Nonlinear Functional Analysis and Applications
Vol. 29, No. 1 (2024), pp. 131-164
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.01.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

ALTERNATED INERTIAL RELAXED TSENG METHOD FOR SOLVING FIXED POINT AND QUASI-MONOTONE VARIATIONAL INEQUALITY PROBLEMS

A. E. Ofem[1], A. A. Mebawondu[2], C. Agbonkhese[2], G. C. Ugwunnadi[3,4] and O. K. Narain[1]

[1]School of Mathematics, Statistics and Computer Science,
University of KwaZulu-Natal, Durban, South Africa
e-mail: `222128007@stu.ukzn.ac.za (A.E.O)`; `naraino@ukzn.ac.za (O.K.N)`

[2]Department of Computer Science and Mathematics, Mountain Top University,
Prayer City, Ogun State, Nigeria
e-mail: `dele@aims.ac.za (A.A.M)`; `cagbonkhese@mtu.edu.ng (C.A)`

[3]Department of Mathematics, University of Eswatini, Private Bag 4, Kwaluseni, Eswatini
e-mail: `gcugwunnadi@uniswa.sz`

[4]Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Sciences University, Medunsa 0204, Pretoria, South Africa
e-mail: `ugwunnadi4u@yahoo.com`

Abstract. In this research, we study a modified relaxed Tseng method with a single projection approach for solving common solution to a fixed point problem involving finite family of τ-demimetric operators and a quasi-monotone variational inequalities in real Hilbert spaces with alternating inertial extrapolation steps and adaptive non-monotonic step sizes. Under some appropriate conditions that are imposed on the parameters, the weak and linear convergence results of the proposed iterative scheme are established. Furthermore, we present some numerical examples and application of our proposed methods in comparison with other existing iterative methods. In order to show the practical applicability of our method to real word problems, we show that our algorithm has better restoration efficiency than many well known methods in image restoration problem. Our proposed iterative method generalizes and extends many existing methods in the literature.

---

[0]Received June 22, 2023. Revised July 12, 2023. Accepted July 16, 2023.
[0]2020 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Variational inequality problem, fixed point, quasimonotone operator, strong convergence and linear convergence, Tseng method.
[0]Corresponding author: A. E. Ofem(`222128007@stu.ukzn.ac.za`).

1. Introduction

Let $\mathcal{G}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator. The notion of variational inequality problem (VIP) is formulated as:

$$\text{find } u \in \mathcal{G} \quad \text{such that} \quad \langle \mathcal{K}u, v - u \rangle \geq 0, \ \forall \ v \in \mathcal{G}. \tag{1.1}$$

Stampacchia [32] and Fichera [12, 13] independently introduced the notion of VIP (1.1) to solve the Signorini problem for modeling problems in mechanics. Vast problems in mathematical physics, economics and the mathematical sciences can easily be formulated as VIPs. We represent the solution set of VIP (1.1) by $\mathcal{V}$. In recent years, many authors in this field have constructed different iterative approaches for solving VIP (1.1) as a result of its important applications. The simplest method for solving VIP (1.1) is the gradient-projection technique which only computes one projection on the feasible set in each iteration. The limitation of this technique is that it has stringent and strong condition since the operator requires to be substantially monotonic for convergence.

This drawback motivated Korpelevich [18] to introduce and study the extragradient method (EM) as follows:

$$\begin{cases} u_1 \in \mathcal{G}, \\ v_m = P_{\mathcal{G}}(u_m - \lambda_m \mathcal{K} u_n), \\ u_{m+1} = P_{\mathcal{G}}(u_m - \lambda_m \mathcal{K} v_m), \ \forall m \geq \mathbb{N}, \end{cases} \tag{1.2}$$

where $P_{\mathcal{G}}$ is a metric projection defined from $\mathcal{H}$ onto $\mathcal{G}$, $\lambda_m \in (0, \frac{1}{L})$ and $\mathcal{K}$ is monotone and L-Lipschitz continuous. Obviously, the computation of the λ_m will be difficult since it involves calculation of two projections. Also, this method has the disadvantage of being computationally expensive and this slows down the rate at which the iterative process converges. After the development of EM, several researcher have developed, modified and generalized the EM such that the cost operator $\mathcal{K}$ possesses monotonicity and pseudomonotonicity.

Particularly, Tseng [37] developed and studied the following the iterative process:

$$\begin{cases} u_1 \in \mathcal{G}, \\ v_m = P_{\mathcal{G}}(u_m - \lambda_m \mathcal{K} u_m), \\ u_{m+1} = v_m + \lambda_m (\mathcal{K} u_m - \mathcal{K} v_n), \end{cases} \tag{1.3}$$

for all $m \in \mathbb{N}$, where $\lambda_m \in (0, \frac{1}{L})$. Some recent literatures include: Apostol et al. [7], Ceng et al. [8], Ceng et al. [9], He et al. [16], He et al. [17], and Nadezhkina and Takahashi [19] and several others.

The inertial technique is based on a discrete representation of a second-order dissipative dynamical system (see [27]) and has recently received a lot of attention from researcher as one of the methods to speed up algorithm convergence. The primary concept behind inertial-type approaches is that the combination of the past two (or more) iterations determines what happens in the subsequent iteration. The notion of the inertial has been improved, generalized and extended by researchers in this field (see [20, 22, 23]). As helpful as the concept of the inertial technique is, it has been observed that with the inertial steps the iterative methods lose the Fejer monotonicity of the sequence with respect to the solution which is being enjoyed by their corresponding non-inertial projection methods for variational inequalities. This lack of Fejer monotonicity makes projection methods with inertial extrapolation step for variational inequalities not to converge faster than their corresponding non-inertial projection methods at times.

In the light of this development, Shehu and Iyiola [28], introduced and studied the following iterative method:

Algorithm 1.1. Initialization Step: *Choose* $\lambda_1 > 0$, $\gamma \in (0, 2)$, $0 \le \alpha_m \le \alpha < \frac{2-\gamma}{\gamma}$, $\mu \in (0, 1)$ *and let* $u_0, u_1 \in \mathcal{H}$, *given the iterates* u_{m-1} *and* u_m *for all* $m \in \mathbb{N}$.

Step 1: *Compute*

$$w_m = \begin{cases} u_m, & \text{if } m = \text{even} \\ u_m + \alpha_m(u_m - u_{m-1}), & \text{if } m = \text{odd.} \end{cases} \tag{1.4}$$

Step 2: *Compute*

$$v_m = P_{\mathcal{G}}(w_m - \lambda_m \mathcal{K} w_m), \tag{1.5}$$

where

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|w_m - v_m\|}{\|\mathcal{K}w_m - \mathcal{K}v_m\|}, \lambda_m\right\}, & \text{if } \mathcal{K}w_m \neq \mathcal{K}v_m \\ \lambda_m, & \text{otherwise,} \end{cases} \tag{1.6}$$

If $w_m = v_m$, *then stop* v_m *is a solution of the VIP* (1.1). *Else, go to :*

Step 3: *Compute*

$$u_{m+1} = w_m - \gamma \eta_m d_m, \tag{1.7}$$

where $d_m = w_m - v_m - \lambda_m(\mathcal{K}w_m - \mathcal{K}v_n)$ *and*

$$\eta_m = \begin{cases} \min\left\{\frac{\langle w_m - v_m, d_m\rangle}{\|d_m\|^2}, \lambda_m\right\}, & \text{if } d_m \neq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{1.8}$$

The authors showed the Algorithm 1.1 weakly converges to an element in the solution set of the VIP (1.1). Furthermore, very recently, Ogbuisi et al. [24] developed a new alternating inertial Tseng exragradient approach with relaxation effects and adaptable step sizes for solving pseudo-monotone variational inequality problem in real Hilbert spaces as follows:

Algorithm 1.2. Initialization Step: *Choose* $\theta \in (0, 1]$, $0 \leq \alpha_m \leq \frac{(1-\mu)^2}{(1+\mu)^2}$, $\mu \in (0, 1)$ *and* $\lambda_1 > 0$. *Let* $u_0, u_1 \in \mathcal{H}$, *given the iterates* u_{m-1} *and* u_m *for all* $m \in \mathbb{N}$.

Step 1: *Compute*

$$w_m = \begin{cases} u_m, & \text{if } m = \text{even} \\ u_m + \alpha_m(u_m - u_{m-1}), & \text{if } m = \text{odd.} \end{cases} \tag{1.9}$$

Step 2: *Compute*

$$v_m = P_{\mathcal{G}}(w_m - \lambda_m \mathcal{K} w_m), \tag{1.10}$$

where

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu\|w_m - v_m\|}{\|\mathcal{K}w_m - \mathcal{K}v_m\|}, \lambda_m\right\}, & \text{if } \mathcal{K}w_m \neq \mathcal{K}v_m \\ \lambda_m, & \text{otherwise,} \end{cases} \tag{1.11}$$

If $w_m = v_m$, *then stop* v_m *is a solution of the VIP* (1.1). *Else, go to:*

Step 3: *Compute*

$$u_{m+1} = (1-\theta)w_m + \theta z_m, \tag{1.12}$$

where $z_m = v_m + \lambda_m(\mathcal{K}w_m - \mathcal{K}v_m)$.

There are many inertial projection-based algorithms for approximating the solution to variational inequality problems in the literature (see, for examples, [1, 2, 3, 4, 5]).

A fixed point of an operator S is a point $u \in \mathcal{H}$ such that $u = Su$. There exist many results in the literature concerning fixed point theory, see, for examples [21, 25]. As far as we know, there is no result in the literature involving the concept of alternated inertial algorithm for approximating the common solution to demimetric fixed point problem and quasi-monotone variational inequality problem. That is,

$$\text{find } u \in \mathcal{H} \quad \text{such that} \quad u \in \mathcal{V} \cap F(S), \tag{1.13}$$

where $S : \mathcal{H} \to \mathcal{H}$ and the set of fixed points of S is denoted as $F(S)$. Therefore, it is natural to ask the following question:

Question: Is it possible to construct a modified self-adaptive alternated inertial Tseng algorithm for approximating the common solution to demimetric fixed point problem and quasi-monotone variational inequality problem?

Motivated by the ongoing research in these directions, the purpose of this work is to provide an affirmative answer to the above question by introducing a modified inertial Tseng iterative technique with self-adaptive step size for approximating the common solution to demimetric fixed point problem and quasimonotone variational inequality problem. Our results improve and generalize many results in the literature in the sense that the classes of operators considered in our method are more general than those considered in several existing results. Precisely, the class of quasi-monontone operators are more general than that of monotone operators which has been studied by many authors (see, for example [2, 6, 7, 10, 11, 15, 16, 24, 31]) and also, the class of demimetric operators properly includes the classes of nonexpansive, quasi-nonexpansive, strictly pseudocontractive and demicontractive operators which have been considered by several researchers (see, for examples, [35, 36] and the references in them).

In addition, we establish the weak and linear convergence results of the proposed iterative technique under some mild assumptions. We present some numerical examples to show that our proposed methods are more efficient than many prominent existing algorithms in the literature. Furthermore, we apply our iterative technique to the solution of image restoration problem.

The remaining parts of this article is organized as follows: In Section 2, we state and define some results that will be important in our study. In Section 3, we establish our proposed algorithm. In Section 4, we prove the weak convergence results of our algorithm. In Section 5, we show that our algorithm is linearly convergent to a common solution. In section 6, we carry out numerical experiments to show the advantage of new method over some existing methods in the literature. In Section 7, we use our algorithm to solve real-world problems involving image restoration and in Section 8, we give the conclusion of this study.

