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ANALYSIS OF EXISTENCE AND STABILITY RESULTS FOR FRACTIONAL IMPULSIVE $\Im$-HILFER FREDHOLM-VOLTERRA MODELS

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Abstract. In this paper, we investigate the suitable conditions for the existence results for a class of $\Im$-Hilfer fractional nonlinear Fredholm-Volterra models with new conditions. The findings are based on Banach contraction principle and Schauder's fixed point theorem. Also, the generalized Hyers-Ulam stability and generalized Hyers-Ulam-Rassias stability for solutions of the given problem are provided.

1. Introduction

Science and progress have advanced significantly in recent decades thanks to the fractional calculus theory, which comprises fractional equations. This is mostly because it has produced adequate models, especially for real-world issues. Furthermore, generalized differential equations are thought to include fractional differential equations (FDEs) [1, 6, 8, 14, 16, 17, 18, 20].

The integro-differential equation (IDE) is an operator that contains both integer-order derivatives and integer-order integrals as special cases, which is the reason why in the present, fractional calculus gains enormous popularity and many applications arise from the term in the field of mathematical

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physics where The integrals are essentially power law and have a convolution or logarithm type single kernels [3, 4, 5, 7, 8, 13, 23, 24].

The initial-boundary value difficulties posed by nonlinear parabolic differential and IDEs may be used to mimic many real phenomena; for examples, see [8, 15, 16, 17, 25, 26, 27, 28]. As a result, the Caputo fractional operator's application and generalization in IDEs are justified.

Through the use of Caputo, Hilfer operators and their generalization, several scholars have recently examined the stability, uniqueness, and existence of various boundary value problems (BVPs). Non-instantaneous impulsive BVPs and the $\Im$-Caputo (or, more precisely, $\Im$-Liouville-Caputo) fractional derivative were explored by Asawasamrit et al. in [2]. The $\Im$-Hilfer fractional derivative including boundary conditions was explored by Ivaz et al. in [12].

The fractional Hilfer derivative under nonlocal boundary constraints was studied by Nuchpong et al. [19]

$$\begin{cases} {}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q}}X(\phi) = \Xi\left(\phi, X(\phi), \mathcal{I}^{\delta}X(\phi)\right), & \phi \in [\alpha_1, \alpha_2], \\ X(\alpha_1) = 0, \quad \nabla + \int_{\alpha_1}^{\alpha_2} X(l)\mathrm{d}l = \sum_{\mathrm{k}=1}^{\varrho-2} \varsigma_{\mathrm{k}} X(\vartheta_{\mathrm{k}}), \end{cases}$$

where they have used the the $\mathcal{I}^{\delta}$-R-L and ${}^{\mathrm{H}}\mathcal{D}^{\mathrm{p},\mathrm{q}}$-Hilfer fractional derivative, and Ξ is continuous function.

In [22], Salim et al. investigated the BVP for the implicit fractional-order generalised fractional impulses Hilfer-type of the following form:

$$\begin{cases} \left({}^{\wp}\mathcal{D}_{\tau^+}^{\mathrm{p},\mathrm{q}}X\right)(\phi) = \Xi\left(\phi, X(\phi), \left({}^{\wp}\mathcal{D}^{\mathrm{p},\mathrm{q}}X\right)(\phi)\right), & \phi \in \phi_c, \\ X(\phi) = \mathcal{B}_{\mathrm{c}}(\phi, X(\phi)), \quad \phi \in (\phi_{\mathrm{c}}, \mathrm{r}_{\mathrm{c}}], & \mathrm{c} = 1, \cdots, \varrho, \\ \varphi_1\left({}^{\wp}\mathcal{I}_{\alpha_1^+}^{1-\epsilon}\right)(\alpha_1) + \varphi_2\left({}^{\wp}\mathcal{I}_{\tau^+}^{1-\epsilon}\right)(\alpha_2) = \varphi_3, \end{cases}$$

where ${}^{\wp}\mathcal{I}_{\alpha_1^+}^{1-\epsilon}$ and ${}^{\wp}\mathcal{D}_{\tau^+}^{\mathrm{p},\mathrm{q}}$ are the fractional generalized Hilfer-type integral and derivative and the function Ξ is continuous.

