambda(\sigma),\Psi\Lambda(\sigma),\Upsilon\Lambda(\sigma))d\sigma\Big) \\
&\quad - \Big(\int_0^{\Omega} \frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\vartheta(\sigma,v(\sigma))d\sigma - \frac{B(\alpha)}{1-\alpha}\left(E^1_{\alpha,1,-\alpha/1-\alpha;0+}v\right)(\Xi) \\
&\quad + \int_0^{\Xi} \Im(\sigma,v(\sigma),\Psi v(\sigma),\Upsilon v(\sigma))d\sigma\Big) \\
&\le \frac{\Omega^{\alpha}}{\Gamma(\alpha+1)}\left(L_{\vartheta}|\Lambda(\sigma)-v(\sigma)|\right) \\
&\quad + \frac{B(\alpha)}{1-\alpha}\int_0^{\Xi} \mathbb{E}_{\alpha}\left(\left|\frac{-\alpha}{(1-\alpha)}(\Omega-\sigma)^{\alpha}\right|\right)|\Lambda(\sigma)-v(\sigma)|d\sigma \\
&\quad + \int_0^{\Xi} pl_1(\sigma)|\Lambda(\sigma)-v(\sigma)|d\sigma \\
&\le \frac{\Omega^{\alpha}}{\Gamma(\alpha+1)}\left(L_{C}|\Lambda(\sigma)-v(\sigma)|\right) \\
&\quad + \frac{B(\alpha)}{1-\alpha}\int_0^{\Xi} \mathbb{E}_{\alpha}\left(\frac{\alpha}{(1-\alpha)}(\Omega)^{\alpha}\right)|\Lambda(\sigma)-v(\sigma)|d\sigma \\
&\quad + \int_0^{\Xi} pl_1(\sigma)|\Lambda(\sigma)-v(\sigma)|d\sigma \\
&\le \int_0^{\Xi}\left[\frac{(\Omega-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L_{\vartheta}+\frac{B(\alpha)}{1-\alpha}\mathbb{E}_{\alpha}\left(\frac{\alpha}{(1-\alpha)}(\Omega)^{\alpha}+pl_1(\sigma)\right)\right]|\Lambda(\sigma)-v(\sigma)|d\sigma.
\end{aligned}
$$

By Lemma 2.12, we have

$$|\Lambda(\Xi)-v(\Xi)| \le 0,\ \Xi\in J.$$

This demonstrates that for any $\Xi\in J$, $\Lambda(\Xi)=v(\Xi)$. This proves that the system (1.1) has a unique solution. □

4. Illustrative example

Example 4.1. Let us consider the following integro-differential system:

$$
\begin{cases}
{}^{*}_{0}D^{1/2}_{\Xi}\Lambda(\Xi) = \frac{\cos(|\Lambda(t)|)}{35} + \frac{1}{25}\int_0^t \left(t^2+s^2\right)\Lambda(s)\mathrm{d}s + \frac{1}{15}\int_0^1 s^2\Lambda(s)\mathrm{d}s, \quad t\in[0,1], \\
\Delta\Lambda(t) = \frac{\Lambda\left(1/2^-\right)}{3+\Lambda(1/2^-)}, \\
\Lambda(0) = \int_0^1 \frac{1}{30}\sigma^2\Lambda(\sigma)d\sigma,
\end{cases}
$$

where

$$\Im(\Xi,\Lambda(\Xi),\Psi\Lambda(\Xi),\Upsilon\Lambda(\Xi)) = \frac{\cos(|\Lambda(t)|)}{35} + \frac{1}{25}\int_0^t \left(t^2+s^2\right)\Lambda(s)ds + \frac{1}{15}\int_0^1 s^2\Lambda(s)\mathrm{d}s,$$

$$\vartheta(\sigma,\Lambda(\sigma)) = \frac{1}{30}\sigma^2\Lambda(\sigma).$$

As $\Omega = 1, \alpha = 1/2$, let $\Lambda, v \in PC(J)$ with integral boundary condition

$$\begin{aligned}
&|\Im(\Xi,\Lambda(\Xi),\Psi\Lambda(\Xi),\Upsilon\Lambda(\Xi)) - \Im(\Xi,v(\Xi),\Psi v(\Xi),\Upsilon v(\Xi))| \\
&\leq \Big|\frac{\cos(|\Lambda(t)|)}{35} + \frac{1}{25}\int_0^t \left(t^2+s^2\right)\Lambda(s)ds + \frac{1}{15}\int_0^1 s^2\Lambda(s)\mathrm{d}s \\
&\quad - \frac{\cos(|v(t)|)}{35} - \frac{1}{25}\int_0^t \left(t^2+s^2\right)v(s)ds + \frac{1}{15}\int_0^1 s^2 v(s)\mathrm{d}s\Big| \\
&\leq \left[\frac{1}{35}+\frac{2}{75}+\frac{1}{45}\right](\Lambda(t)-v(t)),
\end{aligned}$$

$$\begin{aligned}
|\vartheta(\sigma,\Lambda(\sigma)) - \vartheta(\sigma,v(\sigma))| &\leq \left|\frac{1}{30}\sigma^2\Lambda(\sigma) - \frac{1}{30}\sigma^2 v(\sigma)\right| \\
&\leq \frac{1}{30}|\Lambda(t)-v(t)|,
\end{aligned}$$

$$\begin{aligned}
|\vartheta(\sigma,\Lambda(\sigma))| &\leq \left|\frac{1}{30}\sigma^2\Lambda(\sigma)\right| \\
&\leq \frac{1}{30}|\Lambda(t)| + \frac{1}{30}.
\end{aligned}$$

Then, we get $L = 0.077, C_\vartheta = 1/30, M_\vartheta = 1/30, M_\Im = 1/35, M^* = 1/3$, let $k = 1$,

$$\left(\frac{B(\alpha)\mathbb{E}_{\alpha,2}(\alpha/1-\alpha)\Omega^{\alpha+1}}{1-\alpha} + \frac{\Omega^\alpha}{\Gamma(\alpha+1)}C_\vartheta\right) = 0.73 < 1,$$

$$R = \frac{\Omega^\alpha/\Gamma(\alpha+1)M_\vartheta + M_\Im\Omega + M^*}{1 - L\Omega - \Omega^\alpha/\Gamma(\alpha+1)C_\vartheta - B(\alpha)_{\alpha,2}(\alpha/1-\alpha)\Omega^{\alpha+1}/1-\alpha} = 1.92.$$

As a result, the requirement of Theorem 3.1 is met.

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