2. Preliminaries

In what follows, we recall some results which will be useful in the sequel. Let H be a real Hilbert space. The strong and weak convergence will denoted by "$\to$" and "$\rightharpoonup$", respectively. For each $u, v \in \mathcal{H}$ and $\alpha \in [0, 1]$, the following concepts are well known:

$$\|u-v\|^2 = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2, \tag{2.1}$$

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2, \tag{2.2}$$

$$\|u-v\|^2 \leq \|u\|^2 + 2\langle v, u-v\rangle, \tag{2.3}$$

$$\|\alpha u + (1-\alpha)v\|^2 = \alpha\|u\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|u-v\|^2. \tag{2.4}$$

Definition 2.1. Let $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be an operator. Then $\mathcal{K}$ is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|\mathcal{K}u - \mathcal{K}v\| \leq L\|u-v\|$$

for each $u, v \in \mathcal{H}$. If $L = 1$, then $\mathcal{K}$ is said to be nonexpansive and we call $\mathcal{K}$ quasi-nonexpansive if for all $v \in F(\mathcal{K})$, and $u \in \mathcal{H}$, we have

$$\|\mathcal{K}u - v\| \leq \|u - v\|,$$

(b) α- strongly monotone if there exists $\alpha > 0$ such that

$$\langle \mathcal{K}u - \mathcal{K}v, u - v\rangle \geq \alpha\|u-v\|^2, \ \forall u, v \in \mathcal{H};$$

(c) monotone if

$$\langle \mathcal{K}u - \mathcal{K}v, u - v\rangle \geq 0, \ \forall u, v \in \mathcal{H};$$

(d) pseudomonotone if

$$\langle \mathcal{K}u, v - u\rangle \geq 0 \ \ \Rightarrow \ \ \langle \mathcal{K}v, v - u\rangle \geq 0, \ \forall u, v \in \mathcal{H};$$

(e) quasi-monotone if

$$\langle \mathcal{K}u, u - v\rangle > 0 \ \ \Rightarrow \ \ \langle \mathcal{K}v, u - v\rangle \geq 0, \ \forall \ u, v \in \mathcal{H};$$

(f) sequentially weakly continuous if for each sequence $\{u_m\}$, we have $\{u_m\} \rightharpoonup u \implies \mathcal{K}u_m \rightharpoonup \mathcal{K}u$.

Remark 2.2. It is well known (b) $\implies$ (c) $\implies$ (d) $\implies$ (e). However, the converses are not generally true.

Let $\mathcal{G}$ be a nonempty, closed and convex subset of $\mathcal{H}$. For any $u \in \mathcal{H}$, a unique point $P_{\mathcal{G}}u \in \mathcal{G}$ exists such that

$$\|u - P_{\mathcal{G}}u\| \leq \|u - v\|, \ \forall v \in \mathcal{G}.$$

The operator $P_{\mathcal{G}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{G}$ and it is well known to be nonexpansive satisfying

$$\langle u - v, P_{\mathcal{G}}u - P_{\mathcal{G}}v\rangle \geq \|P_{\mathcal{G}}u - P_{\mathcal{G}}v\|^2 \tag{2.5}$$

for all $u, v \in \mathcal{H}$. Furthermore, $P_{\mathcal{G}}$ possesses the following properties:

$$\|u - v\|^2 \geq \|u - P_{\mathcal{G}}u\|^2 + \|v - P_{\mathcal{G}}u\|^2$$

and

$$\langle u - P_{\mathcal{G}}u, v - P_{\mathcal{G}}u\rangle \leq 0 \tag{2.6}$$

for all $u \in \mathcal{H}$ and $v \in \mathcal{G}$.

Lemma 2.3. ([14, 39]) *Let $\mathcal{G}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ be a quasi-monotone and L-Lipschitzian operator. Assume that $v \in \mathcal{G}$ and for some $p^* \in \mathcal{G}$, we have $\langle \mathcal{K}v, p^* - v\rangle \geq 0$. Then either $\langle \mathcal{K}p^*, p^* - v\rangle \geq 0$ or $\langle \mathcal{K}v, q^* - v\rangle \leq 0$ for all $q^* \in \mathcal{G}$.*

Lemma 2.4. ([26]) *Let $\{\alpha_m\}, \{\beta_m\}$ and $\{\gamma_m\}$ be positive real sequences such that*

$$\alpha_m \leq \beta_m \alpha_m + \gamma_m$$

for all $m \in \mathbb{N}$. If $\{\beta_m\} \subset [1, \infty)$, $\sum_{m=1}^{\infty}(\beta_m - 1) < \infty$ and $\sum_{m=1}^{\infty} \gamma_m < \infty$, then $\lim_{m\to\infty} \lambda_m$ exists.

Lemma 2.5. ([38]) *Let $\{\alpha_m\}, \{\beta_m\} \subset \mathbb{R}_+$, $\{\eta_m\} \subset (0, 1)$ and $\{\chi_m\}$ be a real sequence such that*

$$\alpha_{m+1} \leq (1 - \eta_m)\alpha_m + \chi_m + \beta_m$$

for all $m \in \mathbb{N}$. Assume that $\sum_{m=0}^{\infty} \beta_m < \infty$. Then the following results hold:

1. *If $\beta_m < \eta_m M$ for some $M > 0$, then the sequence $\{\alpha_m\}$ is bounded;*
2. *If $\sum_{m=0}^{\infty} \eta_m = \infty$ and $\limsup_{m\to\infty} \frac{\chi_m}{\eta_m} = 0$, then $\alpha_m \to 0$ as $m \to \infty$.*

Lemma 2.6. ([33, 34]) *Let $\mathcal{G}$ be a nonempty, closed convex subset of a real Hilbert space $\mathcal{H}$ and let $\mathcal{S} : \mathcal{G} \to \mathcal{G}$ be a τ-demimetric operator such that $\tau \in (-\infty, 1)$ and $F(\mathcal{S}) \neq \emptyset$. Let ψ be a real number with $0 < \psi < 1 - \tau$ and $K = (1 - \psi)I + \psi\mathcal{S}$. Then K is a quasi-nonexpansive operator.*

3. Proposed algorithm

In this section, we present our proposed algorithm for solving fixed point problem and variational inequality problem.

Assumptiom 3.1. *Now we assume that the following conditions:*
Condition A.

1. *Let $\mathcal{G}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$.*
2. *$\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is quasi-monotone and Lipschitz continuous with Lipschitz constant L.*
3. *For each $i \in \{1, 2, \cdots, N\}$, $S_i : H \to H$ is a τ-demimetric operator with $\tau \in (-\infty, 1)$ and such that $I - S_i$ is demiclosed at zero.*
4. *The common solution set is denoted by $\Gamma = \mathcal{V} \cap \bigcap_{i=1}^{N} F(S_i)$ is nonempty.*

Condition B.

(1) $\lambda_1 > 0,\ \theta > 0,\ \mu, \phi, \in (0,1),\ \beta \in (0, \frac{1}{1+\mu})$.
(2) $\{\psi_m\}, \{\beta_m\}, \{\alpha_m\} \subset (0,1)$, $\{p_m\} \subset [1, \infty)$ *such that* $\liminf_{m\to\infty} p_m = 1$, *and* $0 < \lim_{n\to\infty} \alpha_m < 1$.
(3) $\{\delta_m\} \subset [1, \infty)$ *such that* $\sum_{m=1}^{\infty}(\delta_m - 1) < \infty$ *and* $\chi_m \subset [0, \infty)$ *such that* $\sum_{m=1}^{\infty} \chi_m < \infty$.

Algorithm 3.2. Initialization Step: *Choose* $u_0, u_1 \in \mathcal{H}$, *given the iterates* u_{m-1} *and* u_m *for all* $m \in \mathbb{N}$.

$$\theta_m = \begin{cases} \min\left\{\theta, \frac{\epsilon_m}{\|u_m - u_{m-1}\|}\right\}, & \text{if } u_m \neq u_{m-1} \\ \\ \theta, & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\{\epsilon_m\}$ *is a positive sequence such that* $\epsilon_m = \circ(\alpha_m)$.

Step 1: *Compute*

$$w_m = \begin{cases} u_m, & \text{if } m = \text{even} \\ \\ u_m + \theta_m(u_m - u_{m-1}), & \text{if } m = \text{odd}, \end{cases} \tag{3.2}$$

and

$$v_m = P_{\mathcal{G}}(w_m - \beta\lambda_m \mathcal{K} w_m), \tag{3.3}$$

where

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu p_m \|w_m - v_m\|}{\|\mathcal{K}w_m - \mathcal{K}v_m\|}, \chi_m + \delta_m\lambda_m\right\}, & \text{if } \mathcal{K}w_m \neq \mathcal{K}v_m \\ \\ \chi_m + \delta_m\lambda_m, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 2: *Compute*

$$\begin{aligned} t_m = (1-\beta_m)&[(1-\phi)w_m + \phi v_m + \phi\beta\lambda_m(\mathcal{K}w_m - \mathcal{K}v_m)] \\ &+ \beta_m T_m[(1-\phi)w_m + \phi v_m + \phi\beta\lambda_n(\mathcal{K}w_m - \mathcal{K}v_m)], \end{aligned} \tag{3.5}$$

where $T_m := \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_m)I + \psi_m S_i)$.

Step 3: *Compute*

$$u_{m+1} = (1-\alpha_m)w_m + \alpha_m t_m. \tag{3.6}$$

Remark 3.3. (1) Suppose that the Assumptions 3.1 hold. Then the sequence $\{\lambda_m\}$ generated by (3.4) is well defined and $\lim_{m\to\infty}\lambda_m$ exists. In fact, by the Lipschitz continuity of $\mathcal{K}$ with $L>0$ and $p_m \geq 1$, we get

$$\frac{\mu p_m\|w_m - v_m\|}{\|\mathcal{K}w_m - \mathcal{K}v_m\|} \geq \frac{\mu p_m\|w_m - v_m\|}{L\|w_m - v_m\|} \geq \frac{\mu}{L}, \tag{3.7}$$

$$\lambda_{m+1} = \min\left\{\frac{\mu p_m\|w_m - v_m\|}{\|\mathcal{K}w_m - \mathcal{K}v_m\|}, \chi_m + \delta_m\lambda_m\right\} \geq \left\{\frac{\mu}{L}, \lambda_m\right\}, \tag{3.8}$$

where $\delta_m, \chi_m > 0$. Inductively, we have that the sequence $\{\lambda_m\}$ has a lower bound $\{\frac{\mu}{L}, \lambda_1\}$. It is easy to see from (3.4)

$$\lambda_{m+1} \leq \delta_m\lambda_m + \chi_m.$$

Thus, using our Assumptions 3.1 and Lemma 2.4, we have $\lim_{m\to\infty}\lambda_m$ exists.
(2) We note that $T_m := \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_m)I + \psi_m S_i)$ is a quasi-nonexpansive mapping. To see this, let $p \in \Gamma$, and using Lemma 2.6, we have

$$\begin{aligned}
\|T_m u - p^*\| &= \|\frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_m)I + \psi_m S_i)u - p^*\| \\
&\leq \frac{1}{N}\sum_{i=0}^{N-1}\|((1-\psi_m)I + \psi_m S_i)u - p^*\| \\
&\leq \frac{1}{N}\sum_{i=0}^{N-1}\|u - p^*\| \\
&= \|u - p^*\|.
\end{aligned} \tag{3.9}$$

Thus, T_m is quasi-nonexpansive.

4. Convergence analysis

In section, we establish weak convergence result of our proposed algorithm.