As started by Sousa et al. in [25], we examine the existence, and stability of fractional IDEs using the $\Im$-Hilfer fractional derivative in this work. A remark on the transformation may be found in [25]. Multiple fractional derivatives are combined into the fractional $\Im$-Hilfer derivative.

$$\begin{cases} D_{\alpha+}^{\wp,\beta;\Im}\xi(\upsilon) = \varpi\left(\upsilon, \xi(\upsilon), G\xi(\upsilon), K\xi(\upsilon)\right), & \upsilon \in J := (\alpha, \chi], \\ I_{\alpha+}^{1-\gamma;\Im}\xi(\alpha) = \xi_\alpha, \quad \gamma = \wp + \beta - \wp\beta, \end{cases} \tag{1.1}$$

where $I_{\alpha+}^{1-\gamma;\Im}$ is $\Im$-fractional integral and $D_{\alpha+}^{\wp,\beta;\Im}$ is fractional $\Im$-Hilfer derivative of order $\wp$ and type β, where $\varpi : J \times \mathbb{R}^3 \to \mathbb{R},\ g, k : \Delta \times R \to \mathbb{R}$ are continuous, with $\Delta = \{(\upsilon,\sigma) : \alpha \le \sigma \le \upsilon \le \chi\}$, and

$$G\xi(\upsilon) = \int_\alpha^\upsilon g(\upsilon,\sigma,\xi(\sigma))d\sigma, \quad K\xi(\upsilon) = \int_\alpha^\chi k(\upsilon,\sigma,\xi(\sigma))d\sigma.$$

2. An auxiliary result

We review several definitions and outcomes from fractional calculus in this section. The observations that follow are drawn from [8, 11, 12, 16, 20]. Let $\xi : J \longrightarrow \mathbb{R}$ in $\zeta[\alpha,\chi]$ with

$$\|\xi\| = \sup\{|\xi(\upsilon)| : \upsilon \in J\}.$$

The $\zeta_{\gamma,\Im}[\alpha,\chi]$ is weighted space, let ϖ on $(\alpha,\chi]$ is given by

$$\zeta_{\gamma,\Im}[\alpha,\chi] = \{\varpi : (\alpha,\chi] \to R : (\Im(\upsilon) - \Im(\alpha))^\gamma \varpi(\upsilon) \in \zeta[\alpha,\chi]\}, \quad 0 \le \gamma < 1,$$

and the norm

$$\|\varpi\|_{\zeta_{\gamma,\Im}} = \|(\Im(\upsilon) - \Im(\alpha))^\gamma \varpi(\upsilon)\|_{\zeta[\alpha,\chi]} = \max_{\upsilon\in J} |(\Im(\upsilon) - \Im(\alpha))^\gamma \varpi(\upsilon)|.$$

The weighted space $\zeta^n_{\gamma,\Im}[\alpha,\chi]$ of ϖ on $(\alpha,\chi]$ is given by

$$\zeta^n_{\gamma,\Im}[\alpha,\chi] = \left\{\varpi : J \to R : \varpi(\upsilon) \in \zeta^{n-1}[\alpha,\chi]; \varpi(\upsilon) \in \zeta_{\gamma,\Im}[\alpha,\chi]\right\}, 0 \le \gamma < 1,$$

and the norm

$$\|\varpi\|_{\zeta^n_{\gamma,\Im}[\alpha,\chi]} = \sum_{k=0}^{n-1} \left\|\varpi^k\right\|_{\zeta[\alpha,\chi]} + \|\varpi^n\|_{\zeta_{\gamma,\Im}[\alpha,\chi]}.$$

For $n = 0$, we get $\zeta^0_\gamma[\alpha,\chi] = \zeta_\gamma[\alpha,\chi]$.