Lemma 4.1. *If $\{u_m\}$ is a sequence defined by Algorithm 3.2 and Assumption 3.1 holds such that there exists a subsequence $\{u_{2m_k}\}$ of $\{u_{2m}\}$ with $u_{2m_k} \rightharpoonup u^* \in \mathcal{H}$ and $\lim_{k\to\infty}\|w_{2m_k} - v_{2m_k}\| = 0$. Then $u^* \in \mathcal{V}$.*

Proof. Since $v_{2m_k} = P_{\mathcal{G}}(w_{2m_k} - \beta\lambda_{2m_k}\mathcal{K}w_{2m_k})$, we have

$$\langle w_{2m_k} - \beta\lambda_{2m_k}\mathcal{K}w_{2m_k} - v_{2m_k}, u - v_{2m_k}\rangle \leq 0, \ \forall\ u \in \mathcal{G} \tag{4.1}$$

and

$$\begin{aligned}\langle w_{2m_k} - v_{2m_k}, u - v_{2m_k}\rangle &\leq \beta\lambda_{2m_k}\langle \mathcal{K}w_{2m_k}, u - v_{2m_k}\rangle \\ &= \beta\lambda_{2m_k}\langle \mathcal{K}w_{2m_k}, w_{2m_k} - v_{2m_k}\rangle \\ &\quad + \beta\lambda_{2m_k}\langle \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle. \end{aligned} \tag{4.2}$$

Since $\beta, \lambda_{2m_k} > 0$, we have

$$\frac{1}{\beta\lambda_{2m_k}}\langle w_{2m_k} - v_{2m_k}, u - v_{2_k}\rangle + \langle \mathcal{K}w_{2m_k}, v_{2m_k} - w_{2m_k}\rangle \leq \langle \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle. \tag{4.3}$$

Using our hypothesis, we get

$$0 \leq \liminf_{k\to\infty}\langle \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle \leq \limsup_{k\to\infty}\langle \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle. \tag{4.4}$$

Now, observe that

$$\begin{aligned}\langle \mathcal{K}v_{2m_k}, u - v_{2m_k}\rangle &= \langle \mathcal{K}v_{2m_k}, u - w_{2m_k}\rangle + \langle \mathcal{K}v_{2m_k}, w_{2m_k} - v_{2m_k}\rangle \\ &= \langle \mathcal{K}v_{2m_k} - \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle + \langle \mathcal{K}w_{2m_k}, u - w_{2m_k}\rangle \\ &\quad + \langle \mathcal{K}v_{2m_k}, w_{2m_k} - v_{2m_k}\rangle. \end{aligned} \tag{4.5}$$

Using (3.4), $\lim_{k\to\infty} p_{2m_k} = 1, \lim_{k\to\infty}\lambda_{2m_k+1} > 0$ and our hypothesis $\lim\limits_{k\to\infty}\|w_{2m_k} - v_{2m_k}\| = 0$, we have

$$\begin{aligned}\lim_{k\to\infty}\|\mathcal{K}w_{2m_k} - \mathcal{K}v_{2m_k}\| &= \lim_{k\to\infty}\frac{\mu p_{2m_k}}{\lambda_{2m+1}}\|w_{2m_k} - v_{2m_k}\| \\ &= 0. \end{aligned} \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we have

$$\begin{aligned} 0 &\leq \liminf_{k\to\infty}\langle \mathcal{K}v_{m_k}, u - v_{m_k}\rangle \\ &\leq \limsup_{k\to\infty}\langle \mathcal{K}v_{m_k}, u - v_{m_k}\rangle. \end{aligned} \tag{4.7}$$

Now, we show that $u^* \in \mathcal{V}$. For this, the case $\limsup_{k\to\infty}\langle \mathcal{K}v_{2m_k}, u - v_{2m_k}\rangle > 0$ will be considered for all $u \in \mathcal{G}$. Then, a subsequence $\{v_{2m_{k_j}}\}$ of $\{v_{2m_k}\}$ exists such that $\limsup_{j\to\infty}\langle \mathcal{K}v_{2m_{k_j}}, u - v_{2m_{k_j}}\rangle > 0$ for all $u \in \mathcal{G}$. This implies that one can find N_0 such that

$$\langle \mathcal{K}v_{2m_{k_j}}, u - v_{2m_{k_j}}\rangle > 0, \ \forall j > N_0. \tag{4.8}$$

Since $\mathcal{K}$ is quasi-monotone, it follows that

$$\langle \mathcal{K}u, u - v_{2m_{k_j}}\rangle > 0, \ \forall j > N_0. \tag{4.9}$$

Now, observe that

$$\|w_{2m_{k_j}} - u_{2m_{k_j}}\| = \|u_{2m_{k_j}} - u_{2m_{k_j}}\| \to 0 \text{ as } j \to \infty. \tag{4.10}$$

Since $\{u_{2m_k}\}$ is a subsequence of $\{u_{2m}\}$ which converges weakly to $u^* \in \mathcal{H}$, by our hypothesis we have $\lim_{j\to\infty} \|w_{2m_{k_j}} - v_{2m_{k_j}}\| = 0$, we also obtain

$$\|v_{2m_j} - u_{2m_j}\| \leq \|v_{2m_j} - w_{2m_j}\| + \|w_{2m_j} - u_{2m_j}\| \to 0$$

as $j \to \infty$, it implies that $v_{2m_{k_j}} \rightharpoonup u^*$.

Now, passing the limit as $j \to \infty$ in (4.9), we have

$$\lim_{j\to\infty} \langle \mathcal{K}u, u - v_{2m_{k_j}} \rangle = \langle \mathcal{K}u, u - u^* \rangle \geq 0. \tag{4.11}$$

Hence, $u^* \in \mathcal{V}$.

Secondly, we consider the case in which $\limsup_{k\to\infty} \langle \mathcal{K}v_{2m_k}, u - v_{2m_k} \rangle = 0$ for each $u \in \mathcal{G}$. Now we define a non-increasing function δ_k by

$$\delta_k = |\langle \mathcal{K}v_{2m_k}, u - v_{2m_k} \rangle| + \frac{1}{k+1}. \tag{4.12}$$

It follows that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \langle \mathcal{K}v_{2m_k}, u - v_{2m_k} \rangle + \lim_{k\to\infty} \frac{1}{k+1} = 0. \tag{4.13}$$

By our hypothesis and (4.12), we have

$$\langle \mathcal{K}v_{2m_k}, u - v_{2m_k} \rangle + \delta_k > 0 \tag{4.14}$$

for all $k \geq 1$. Since $\{v_{2m_k}\} \subset \mathcal{G}$, it implies that $\{\mathcal{K}v_{2m_k}\}$ is strictly non-zero and $\liminf_{k\to\infty} \|\mathcal{K}v_{2m_k}\| = N_0 > 0$. Therefore, we infer that

$$\|\mathcal{K}v_{2m_k}\| > \frac{N_0}{2}. \tag{4.15}$$

Furthermore, let the sequence $\{\epsilon_{2m_k}\}$ be defined by $\epsilon_{2m_k} = \frac{\mathcal{K}v_{2m_k}}{\|\mathcal{K}v_{2m_k}\|^2}$. Then, it is easy to see that

$$\langle \mathcal{K}v_{2m_k}, \epsilon_{2m_k} \rangle = 1. \tag{4.16}$$

Combining (4.14) and (4.16), we have

$$\langle \mathcal{K}v_{2m_k}, u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle > 0. \tag{4.17}$$

Since $\mathcal{K}$ is quasi-monotone operator defined on $\mathcal{H}$, we know

$$\langle \mathcal{K}(u + \delta_k \epsilon_{2m_k}), u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle \geq 0. \tag{4.18}$$

Now, observe that

$$\begin{aligned} \langle \mathcal{K}u, u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle &= \langle \mathcal{K}u - \mathcal{K}(u + \delta_k \epsilon_{m_k}) \\ &\quad + \mathcal{K}(u + \delta_k \epsilon_{2m_k}), u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle \\ &= \langle \mathcal{K}u - \mathcal{K}(u + \delta_k \epsilon_{2m_k}), u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle \\ &\quad + \langle \mathcal{K}(u + \delta_k \epsilon_{2m_k}), u + \delta_k \epsilon_{2m_k} - v_{2m_k} \rangle. \end{aligned} \tag{4.19}$$

Using (4.18), (4.19) and Cauchy Schwartz inequality, we obtain

$$\begin{aligned}\langle \mathcal{K}u, u + \delta_k \epsilon_{2m_k} - v_{2m_k}\rangle &\geq \langle \mathcal{K}u - \mathcal{K}(u + \delta_k \epsilon_{2m_k}), u + \delta_k \epsilon_{2m_k} - v_{2m}\rangle \\ &\geq -\|\mathcal{K}u - \mathcal{K}(u + \delta_k \epsilon_{2m_k})\| \|u + \delta_k \epsilon_{2m_k} - v_{2m_k}\|. \end{aligned} \tag{4.20}$$

By the Lipschitz continuity of $\mathcal{K}$, we have

$$\langle \mathcal{K}u, u + \delta_k \epsilon_{2m_k} - v_{2m_k}\rangle + L\|\delta_k \epsilon_{2m_k}\| \|u + \delta_k \epsilon_{2m_k} - v_{2m_k}\| \geq 0. \tag{4.21}$$

Using (4.15), (4.21) and the definition of $\{\epsilon_{2m_k}\}$, we obtain

$$\langle \mathcal{K}u, u + \delta_k \epsilon_{2m_k} - v_{2m_k}\rangle + \frac{2L}{N_0}\delta_k \|u + \delta_k \epsilon_{2m_k} - v_{2m_k}\| \geq 0. \tag{4.22}$$

Since the subsequence $\{u_{2m_k}\}$ of $\{u_{2m}\}$ converges weakly to $u^* \in \mathcal{H}$, by the fact that $\lim_{k\to\infty} \|w_{2m_k} - v_{2m_k}\| = 0$, using (4.10), it follows that $\{v_{2m_k}\}$ also converges to u^*. If we take the limit as $k \to \infty$, since $\delta_k \to 0$, we obtain

$$\begin{aligned}\lim_{k\to\infty} \left[\langle \mathcal{K}u, u + \delta_k \epsilon_{2m_k} - v_{2m_k}\rangle + \frac{2L}{N_0}\delta_k \|u + \delta_k \epsilon_{2m_k} - v_{2m_k}\|\right] &= \langle \mathcal{K}u, u - u^*\rangle \\ &> 0. \end{aligned} \tag{4.23}$$

Hence, $u^* \in \mathcal{V}$. □

Lemma 4.2. *If $\{u_m\}$ is a sequence defined by Algorithm* 3.2 *and Assumption* 3.1 *holds. Then, $\{u_m\}$ converges weakly to a point in $\mathcal{V}$.*

Proof. Let $m = 2m + 1$ and $p^* \in \Gamma$. Observe that, $v_{2m+1} = P_{\mathcal{G}}(w_{2m+1} - \lambda_{2m+1}\beta\mathcal{K}w_{2m+1})$ and $p^* \in \mathcal{V}$, then by the characteristics of $P_{\mathcal{G}}$, we have that

$$\langle w_{2m+1} - v_{2m+1} - \lambda_{2m+1}\beta\mathcal{K}w_{2m+1}, v_{2m+1} - p^*\rangle \geq 0,$$

which is equivalent to

$$\begin{aligned}&2\langle w_{2m+1} - v_{2m+1}, v_{2m+1} - p^*\rangle - 2\lambda_{2m+1}\beta\langle \mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle \\ &- 2\lambda_{2m+1}\beta\langle \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle \geq 0. \end{aligned} \tag{4.24}$$

Since

$$\begin{aligned}2\langle w_{2m+1} - v_{2m+1}, v_{2m+1} - p^*\rangle = \|w_{2m+1} - p^*\|^2 - \|w_{2m+1} - v_{2m+1}\|^2 \\ - \|v_{2m+1} - p^*\|^2, \end{aligned}$$

(4.24) becomes

$$\begin{aligned}&\|w_{2m+1} - p^*\|^2 - \|w_{2m+1} - v_{2m+1}\|^2 - \|v_{2m+1} - p^*\|^2 \\ &\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle \\ &\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle \geq 0. \end{aligned} \tag{4.25}$$

Since $p^* \in \Gamma$ and $v_{2m+1} \in \mathcal{G}$, we have

$$\langle \mathcal{K}p^*, v_{2m+1} - p^* \rangle \geq 0$$

and by Lemma 2.3, we have

$$\langle \mathcal{K}v_{2m+1}, v_{2m+1} - p^* \rangle \geq 0.$$

Thus, (4.25) becomes

$$\begin{aligned}\|v_{2m+1} - p^*\|^2 &\leq \|w_{2m+1} - p^*\|^2 - \|w_{2m+1} - v_{2m+1}\|^2 \\ &\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}, v_{2m+1} - p\rangle \\ &\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle \\ &\leq \|w_{2m+1} - p^*\|^2 - \|w_{2m+1} - v_{2m+1}\|^2 \\ &\quad - 2\lambda_m\beta\langle \mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|v_{2m+1} - p^*\|^2 &\leq \|w_{2m+1} - p^*\|^2 - \|w_{2m+1} - v_{2m+1}\|^2 \\ &\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}, v_{2m+1} - p^*\rangle. \qquad (4.26)\end{aligned}$$

For easy computation, we suppose that

$$z_m = (1-\phi)w_m + \phi v_m + \phi\beta\lambda_m(\mathcal{K}w_m - \mathcal{K}v_m)$$

as such, we have

$$t_m = (1-\beta_m)z_m + \beta_m T_m z_m.$$

Using (4.26) and Algorithm 3.2, we have

$$\begin{aligned}\|z_{2m+1} - p\|^2 &= \|(1-\phi)w_{2m+1} + \phi v_{2m+1} \\ &\qquad + \phi\beta\lambda_{2m+1}(\mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1}) - p^*\|^2 \\ &= \|(1-\phi)(w_{2m+1} - p^*) + \phi(v_{2m+1} - p^*) \\ &\qquad + \phi\beta\lambda_{2m+1}(\mathcal{K}w_{2m+1} - \mathcal{K}v_{2m+1})\|^2\end{aligned}$$

$$
\begin{aligned}
&= (1-\phi)^2\|w_{2m+1}-p^*\|^2 + \phi^2\|v_{2m+1}-p^*\|^2 \\
&\quad + \phi^2\beta^2\lambda_{2m+1}^2\|\mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\|^2 \\
&\quad + 2\phi(1-\phi)\langle w_{2m+1}-p^*, v_{2m+1}-p^*\rangle \\
&\quad + 2\beta\lambda_{2m+1}\phi(1-\phi)\langle w_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\quad + 2\lambda_{2m+1}\beta\phi^2\langle v_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&= (1-\phi)^2\|w_{2m+1}-p^*\|^2 + \phi^2\|v_{2m+1}-p^*\|^2 \\
&\quad + \phi^2\beta^2\lambda_{2n+1}^2\|\mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\|^2 \\
&\quad + \phi(1-\phi)[\|w_{2m+1}-p^*\|^2 + \|v_{2m+1}-p^*\|^2 - \|w_{2m+1}-v_{2m+1}\|^2] \\
&\quad + 2\lambda_{2m+1}\beta\phi(1-\phi)\langle w_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\quad + 2\lambda_{2m+1}\beta\phi^2\langle v_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&= (1-\phi)\|w_{2m+1}-p^*\|^2 + \phi\|v_{2m+1}-p^*\|^2 \\
&\quad - \phi(1-\phi)\|w_{2m+1}-v_{2m+1}\|^2 + \phi^2\beta^2\lambda_{2m+1}^2\|\mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\|^2 \\
&\quad + 2\lambda_{2m+1}\beta\phi(1-\phi)\langle w_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\quad + 2\lambda_{2m+1}\beta\phi^2\langle v_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\leq (1-\phi)\|w_{2m+1}-p^*\|^2 + \phi[\|w_{2m+1}-p^*\|^2 - \|w_{2m+1}-v_{2m+1}\|^2 \\
&\quad - 2\lambda_{2m+1}\beta\langle \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}, v_{2m+1}-p^*\rangle] \\
&\quad - \phi(1-\phi)\|w_{2m+1}-v_{2m+1}\|^2 + \phi^2\beta^2\lambda_{2m+1}^2\|\mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\|^2 \\
&\quad + 2\lambda_{2m+1}\beta\phi(1-\phi)\langle w_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\quad + 2\lambda_{2m+1}\beta\phi^2\langle v_{2m+1}-p^*, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&= \|w_{2m+1}-p^*\|^2 - \phi(2-\phi)\|w_{2m+1}-v_{2m+1}\|^2 \\
&\quad + \phi^2\beta^2\lambda_{2m+1}^2\|\mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\|^2 \\
&\quad + 2\lambda_{2m+1}\beta\phi(1-\phi)\langle w_{2m+1}-v_{2m+1}, \mathcal{K}w_{2m+1}-\mathcal{K}v_{2m+1}\rangle \\
&\leq \|w_{2m+1}-p^*\|^2 - \phi(2-\phi)\|w_{2m+1}-v_{2m+1}\|^2 \\
&\quad + \frac{\phi^2\mu\beta^2\lambda_{2m+1}^2 v_{2m+1}}{\lambda_{2m+2}^2}\|w_{2m+1}-v_{2m+1}\|^2 \\
&\quad + 2\frac{\phi\mu\beta\lambda_{2m+1}v_{2m+1}}{\lambda_{2m+2}}(1-\phi)\|w_{2m+1}-v_{2m+1}\|^2 \\
&= \|w_{2m+1}-p\|^2 - \phi[2-\phi-\frac{\phi\mu^2 v_{2m+1}\beta^2\lambda_{2m+1}^2}{\lambda_{2m+2}^2} \\
&\quad - 2\frac{(1-\phi)\mu\beta\lambda_{2m+1}v_{2m+1}}{\lambda_{2m+2}}]\|w_{2m+1}-v_{2m+1}\|^2. \qquad (4.27)
\end{aligned}
$$

Using the fact that $\lim_{m\to\infty}\lambda_{2m+1}=\lim_{m\to\infty}\lambda_{2m+2}$ and $\lim_{m\to\infty}p_{2m+1}=1$, we have that

$$\phi\Big[2-\phi-\frac{\mu^2p_m\beta^2\lambda_{2m+1}^2}{\lambda_{2m+2}^2}-2\frac{(1-\phi)\mu\beta\lambda_{2m+1}p_{2m+1}}{\lambda_{2m+2}}\Big]$$
$$\to\phi[2-\phi-\mu^2\beta^2-2(1-\phi)\mu\beta]>0$$

as $m\to\infty$. Thus, this implies that

$$\|z_{2m+1}-p^*\|\leq\|w_{2m+1}-p^*\|. \tag{4.28}$$

In addition, we have

$$\begin{aligned}\|t_{2m+1}-p^*\|^2&=\|(1-\beta_{2m+1})z_{2m+1}+\beta_{2m+1}T_{2m+1}z_{2m+1}-p^*\|^2\\&=\|(1-\beta_{2m+1})(z_{2m+1}-p^*)+\beta_{2m+1}(T_{2m+1}z_{2m+1}-p^*)\|^2\\&\leq(1-\beta_{2m+1})\|z_{2m+1}-p^*\|^2+\beta_{2m+1}\|T_{2m+1}z_{2m+1}-p^*\|^2\\&\quad-(1-\beta_{2n+1})\beta_{2m+1}\|z_{2m+1}-T_{2m+1}z_{2m+1}\|^2\\&\leq(1-\beta_{2m+1})\|z_{2m+1}-p^*\|^2+\beta_{2m+1}\|z_{2m+1}-p^*\|^2\\&=\|z_{2m+1}-p^*\|^2.\end{aligned} \tag{4.29}$$

Finally, we obtain

$$\begin{aligned}\|u_{2m+2}-p^*\|^2&=\|(1-\alpha_{2m+1})w_{2m+1}+\alpha_{2m+1}t_{2m+1}-p^*\|^2\\&=\|(1-\alpha_{2m+1})(w_{2m+1}-p^*)+\alpha_{2m+1}(t_{2m+1}-p^*)\|^2\\&=(1-\alpha_{2m+1})\|w_{2m+1}-p^*\|^2+\alpha_{2m+1}\|t_{2m+1}-p^*)\|^2\\&\quad-\alpha_{2m+1}(1-\alpha_{2m+1})\|w_{2m+1}-t_{2m+1}\|^2\\&\leq(1-\alpha_{2m+1})\|w_{2m+1}-p^*\|^2+\alpha_{2m+1}\|z_{2m+1}-p^*)\|^2\\&\quad-\alpha_{2m+1}(1-\alpha_{2m+1})\|w_{2m+1}-t_{2m+1}\|^2\\&\leq(1-\alpha_{2m+1})\|w_{2m+1}-p^*\|^2+\alpha_{2m+1}\|w_{2m+1}-p^*)\|^2\\&\quad-\alpha_{2m+1}(1-\alpha_{2m+1})\|w_{2m+1}-t_{2m+1}\|^2\\&\leq\|w_{2m+1}-p^*\|^2-\alpha_{2m+1}(1-\alpha_{2m+1})\|w_{2m+1}-t_{2m+1}\|^2.\end{aligned} \tag{4.30}$$

It is not hard to see that

$$\begin{aligned}\|u_{2m+1}-p^*\|^2 &\le (1-\alpha_{2m})\|w_{2m}-p^*\|^2+\alpha_{2m}\|z_{2m}-p^*)\|^2\\ &\quad -\alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2\\ &\le (1-\alpha_{2m})\|w_{2m}-p^*\|^2+\alpha_{2m}\|w_{2m}-p^*)\|^2\\ &\quad -\alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2\\ &= \|w_{2m}-p^*\|^2-\alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2\\ &= \|u_{2m}-p^*\|^2-\alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2. \end{aligned} \tag{4.31}$$

Now, from (4.31), we have

$$\begin{aligned}\|w_{2m+1}-p^*\|^2 &= \|u_{2m+1}+\theta_{2m+1}(u_{2m+1}-u_{2m})-p^*\|^2\\ &= \|(1+\theta_{2m+1})(u_{2m+1}-p^*)-\theta_{2m+1}(u_{2m}-p^*)\|^2\\ &= (1+\theta_{2m+1})\|u_{2m+1}-p^*\|^2-\theta_{2m+1}\|u_{2m}-p^*\|^2\\ &\quad +\theta_{2m+1}(1+\theta_{2m+1})\|u_{2m+1}-u_{2m}\|^2\\ &\le (1+\theta_{2m+1})[\|u_{2m}-p^*\|^2-\alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2]\\ &\quad -\theta_{2m+1}\|u_{2m}-p^*\|^2+\theta_{2m+1}(1+\theta_{2m+1})\|u_{2m+1}-u_{2m}\|^2\\ &= \|u_{2m}-p^*\|^2-(1+\theta_{2m+1})\alpha_{2m}(1-\alpha_{2m})\|u_{2m}-t_{2m}\|^2\\ &\quad +\theta_{2m+1}(1+\theta_{2m+1})\|u_{2m+1}-u_{2m}\|^2. \end{aligned} \tag{4.32}$$

Also, observe that

$$\begin{aligned}\|u_{2m+1}-u_{2m}\|^2 &= \|(1-\alpha_{2m})w_{2m}+\alpha_{2m}t_{2m}-u_{2m}\|^2\\ &= \|(1-\alpha_{2m})u_{2m}+\alpha_{2m}t_{2m}-u_{2m}\|^2\\ &= \|\alpha_{2m}(t_{2m}-u_{2m})\|^2\\ &= \alpha_{2m}^2\|t_{2m}-u_{2m}\|^2. \end{aligned} \tag{4.33}$$

Putting (4.33) into (4.32), we have

$$\begin{aligned}\|w_{2m+1}-p^*\|^2 &\le \|u_{2m}-p^*\|^2-(1+\theta_{2m+1})\alpha_{2m}(1-\alpha_{2m})\|u_{2m}-t_{2m}\|^2\\ &\quad +\theta_{2m+1}(1+\theta_{2m+1})\alpha_{2m}^2\|t_{2m}-u_{2m}\|^2\\ &= \|u_{2m}-p^*\|^2-\alpha_{2m}(1+\theta_{2m+1})\\ &\quad \times[1-\alpha_{2m}+\theta_{2m+1}\alpha_{2m}]\|t_{2m}-u_{2m}\|^2\\ &\le \|u_{2m}-p^*\|^2. \end{aligned} \tag{4.34}$$

Combining (4.30) and (4.34), we have

$$\begin{aligned}\|u_{2m+2}-p^*\|^2 &\leq \|u_{2m}-p^*\|^2-\alpha_{2m}(1+\theta_{2m+1})\\ &\quad\times[1-\alpha_{2m}+\theta_{2m+1}\alpha_{2m}]\|t_{2m}-u_{2m}\|^2\\ &\quad-\alpha_{2m+1}(1-\alpha_{2m+1})\|w_{2m+1}-t_{2m+1}\|^2\\ &\leq \|u_{2m}-p^*\|^2-\alpha_{2m}(1+\theta_{2m+1})\\ &\quad\times[1-\alpha_{2m}+\theta_{2n+1}\alpha_{2m}]\|t_{2m}-u_{2m}\|^2\\ &\leq \|u_{2m}-p^*\|^2. \end{aligned} \tag{4.35}$$

It is easy to see from (4.35) that

$$\begin{aligned}&\alpha_{2m}(1+\theta_{2m+1})[1-\alpha_{2m}+\theta_{2n+1}\alpha_{2m}]\|t_{2m}-u_{2m}\|^2\\ &\quad\leq \|u_{2m}-p^*\|^2-\|u_{2m+2}-p^*\|^2. \end{aligned} \tag{4.36}$$