Definition 2.1. ([21]) The fractional integral left-sided of ϖ in relation to $\Im$ on $[\alpha,\chi]$ is given by

$$\left(I^{\wp;\Im}_{\alpha^+}\right)\varpi(\upsilon) = \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}\varpi(\sigma)d\sigma, \quad \upsilon > \alpha. \tag{2.1}$$

Definition 2.2. ([21]) Let $\Im'(\xi) \ne 0(-\infty < \upsilon < \chi < \infty)$. The fractional Liouville-Riemann derivative of ϖ in relation to $\Im$ of order $\wp > 0$ correspondent to the Liouville-Riemann, is given by

$$\left(D^{\wp;\Im}_{\alpha^+}\varpi\right)(\upsilon) = \frac{1}{\Gamma(n-\wp)}\left(\frac{1}{\Im'(\upsilon)}\frac{d}{d\upsilon}\right)^n\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{n-\wp-1}\varpi(\sigma)d\sigma, \tag{2.2}$$

where $n = [\wp] + 1, \ n \in \mathbb{N}$.

Definition 2.3. ([21]) Let $\wp > 0$, $I = [\alpha, \chi]$, $(-\infty < \upsilon < \chi < \infty)$, $\varpi, \Im \in \zeta^n([\alpha, \chi], R)$ be two functions, $\Im$ be increasing and $\Im'(\xi) \neq 0$ for all $\xi \in I$. The left $\Im$-Caputo derivative of ϖ of order $\wp$ as

$$\left(D_{\alpha^+}^{\wp;\Im}\varpi\right)(\upsilon) = I_{\alpha^+}^{n-\wp;\Im}\left(\frac{1}{\Im'(\upsilon)}\frac{d}{d\upsilon}\right)^n \varpi(\upsilon), \tag{2.3}$$

where $n = [\wp] + 1$, for $\wp \notin N$ and $\wp = n$ for $\wp \in N$.

Definition 2.4. ([21]) According to function ϖ of order $\wp$ has the following fractional $\Im$-Hilfer derivative:

$$D_{\alpha^+}^{\wp,\beta;\Im}\varpi(\upsilon) = I_{\alpha^+}^{\beta(1-\wp);\Im}\left(\frac{1}{\Im'(\upsilon)}\frac{d}{dt}\right) I_{\alpha^+}^{(1-\beta)(1-\wp);\Im}\varpi(\upsilon). \tag{2.4}$$

As previously stated, the $\Im$-Hilfer fractional derivative may be expressed as follows:

$$D_{\alpha^+}^{\wp,\beta;\Im}\varpi(\upsilon) = I_{\alpha^+}^{\gamma-\wp;\Im} D_{\alpha^+}^{\gamma;\Im}\varpi(\upsilon).$$

Lemma 2.5. ([8]) *Assume $\wp, \beta > 0$. Then we get*

$$\left(I_{\alpha^+}^{\wp;\Im} I_{\alpha^+}^{\beta;\Im}\varpi\right)(\upsilon) = \left(I_{\alpha^+}^{\wp+\beta;\Im}\right)(\upsilon)$$

and

$$\left(D_{\alpha^+}^{\wp;\Im} I_{\alpha^+}^{\wp;\Im}\varpi\right)(\upsilon) = \varpi(\upsilon).$$

Lemma 2.6. ([8]) *Let $\wp, \beta > 0$.*

(1) *If $\varpi(\xi) = (\Im(\upsilon)\Im(\alpha))^{\beta-1}$, then*

$$I_{\alpha^+}^{\wp;\Im}(\Im(\upsilon) - \Im(\alpha))^{\beta-1}(\upsilon) = \frac{\Gamma(\beta)}{(\wp+\beta)}(\Im(\upsilon) - \Im(\alpha))^{\wp+\beta-1}.$$