Thus, we have

$$\|u_{2m+2}-p^*\|\leq\|u_{2m}-p^*\|. \tag{4.37}$$

This means that the sequences $\{\|u_{2m}-p^*\|\}$ and $\{u_{2m}\}$ are bounded and also, $\lim_{m\to\infty}\|u_{2m}-p^*\|$ exists. More so, we obtain that $\lim_{m\to\infty}\|u_{2m}-t_{2m}\|=0$ from (4.36). Therefore, $\{t_{2m}\}$ is bounded, thus, $\lim_{m\to\infty}\|w_{2m}-t_{2m}\|=\lim_{m\to\infty}\|u_{2m}-t_{2m}\|=0$. In addition, we have that $\lim_{m\to\infty}\|u_{2m+1}-u_{2m}\|=0$ using (4.33). Also, from (4.29) and (4.27), we have

$$\begin{aligned}\|t_{2m}-p^*\|^2 &\leq \|z_{2m}-p^*\|^2\\ &\leq \|w_{2m}-p^*\|^2-\phi[2-\phi-\frac{\mu^2p_{2m}\beta^2\lambda_{2m}^2}{\lambda_{2m+2}^2}\\ &\quad-2\frac{(1-\phi)\mu\beta\lambda_{2m}p_{2m}}{\lambda_{2m+2}}]\|w_{2m}-v_{2n}\|^2\\ &\leq \|u_{2m}-p^*\|^2-\phi[2-\phi-\frac{\mu^2p_{2m}\beta^2\lambda_{2m}^2}{\lambda_{2m+2}^2}\\ &\quad-2\frac{(1-\phi)\mu\beta\lambda_{2m}p_{2m}}{\lambda_{2m+2}}]\|w_{2m}-v_{2m}\|^2, \end{aligned} \tag{4.38}$$

which follows that

$$\begin{aligned}&\phi[2-\phi-\frac{\mu^2p_{2m}\beta^2\lambda_{2m}^2}{\lambda_{2m+2}^2}-2\frac{(1-\phi)\mu\beta\lambda_{2m}p_{2m}}{\lambda_{2m+2}}]\|w_{2m}-v_{2m}\|^2\\ &\quad\leq \|u_{2m}-p^*\|^2-\|t_{2m}-p^*\|^2\\ &\quad\leq (\|u_{2m}-p^*\|+\|t_{2m}-p^*\|)\|u_{2m}-t_{2m}\|, \end{aligned} \tag{4.39}$$

we obtain that

$$\lim_{m\to\infty}\|w_{2m}-v_{2m}\| = \lim_{m\to\infty}\|u_{2m}-v_{2m}\| = 0, \tag{4.40}$$

$$\begin{aligned}\|z_{2m}-w_{2m}\| &\le \phi\|v_{2m}-w_{2m}\| + \phi\beta\lambda_{2m}\|\mathcal{K}v_{2m}-\mathcal{K}w_{2m}\| \\ &\le \phi\|v_{2m}-w_{2m}\| + \frac{\beta\phi\mu p_{2m}\lambda_{2m}}{\lambda_{2m+2}}\|v_{2m}-w_{2m}\| \\ &\to\ 0 \text{ as } m\to\infty. \end{aligned}\tag{4.41}$$

Furthermore, we have

$$\|v_{2m}-z_{2m}\| \le \|v_{2m}-w_{2m}\| + \|w_{2m}-z_{2m}\| \to\ 0 \text{ as } m\to\infty \tag{4.42}$$

and

$$\begin{aligned}\|u_{2m+1}-p^*\|^2 &= (1-\alpha_{2m+1})\|w_{2m}-p^*\|^2 + \alpha_{2m}\|t_{2m}-p^*)\|^2 \\ &\quad - \alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2 \\ &\le (1-\alpha_{2m+1})\|w_{2m}-p^*\|^2 + \alpha_{2m}[\|z_{2m}-p^*\|^2 \\ &\quad - (1-\beta_{2m})\beta_{2m}\|z_{2m}-T_{2m}z_{2m}\|^2] \\ &\quad - \alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2 \\ &\le (1-\alpha_{2m+1})\|w_{2m}-p^*\|^2 + \alpha_{2m}[\|w_{2m}-p^*\|^2 \\ &\quad - (1-\beta_{2m})\beta_{2n}\|z_{2m}-T_{2m}z_{2m}\|^2] \\ &\quad - \alpha_{2m}(1-\alpha_{2m})\|w_{2m}-t_{2m}\|^2 \\ &= \|u_{2m}-p^*\|^2 - \alpha_{2m}(1-\beta_{2m})\beta_{2m}\|z_{2m}-T_{2m}z_{2m}\|^2 \\ &\quad - \alpha_{2m}(1-\alpha_{2m})\|u_{2m}-t_{2m}\|^2 \\ &\le \|u_{2m}-p^*\|^2 - \alpha_{2m}(1-\beta_{2m})\beta_{2m}\|z_{2m}-T_{2m}z_{2m}\|^2, \end{aligned}\tag{4.43}$$

which implies

$$\begin{aligned}&\alpha_{2m}(1-\beta_{2m})\beta_{2m}\|z_{2m}-T_{2m}z_{2m}\|^2 \\ &\quad\le \|x_{2n}-p\|^2 - \|u_{2m+1}-p\|^2 \\ &\quad\le (\|u_{2m}-p^*\| + \|u_{2m+1}-p^*\|)\|u_{2m}-u_{2m+1}\|. \end{aligned}\tag{4.44}$$

Therefore, we obtain

$$\lim_{m\to\infty}\|z_{2m}-T_{2m}z_{2m}\| = 0, \tag{4.45}$$

$$\lim_{m\to\infty}\|w_{2m}-u_{2m}\| = \lim_{m\to\infty}\|u_{2m}-u_{2m}\| = 0, \tag{4.46}$$

$$\|z_{2m_k}-u_{2m_k}\| \le \|z_{2m_k}-w_{2m_k}\| + \|w_{2m_k}-u_{2m_k}\|\ \to 0 \text{ as } k\to\infty, \tag{4.47}$$

$$\|t_{2m_k}-z_{2m_k}\| \le \|t_{2m_k}-u_{2m_k}\| + \|u_{2m_k}-z_{2m_k}\|\ \to 0 \text{ as } k\to\infty, \tag{4.48}$$

and

$$\|t_{2m_k} - T_m z_{2m_k}\| \le \|t_{2n_k} - z_{2n_k}\| + \|z_{2m_k} - T_m z_{2m_k}\| \ \to 0 \text{ as } k \to \infty. \quad (4.49)$$

By the boundedness of $\{u_{2m_k}\}$, we know that a subsequence $\{u_{2m_{k_j}}\}$ of $\{u_{2m_k}\}$ exists such that $\{u_{2m_{k_j}}\} \rightharpoonup u^* \in \mathcal{H}$. Furthermore, by (4.47) and the boundedness of $\{z_{2m_k}\}$, a subsequence $\{z_{2m_{k_j}}\}$ of $\{z_{2m_k}\}$ exists with $\{z_{2m_{k_j}}\} \rightharpoonup u^* \in \mathcal{H}$. Due to the demiclosedness of S_i with respect to (4.45), it follows that $u^* \in F(S_i)$. Thus, by (4.40) and Lemma 4.1, we have $u^* \in \Gamma$.

Next, we need to establish that $\{u_{2m+1}\}$ also converges weakly to u^*. It is not hard to see that $\lim_{n\to\infty}\langle u_{2m} - u^*, v\rangle = 0$ for all $v \in \mathcal{H}$. From $\lim_{m\to\infty}\|u_{2m+1} - u_{2m}\| = 0$, we obtain

$$\begin{aligned} \lim_{m\to\infty} |\langle u_{2m+1} - u^*, v\rangle| &\le \lim_{m\to\infty} |\langle u_{2m} - u^*, v\rangle| + \lim_{n\to\infty} |\langle u_{2m+1} - u_{2m}, v\rangle| \\ &\le \lim_{m\to\infty} |\langle u_{2m} - u^*, v\rangle| + \lim_{m\to\infty} \|u_{2m+1} - u_{2m}\|\|v\| \\ &= 0. \end{aligned} \quad (4.50)$$

Thus, we obtain that $\{u_{2m+1}\}$ also converges weakly to u^*.

Lastly, we establish the uniqueness of u^*. Now, suppose that u_{2m} converges weakly to $u^*, v^* \in \Gamma$ such that $u^* \ne v^*$. Now, observe that

$$\begin{aligned} \|u^* - v^*\|^2 = \langle u^* - v^*, u^* - v^*\rangle &= \langle u^*, u^* - v^*\rangle - \langle v^*, u^* - v^*\rangle \\ &= \lim_{m\to\infty} \langle u_{2m}, u^* - v^*\rangle - \lim_{m\to\infty} \langle u_{2m}, u^* - v^*\rangle \\ &= \lim_{m\to\infty} \langle u_{2m} - u_{2m}, u^* - v^*\rangle \\ &= 0. \end{aligned} \quad (4.51)$$

Thus, $u^* = v^*$. This completes the proof. □

5. Linear convergence

If the operator $\mathcal{K}$ is δ-strongly pseudomonotone, then in Algorithm 3.2, we do not need Step 2 and Step 3 to obtain convergence. We propose the following method for the case when $\mathcal{K}$ is δ-strongly pseudomonotone.

Algorithm 5.1. Initialization Step: *Choose* $u_0, u_1 \in \mathcal{H}$,

$$k = \frac{1}{\sqrt{1 + \lambda_m \beta \left(\frac{\lambda_m \mu^2 p_m^2}{\lambda_{m+1}^2} \right)}}$$

and $0 \le \theta_m \le \frac{1-k}{1+k}$ *given the iterates* u_{m-1} *and* u_m *for all* $m \in \mathbb{N}$,

$$\theta_m = \begin{cases} \min\left\{\theta, \frac{\epsilon_m}{\|u_n - u_{m-1}\|}\right\}, & \text{if } u_m \neq u_{m-1} \\ \theta, & \text{otherwise}, \end{cases} \tag{5.1}$$

with $\{\epsilon_m\}$ *is a positive sequence such that* $\epsilon_m = \circ(\alpha_m)$.

Step 1: *Compute*

$$w_m = \begin{cases} u_m, & \text{if } m = \text{even} \\ u_m + \theta_m(u_m - u_{m-1}), & \text{if } m = \text{odd}, \end{cases} \tag{5.2}$$

and

$$u_{m+1} = P_{\mathcal{G}}(w_m - \beta\lambda_m \mathcal{K} w_m), \tag{5.3}$$

where

$$\lambda_{m+1} = \begin{cases} \min\left\{\frac{\mu p_m \|w_m - u_{m+1}\|}{\|\mathcal{K}w_m - \mathcal{K}u_{m+1}\|}, \chi_m + \delta_m\lambda_m\right\}, & \text{if } \mathcal{K}w_m \neq \mathcal{K}u_{m+1} \\ \chi_m + \delta_m\lambda_m, & \text{otherwise}. \end{cases} \tag{5.4}$$

Theorem 5.2. *Suppose* $\{u_m\}$ *is generated by Algorithm* 5.1*. If* $\mathcal{K}$ *is* δ*-strongly pseudomonotone on* H*, then* $\{u_m\}$ *converges at least R-linearly to the unique solution* $p^* \in \mathcal{V}$.

Proof. From (5.3), we have $u_{m+1} = P_{\mathcal{G}}(w_m - \beta\lambda_m \mathcal{K} w_m)$ for all $m \in \mathbb{N}$. Using the characteristics of projection, we have

$$\langle w_m - \beta\lambda_m \mathcal{K} w_m - u_{m+1}, p^* - u_{m+1}\rangle \leq 0, \ \forall \ p^* \in \mathcal{G},$$

which implies

$$2\langle w_m - u_{m+1}, p^* - u_{m+1}\rangle \leq 2\lambda_m\beta\langle \mathcal{K}w_m, p^* - u_{m+1}\rangle. \tag{5.5}$$

It is easy to see that

$$2\langle w_m - u_{m+1}, p^* - u_{m+1}\rangle = \|w_m - u_{m+1}\|^2 + \|u_{m+1} - p^*\|^2 - \|w_m - p^*\|^2.$$

Thus, (5.5) becomes

$$\|w_m - u_{m+1}\|^2 + \|u_{m+1} - p^*\|^2 - \|w_m - p^*\|^2 \leq 2\lambda_m\beta\langle \mathcal{K}w_m, p^* - u_{m+1}\rangle. \tag{5.6}$$

In addition, using the fact that $p^* \in \mathcal{V}$, we obtain

$$\langle \mathcal{K}p^*, u - p^*\rangle \geq 0$$

for all $u \in \mathcal{G}$.

Using the step size, Cauchy-Schwarz inequality, δ-strong monotononicity of $\mathcal{K}$, and the fact that $(2xy \leq x^2 + y^2)$ for all $x, y \in H$, we have

$$\begin{aligned} 2\lambda_m\beta\langle \mathcal{K}w_m, p^* - u_{n+1}\rangle &= 2\lambda_n\beta\langle \mathcal{K}w_m - \mathcal{K}u_{m+1}, p^* - u_{m+1}\rangle \\ &\quad - 2\lambda_m\beta\langle \mathcal{K}u_{m+1}, u_{m+1} - p^*\rangle \\ &\leq 2\lambda_m\beta\|\mathcal{K}w_m - \mathcal{K}u_{m+1}\|\|p^* - u_{n+1}\| \\ &\quad - 2\lambda_m\beta\tau\|u_{m+1} - p^*\|^2 \\ &= 2\frac{\lambda_m\mu p_m\beta}{\lambda_{m+1}}\|w_m - u_{m+1}\|\|u_{m+1} - p^*\| \\ &\quad - 2\lambda_m\beta\tau\|u_{m+1} - p\|^2 \\ &\leq \|w_m - u_{m+1}\|^2 + \left(\frac{\lambda_m\mu p_m\beta}{\lambda_{m+1}}\right)^2\|u_{m+1} - p^*\|^2 \\ &\quad - 2\lambda_m\beta\tau\|u_{m+1} - p^*\|^2. \end{aligned} \tag{5.7}$$

We have used the fact that, $2xy \leq x^2 + y^2$ to obtain the last line. Thus, (5.6) becomes

$$\begin{aligned} \|w_m - u_{m+1}\|^2 + \|u_{m+1} - p^*\|^2 - \|w_m - p^*\|^2 &\leq \|w_m - u_{m+1}\|^2 \\ &\quad + \left(\frac{\lambda_m\mu p_m\beta}{\lambda_{m+1}}\right)^2\|u_{m+1} - p^*\|^2 \\ &\quad - 2\lambda_m\beta\tau\|u_{m+1} - p^*\|^2, \end{aligned} \tag{5.8}$$

which implies that

$$\begin{aligned} \|u_{m+1} - p^*\|^2 &\leq \frac{1}{[1 + \lambda_m\beta\left(\frac{\lambda_m\mu^2 v_n^2}{\lambda_{m+1}^2}\right)]}\|w_m - p^*\|^2 \\ &\leq k^2\|w_m - p^*\|^2. \end{aligned} \tag{5.9}$$

Using similar approach, we have

$$\begin{aligned} \|u_{2m+1} - p\|^2 &\leq \frac{1}{[1 + \lambda_{2m}\beta\left(\frac{\lambda_{2m}\mu^2 p_{2m}^2}{\lambda_{2m+1}^2}\right)]}\|w_{2m} - p^*\|^2 \\ &\leq k^2\|w_{2m} - p^*\|^2 \\ &= k^2\|u_{2m} - p^*\|^2. \end{aligned} \tag{5.10}$$

Furthermore, using (4.34), our hypothesis ($0 \leq \theta_m \leq \frac{1-k}{1+k}$) and a similar approach as in (5.8), we have

$$\begin{aligned}
\|u_{2m+2} - p^*\|^2 &\leq k^2 \|w_{2m+1} - p^*\|^2 \\
&\leq k^2 [(1+\theta_{2m+1})\|u_{2m+1} - p^*\|^2 - \theta_{2m+1}\|u_{2m} - p^*\|^2 \\
&\quad + \theta_{2m+1}(1+\theta_{2m+1})\|u_{2m+1} - u_{2m}\|^2] \\
&\leq k^2 [k^2(1+\theta_{2m+1})\|u_{2m} - p^*\|^2 - \theta_{2m+1}\|u_{2m} - p^*\|^2 \\
&\quad + \theta_{2m+1}(1+\theta_{2m+1})\|u_{2m+1} - u_{2m}\|^2] \\
&\leq k^2 [k^2(1+\theta_{2m+1})\|u_{2m} - p^*\|^2 - \theta_{2m+1}\|u_{2m} - p^*\|^2 \\
&\quad + \theta_{2m+1}(1+\theta_{2m+1})(\|u_{2m+1} - p^*\| + \|u_{2m} - p\|)^2] \\
&\leq k^2 [k^2(1+\theta_{2m+1})\|u_{2m} - p^*\|^2 - \theta_{2m+1}\|u_{2m} - p^*\|^2 \\
&\quad + \theta_{2m+1}(1+\theta_{2m+1})(1+k)^2\|u_{2m} - p^*\|^2], \qquad (5.11)
\end{aligned}$$

which implies that

$$\|u_{2m+2} - p^*\| \leq k\|u_{2m} - p^*\|.$$

Thus, we have

$$\begin{aligned}
\|u_{2m+2} - p^*\| &\leq k\|u_{2m} - p^*\| \\
&\leq k^2\|u_{2m-2} - p^*\| \\
&\leq k^3\|u_{2m-4} - p^*\| \\
&\;\;\vdots \\
&\leq k^m\|u_2 - p^*\|. \qquad (5.12)
\end{aligned}$$

Therefore, we obtain

$$k\|u_{2m} - p^*\| \leq k^m\|u_2 - p^*\| \Rightarrow \|u_{2m} - p^*\| \leq k^{m-1}\|x_2 - p^*\|.$$

Lastly, we have

$$\|u_{2m+1} - p^*\| \leq k\|u_{2m} - p^*\| \leq \|u_2 - p^*\| \leq k^{m-1}\|u_2 - p^*\|.$$

It follows that $\{u_m\}$ converges R-linearly to p^*. □

6. Numerical example

In this section, we provide some numerical experiments to demonstrate the advantage of our suggested algorithm, in comparison with some well-known algorithms in the literature. We implement all programs in MATLAB 2021b on a PC with an Intel(R) Core(TM) i5-8250U CPU running at 1.60GHz and 8.00 GB of RAM.

Example 6.1. Let $\mathcal{H} = l_2$, that is,,

$$\mathcal{H} = \left\{ u = (u_1, u_2, u_3, \cdots, u_i, \cdots) : \sum_{i=1}^{\infty} |u_i|^2 < +\infty \right\}.$$

Let $d, e \in \mathbb{R}$ be such that $d > e > \frac{d}{2} > 0$. Let $\mathcal{G} = \{u \in l_2 : \|u\| \leq e\}$ and $\mathcal{K}u = (d - \|u\|)u$. Obviously, the solution set $\mathcal{V} = \{0\}$.

Now, we show that $\mathcal{K}$ is L–Lipschitz continuous on $\mathcal{H}$ and quasi-monotone on $\mathcal{G}$. Indeed, for any $u, v \in \mathcal{H}$, we have

$$\begin{aligned}
\|\mathcal{K}u - \mathcal{K}v\| &= \|(d - \|u\|)u - (d - \|v\|)v\| \\
&= \|d(u - v) - \|u\|(u - v) - (\|u\| - \|v\|)v\| \\
&\leq d\|u - v\| + \|u\|\|u - v\| + |\|u\| - \|v\||\|v\| \\
&\leq d\|u - v\| + e\|u - v\| + \|u - v\|e \\
&= (d + 2e)\|u - v\|.
\end{aligned}$$

Hence, $\mathcal{K}$ is Lipschitz continuous with $L = d + 2e$.

Now, let $u, v \in \mathcal{G}$ be such that $\langle \mathcal{K}u, v - u \rangle > 0$. Then, we have $(d - \|u\|)\langle u, v - u \rangle > 0$. Since $\|u\| \leq e \leq d$, we have $\langle u, v - u \rangle > 0$. Hence,

$$\begin{aligned}
\langle \mathcal{K}v, v - u \rangle &= (d - \|v\|)\langle v, v - u \rangle \\
&\geq (d - \|v\|)(\langle v, v - u \rangle - \langle u, v - u \rangle \\
&\geq (d - e)\|u - v\|^2 \\
&\geq 0.
\end{aligned}$$

This shows that $\mathcal{K}$ is a quasi-monotone mapping. If we set $e = 3$ and $d = 5$, the projection formula is defined by

$$P_{\mathcal{G}} = \begin{cases} u, & \text{if } \|u\| \leq 3, \\ \frac{3u}{\|u\|}, & \text{otherwise.} \end{cases} \tag{6.1}$$

Now, let $T_m := \frac{u}{2}$. Hence, $\Gamma = \{0\} \neq \emptyset$. In this experiment, we compare numerical convergence of Algorithm 3.2 with Algorithms 1.1 and Algorithm 1.2, respectively. For each algorithm, we set their parameters as follows:

(1) Algorithm 1.1: $\mu = 0.8$, $\gamma = 0.3$, $\lambda_1 = 1$ and $\alpha_m = \frac{m+1}{(m+5)^2}$.

(2) Algorithm 1.2: $\theta = \mu = 0.8$, $\lambda_1 = 1.2$ and $\frac{(1-\mu)^2(m+1)}{[(\mu+1)(m+5)]^2}$.

(3) Algorithm 3.2: $\theta = 0.9$, $\mu = \phi = 0.8$, $\lambda_1 = 1.2$, $\chi_m = \frac{1}{(m+1)^{1.1}}$, $\alpha_m = \frac{1}{m}$, $\beta = 0.6$, $p_m = 1 + \frac{1}{(m+1)^{1.1}}$, $\psi_m = 0.4$ and $\beta_m = 0.5$.

We use the stopping criterion $E_m = \|u_m - u_{m-1}\| < 10^{-5}$ for all algorithms. We consider the following four cases of u_0 and u_1:

Case I: $u_0 = \left(\frac{1}{10}, \frac{1}{3}, \frac{1}{6}, \cdots\right)$, $u_1 = \left(\frac{2}{5}, \frac{1}{4}, \frac{1}{2}, \cdots\right)$.

Case II: $u_0 = \left(\frac{1}{4}, \frac{7}{10}, \frac{1}{10}, \cdots\right)$, $u_1 = \left(\frac{1}{3}, \frac{1}{6}, \frac{3}{4}, \cdots\right)$.

Case III: $u_0 = \left(1, 3, \frac{1}{3}, \cdots\right)$, $u_1 = \left(-2, \frac{-3}{2}, \frac{-1}{2}, \cdots\right)$.

Case IV: $u_0 = \left(\frac{1}{2}, 1, \frac{1}{5}, \cdots\right)$, $u_1 = (-4, 3, 1, \cdots)$.

We present our numerical results in Table 6.1 and Figure 6.1.

TABLE 1. Numerical results of Example 6.1

Cases		Algorithm 3.2	Algorithm 1.1	Algorithm 1.2
Case I	CPU time (sec.)	0.0065	0.0076	0.0104
	No of Iter.	9	13	42
Case II	CPU time (sec.)	0.0087	0.0090	0.0319
	No of Iter.	12	13	30
Case III	CPU time (sec.)	0.0091	0.0201	0.0511
	No of Iter.	9	14	33
Case IV	CPU time (sec.)	0.0122	0.0378	0.0732
	No of Iter.	9	13	49

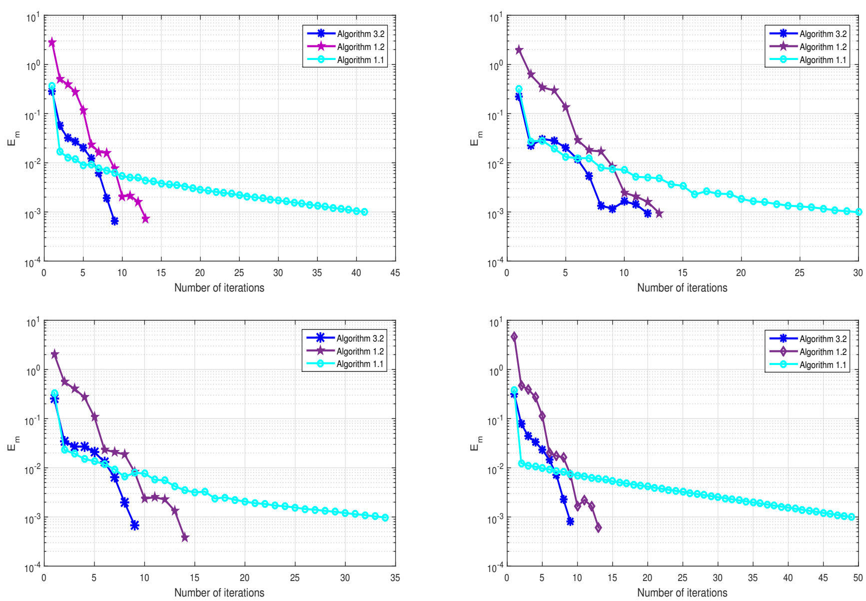

FIGURE 1. Top Left: Case I; Top Right: Case II;
Bottom Left: Case III; Bottom Right: Case IV.

Example 6.2. Let $\mathcal{H} = L^2([0,1])$ with inner product

$$\langle u, v\rangle = \int_0^1 u(z)v(z)dz,\ \forall u, v \in \mathcal{H}$$

and induced norm

$$\|u\| = \left(\int_0^1 |u(z)|^2 dt)^{\frac{1}{2}}\right),\ \forall u \in \mathcal{H}.$$

Let $\mathcal{G} = \{u \in \mathcal{G} : \|u\| \leq 1\}$ and $\mathcal{K} : \mathcal{G} \to \mathcal{H}$ be an operator defined by

$$(\mathcal{K}u)(u) = \int_0^1 (u(z) - D(z,t)d(u(t)))dt + h(z),\ \forall u \in \mathcal{G},$$

where

$$D(z,t) = \frac{2zte^{z+t}}{e\sqrt{e^2-1}},\ d(u) = \cos u,\ h(z) = \frac{2ze^z}{e\sqrt{e^2-1}}.$$

It is not hard to show that $\mathcal{K}$ is monotone (hence, quasi-monotone) and L-Lipschitz continuous, where $L = 2$ and $\mathcal{V} = \{0\}$. The projection on $\mathcal{G}$ is defined by

$$P_{\mathcal{G}} = \begin{cases} \frac{u}{\|u\|}, & \text{if } \|u\| > 1, \\ \\ u, & \text{if } \|u\| \leq 1. \end{cases}$$

Let $T_m : L^2([0,1]) \to L^2([0,1])$ be defined by

$$T_m(u) = \int_0^1 \frac{u(z)}{3} dz,\ z \in [0,1].$$

Then, $\Gamma \neq \emptyset$.

Next, we consider the stopping criterion

$$E_m = \|u_m - u_{m-1}\| < 10^{-6},$$

using the same parameters as above for all algorithms and the following initial values will also be considered:

Case I: $u_0 = z^3 - 2z^2$, $u_1 = (2z+3)^2$;

Case II: $u_0 = e^2 z$, $u_1 = \sin\left(\frac{2z}{3}\right)$;

Case III: $u_0 = \cos\left(\frac{5z}{8}\right)$, $u_1 = \sin(3z)$;

Case IV: $u_0 = z^3 + z - 1$, $u_1 = \frac{e^{(-3z)}}{4}$.

Our numerical results are presented in Table 6.2 and Figure 6.2.

Table 2. Numerical results of Example 6.2

Cases		Algorithm 3.2	Algorithm 1.1	Algorithm 1.2
Case I	CPU time (sec.)	0.0067	0.0079	0.0400
	No of Iter.	12	13	19
Case II	CPU time (sec.)	0.0090	0.0099	0.0209
	No of Iter.	11	12	19
Case III	CPU time (sec.)	0.0067	0.0102	0.0210
	No of Iter.	12	14	19
Case IV	CPU time (sec.)	0.0090	0.0010	0.0210
	No of Iter.	11	13	19

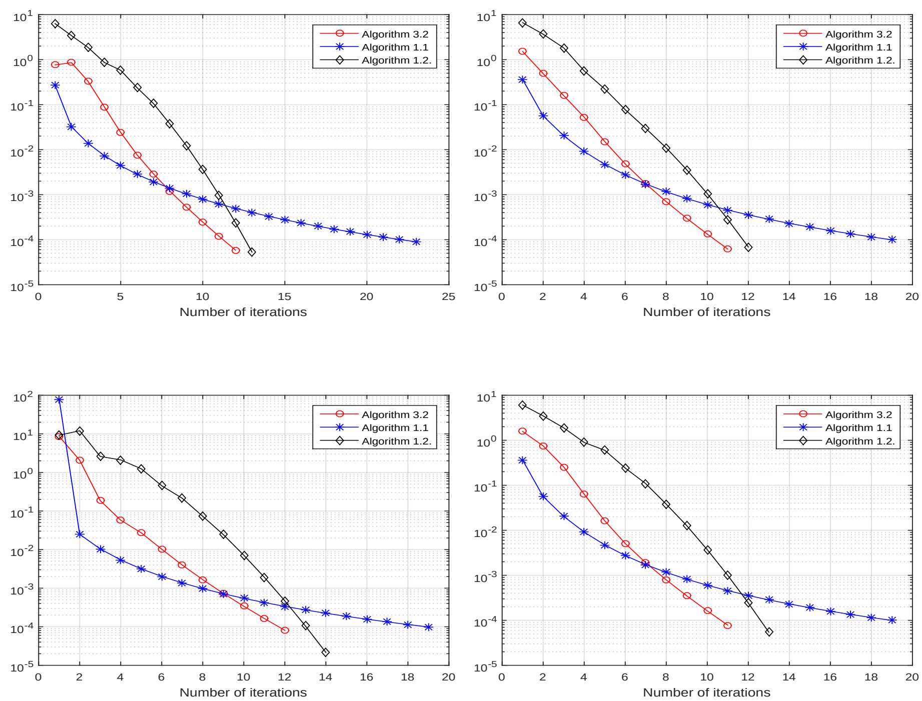

Figure 2. Top Left: Case A; Top Right: Case B;
Bottom Left: Case C; Bottom Right: Case D.

7. Application to image restoration problem

It is noticed that images are in most cases distorted by the process of acquisition. The purpose of restoration technique for distorted image is to restore the original image from noisy observation of it. The image restoration problem can be modeled as the following undetermined system of linear equation:

$$v = Fu + w, \tag{7.1}$$

where $F : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is bounded linear operator, $u \in \mathbb{R}^N$ is an original image and $v \in \mathbb{R}^M$ is the observed image with noise w. It is well known that the solution of the model (7.1) is equivalent the solution of the (LASSO) problem as follows [30]:

$$\min_{u \in \mathbb{R}^N} \left\{ k\|u\|_1 + \frac{1}{2}\|v - Fu\|_2^2 \right\}, \tag{7.2}$$

where $k > 0$.

It is worthy to know that according [29], one can reconstruct the LASSO problem (7.2) as a variational inequality problem by letting

$$\mathcal{K}u = F^T(Fu - v).$$

It is known that $\mathcal{K}$ is monotone hence quasi-monotone and Lipschitz continuous with $L = \|F^T F\|$.

Now, we compare the restoration efficiency of our suggested Algorithm 3.2 with Algorithm 1.1 and Algorithm 1.2. We use *Peacock* and *Lenna* for test images with sizes 256×256 in the Image Processing Toolbox in the MATLAB. The image went through a Gaussian blur of size 8×8 and standard deviation of $\sigma = 4$.

The performances of the algorithms are measured via signal-to-noise ratio (SNR) defined by

$$SNR = 20 \log_{10} \left(\frac{\|u\|_2}{\|u - u^*\|_2} \right), \tag{7.3}$$

where u^* is the restored image and u is the original image. In this experiment, we maintain the same parameters used for all the algorithms in Example 6.1 with stopping criterion

$$E_m = \|u_{m+1} - u_m\| \leq 10^{-5}.$$

The numerical results are shown in Figures 3–6 and Tables 3–4.

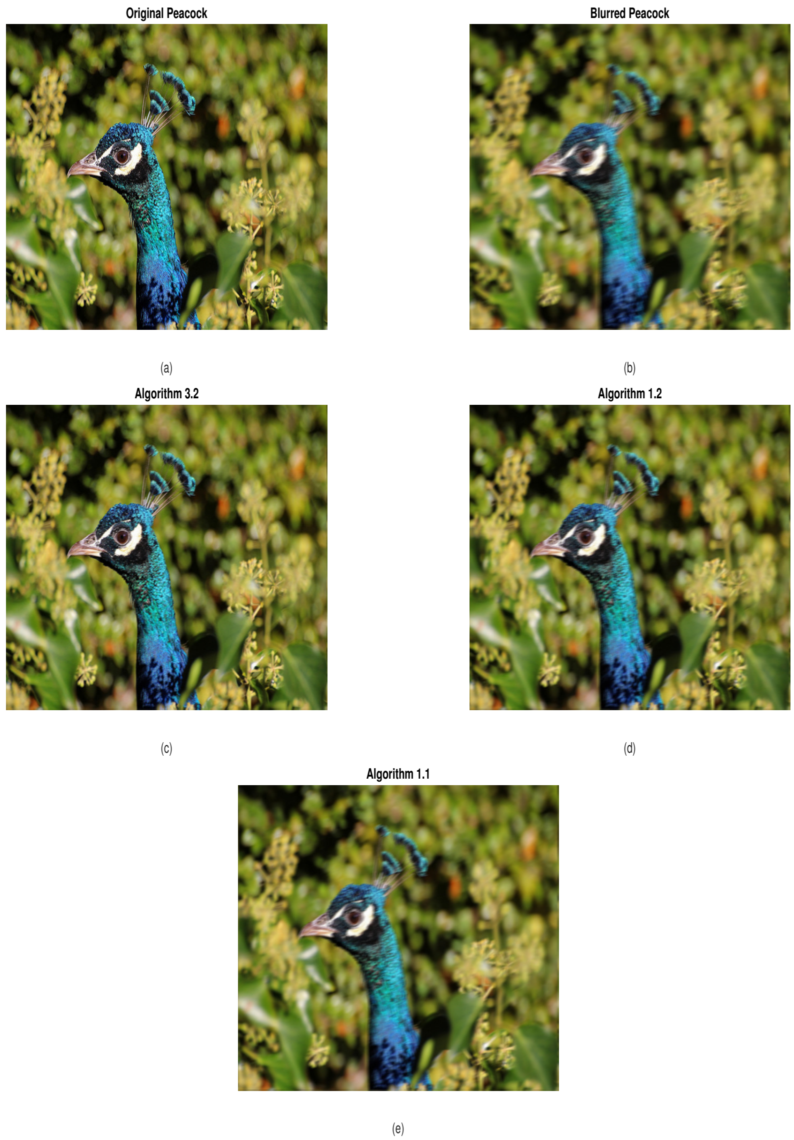

Figure 3. Blurred Peacock and its restorations via various method. (a) Original Peacock; (b) Peacock blurred by motion blur and random noise; (c) Peacock deblurred by Algorithm 3.2; (d) Peacock deblurred by Algorithm 1.2 and (e) Peacock deblurred by Algorithm 1.1.

Figure 4. Blurred Lenna and its restorations via various method. (a) Original Lenna; (b) Lenna blurred by motion blur and random noise; (c) Lenna deblurred by Algorithm 3.2; (d) Lenna deblurred by Algorithm 1.2 and (e) Lenna deblurred by Algorithm 1.1.

Table 3. Comparison of numerical results for Algorithm 3.2, 1.2 and 1.1 using their SNR values for Peacock.

Images	m	Algorithm 3.2	Algorithm 1.2	Algorithm 1.1
Peacock.png		SNR	SNR	SNR
(256 × 256)	300	36.8652	24.5768	21.988
	500	36.9989	24.666	22.8918
	800	37.0548	24.7031	22.9992
	1000	37.0804	24.7201	23.8513

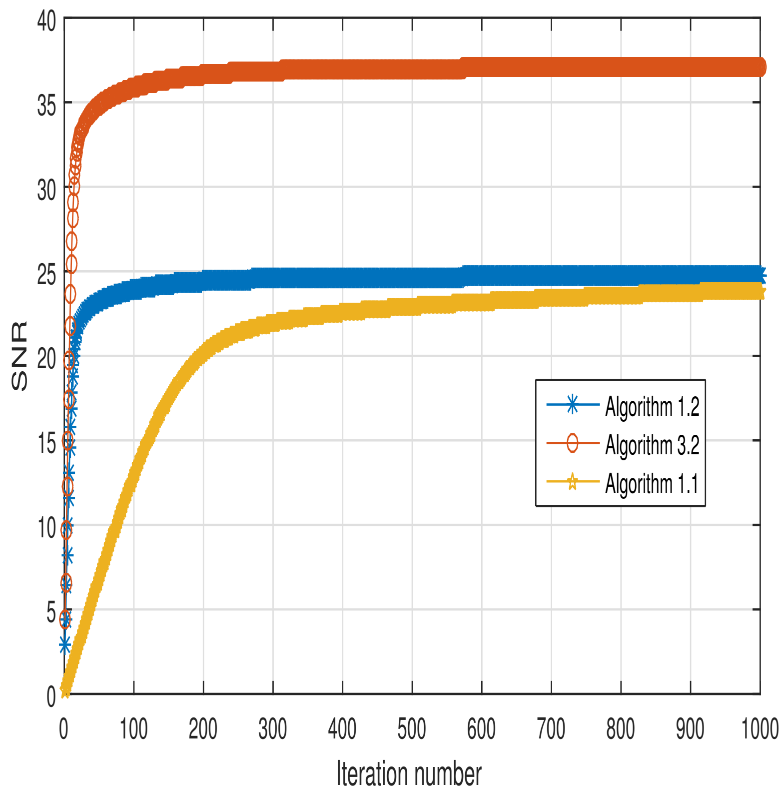

Figure 5. Graph corresponding to Table. 3

Table 4. Comparison of numerical results for Algorithm 3.2, 1.2 and 1.1 using their SNR values for Lenna.

Images	m	Algorithm 3.2	Algorithm 1.2	Algorithm 1.1
Lenna.tif		SNR	SNR	SNR
(256 $\times$ 256)	300	35.2954	23.5302	12.8776
	500	35.7201	23.7134	21.5233
	800	37.0548	24.7031	22.9992
	1000	35.9918	23.9542	23.6006

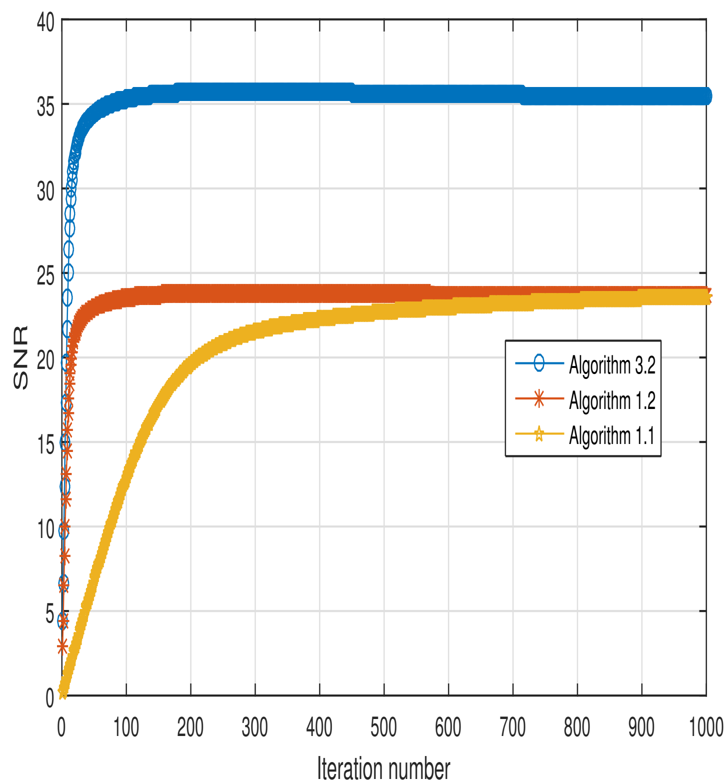

Figure 6. Graph corresponding to Table. 4

Remark 7.1. It is interesting to note that better quality of restored images are obtained with higher value of SNR. From Figures 3–6 and Tables 3–4, it is clear that our new algorithm has better restoration efficiency than the compared methods. Hence, our results are more applicable.

8. Conclusion

In this work, we have introduced a new self-adaptive relaxed Tseng exrtragradient algorithm with alternated inertial for solving variational inequalities and fixed point problems. Our convergence results have been obtained under mild conditions imposed on the control parameters. We have shown in several numerical experiments that our method enjoys better convergence rate than many existing methods. In order to show the practical applicability of our method to real word problems, we showed that our algorithm has better restoration efficiency than many well known methods in image restoration problem.

References

[1] J.A. Abuchu, A.E. Ofem, G.C. Ugwunnadi, O.K. Narain and A. Hussain, *Hybrid Alternated Inertial Projection and Contraction Algorithm for Solving Bilevel Variational Inequality Problems,* J. Math., **2023**, https://doi.org/10.1155/2023/3185746.

[2] F. Akutsah, A.A. Mebawondu, H.A. Abass and O.K. Narain, *A self adaptive method for solving a class of bilevel variational inequalities with split variational inequality and composed fixed point problem constraints in Hilbert spaces,* Numer. Algebra Control Optim., **122** (2023), DOI: 10.3934/naco.2021046.

[3] F. Akutsah, A.A. Mebawondu, G.C. Ugwunnadi and O.K. Narain, *Inertial extrapolation method with regularization for solving monotone bilevel variation inequalities and fixed point problems in real Hilbert space,* J. Nonlinear Funct. Anal., **2022**, Article ID 5,

[4] F. Akutsah, A.A. Mebawondu, G.C. Ugwunnadi, P. Pillay and O.K. Narain, *Inertial extrapolation method with regularization for solving a new class of bilevel problem in real Hilbert spaces,* SeMA J., **80** (2023), 503–524, https://doi.org/10.1007/s40324-022-00293-2.

[5] F. Alvares and H. Attouch, *An inertial proximal monotone operators via discretization of a nonlinear oscillator with damping,* Set Valued Anal., **9** (2001), 3–11.

[6] P.K. Anh, T. Anh and L.D. Muun, *On bilevel split pseudomonotone variational inequality problems with applications,* Acta. Math. Vietnam., **42** (2017), 413–429.

[7] R.Y. Apostol, A.A. Grynenko and V.V. Semenov, *Terative algorithms for monotone bilevel variational inequalities,* J. Comput. Appl. Math., **107** (2012), 3–14.

[8] L.C. Ceng, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space,* J. Optim. Theory Appl., **148** (2011), 318–335.

[9] L.C. Ceng, N. Hadjisavvas and N.C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems,* Glob. Optim., **46** (2020), 635–646.

[10] E.O. Effanga, J.A. Ugboh, E.I. Enang and B.E.A. Eno, *A tool for constructing pair-wise balanced incomplete block design,* Journal of Modern Mathematics and Statistics, **3**(4), 69–72.

[11] I.M. Esuabana, U.A. Abasiekwere, J.A. Ugboh and Z. Lipcsey, *Equivalent Construction of Ordinary Differential Equations from Impulsive System,* Academic Journal of Applied Mathematical Sciences, **4**(8), 77–89.

[12] G. Ficher, *Sur problem elastostatico di signorini con ambigue condizioni al contorno,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **34** (1963), 138–142.

[13] G. Ficher, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno,* Atti Accad. Naz. Lincci, Cl. Sci. Fis. Mat. Nat., Sez., **7** (1964), 91–140.

[14] N. Hadjisavvas and S. Schaible, *Quasimonotone variational inequalities in Banach spaces,* J. Optim. Theory Appl., **90** (1996), 95–111.

[15] Y. He, *A new double projection algorithm for variational inequalities,* J. Comput. Appl. Math., **185**(1) (2006), 166—173.

[16] B.S. He and L.Z. Liao, *Improvements of some projection methods for monotone nonlinear variational inequalities,* Optim. Theory Appl., **112** (2002), 111–128.

[17] H. He, C. Ling and H.K. Xu, *A relaxed projection method for split variational inequalities,* J. Optim. Theory Appl., **166** (2015), 213–233.

[18] G.M. Korpelevich, *An extragradient method for finding saddle points and for other problems,* Ekon. Mat. Metody., **12** (1976), 747–756.

[19] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings,* J. Optim. Theory Appl., **128** (2006), 191–201.

[20] A.E. Ofem, J.A. Abuchu, H.A. Nabwey, G.C. Ugwunnadi and O.K. Narain, *On Bilevel Monotone Inclusion and Variational Inequality Problems,* Mathematics, **11**(22) (2023), 4643, https://0-doi-org.brum.beds.ac.uk/10.3390/math11224643.

[21] A.E. Ofem, H. Işik, F. Ali and J. Ahmad, *A new iterative approximation scheme for Reich–Suzuki type nonexpansive operators with an application,* J. Inequal. Appl., **28** (2022), doi.org/101186/s13660-022-02762-8.

[22] A.E. Ofem, A.A. Mebawondu, G.C. Ugwunnadi, P. Cholamjiak and O.K. Narain, R̂elaxed Tseng splitting method with double inertial steps for solving monotone inclusions and fixed point problems, Numer. Algorithms, (2023), https://doi.org/10.1007/s11075-023-01674-y.

[23] A.E. Ofem, A.A. Mebawondu, G.C. Ugwunnadi, H. Işik and O.K. Narainl, *A modified subgradient extragradient algorithm-type for solving quasimonotone variational inequality problems with applications,* J. Inequal. Appl. **2023(73)** (2023), https://doi.org/10.1186/s13660-023-02981-7.

[24] F. Ogbuisi, Yekini Shehu and Jen-Chih Yao, *An alternated inertial method for pseudomonotone variational inequalities in Hilbert spaces,* Optim. Eng., **23**(1) (2022), 1–29, DOI:10.1007/s11081-021-09615-1.

[25] G.A. Okeke, A.E. Ofem, T. Abdeljawad, M.A. Alqudah and A. Khan, *A solution of a nonlinear Volterra integral equation with delay via a faster iteration method,* AIMS Mathematics, **8**(1) (2023), 102–124.

[26] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings,* Math. Comput. Model., **32**(2000), 1181–1191.

[27] B.T. Polyak, *Some methods of speeding up the convergence of iteration methods,* Politehn Univ Buchar Sci Bull Ser A Appl Math Phys., **4**(5) (1964), 1—17.

[28] Y. Shehu and O.S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence,* Appl. Numer. Math., **57** (2020), 315–337.

[29] Y. Shehu, O.S. Iyiola and F.U. Ogbuisi, *Iterative method with inertial terms for nonexpansive mappings, Applications to compressed sensing,* Numer. Algor., **83** (2020), 1321–1347.

[30] Y. Shehu and P.T. Vuong and P. Cholamjiak, *A self-adaptive projection method with an inertial technique for split feasibility problems in Banach spaces with applications to image restoration problems,* J. Fixed Point Theory Appl., **21** (2019), 1–24.

[31] Y. Shehu, P.T. Vuong and A. Zemkoho, *An inertial extrapolation method for convex simple bilevel optimization,* Optim. Meth. Softw., (2019), https://doi.org/10.1080/10556788.2019.1619729.

[32] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes,* C. R. Math. Acad. Sci., **258** (1964), 4413–4416.

[33] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces,* J. Conv. Anal., **24** (2017), 1015–1028.

[34] W. Takahashi, C.F. Wen and J.C. Yao., *The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space,* Fixed Point Theory, **19** (2018), 407–419.

[35] D.V. Thong and D.V. Hieu, *Inertial subgradient extragradient algorithms with line–search process for solving variational inequality problems and fixed point problems,* Numer. Algor., **80**(4) (209), 1283–1307, DOI: 10.1007/s11075-018-0527-x.

[36] M. Tian and M. Tong, *Self-adaptive subgradient extragradient method with inertial modification for solving monotone variational inequality problems and quasinonexpansive fixed point problems,* J. Inequ. Appl., **2019**(7) (2019).

[37] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings,* SIAM J. Control. Optim., **38** (2000), 431–446.

[38] H.K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc., **66**(2002), 240–256.

[39] L. Zheng, *A double projection algorithm for quasimonotone variational inequalities in Banach spaces,* Inequal. Appl., **123** (2018), 1–20.