(2) *If $g(\xi) = (\Im(\upsilon)\Im(\alpha))^{\wp-1}$, then*

$$D_{\alpha^+}^{\wp;\Im}(\Im(\upsilon) - \Im(\alpha))^{\wp-1}(\upsilon) = 0.$$

Lemma 2.7. ([22]) *For $0 < \wp < 1$, if $\varpi \in \zeta^n[\alpha, \chi]$, then*

$$\left(I_{\alpha^+}^{\wp;\Im} D_{\alpha^+}^{\wp;\Im}\right)(\upsilon) = \varpi(\upsilon) - \frac{\left(I_{\alpha+}^{1-\wp;\Im}\varpi\right)(\alpha)}{\Gamma(\wp)}(\Im(\upsilon) - \Im(\alpha))^{\wp-1}$$

for all $\xi \in (\alpha, \chi]$.

Lemma 2.8. ([11]) *Assume $n-1 \leq \gamma < n$ and $\varpi \in \zeta_\gamma[\alpha,\chi]$. Then*

$$\left(I_{\alpha^+}^{\wp;\Im}\varpi\right)(\alpha) = \lim_{\upsilon\to a^+}\left(I_{\alpha^+}^{\wp;\Im}\right)\varpi(\upsilon) = 0.$$

The weighted space is shown below as follows.

$$\zeta_{1-\gamma;\Im}^{\wp,\beta}[\alpha,\chi] = \left\{\varpi \in \zeta_{1-\gamma;\Im}[\alpha,\chi], D_{\alpha^+}^{\wp,\beta;\Im}\varpi \in \zeta_{\gamma;\Im}[\alpha,\chi]\right\}$$

and

$$\zeta_{1-\gamma;\Im}^{\gamma}[\alpha,\chi] = \left\{\varpi \in \zeta_{1-\gamma;\Im}[\alpha,\chi], D_{\alpha^+}^{\gamma;\Im}\varpi \in \zeta_{1-\gamma;\Im}[\alpha,\chi]\right\}.$$

There is no denying

$$\zeta_{1-\gamma;\Im}^{\gamma}[\alpha,\chi] \subset \zeta_{1-\gamma;\Im}^{\wp,\beta}[\alpha,\chi].$$

Lemma 2.9. ([11]) *Let $0 \leq \beta \leq 1$, $0 < \wp < 1$ and $\gamma = \wp + \beta - \wp\beta$. If $\zeta_{1-\gamma,\Im}^{\gamma}[\alpha,\chi]$, then*

$$I_{\alpha^+}^{\gamma;\Im}D_{\alpha^+}^{\gamma;\Im}\varpi = I_{\alpha^+}^{\wp;\Im}D_{\alpha^+}^{\wp,\beta;\Im}\varpi \tag{2.5}$$

and

$$D_{\alpha^+}^{\gamma;\Im}I_{\alpha^+}^{\wp;\Im}\varpi = D_{\alpha^+}^{\beta(1-\wp);\Im}\varpi. \tag{2.6}$$

Lemma 2.10. ([28]) *Let $\varpi \in L^1(\alpha,\chi)$. If $D_{\alpha^+}^{\beta(1-\wp);\Im}\varpi$ exists on $L^1(\alpha,\chi)$. Then*

$$D_{\alpha^+}^{\wp,\beta;\Im}I_{\alpha^+}^{\wp;\Im}\varpi = I_{\alpha^+}^{\beta(1-\wp);\Im}D_{\alpha^+}^{\beta(1-\wp);\Im}\varpi.$$

Lemma 2.11. ([28]) *Suppose $\varpi \in \zeta^1[\alpha,\chi]$, $\wp > 0$ and $0 \leq \beta \leq 1$. Then we get*

$$D_{\alpha^+}^{\wp,\beta;\Im}I_{\alpha^+}^{\wp;\Im}\varpi = \varpi.$$

Lemma 2.12. *Let $\wp > 0, a(\upsilon)$ be an integrable on $a \leq \upsilon < \chi$ (some $\chi \leq \infty$), and let $g(\upsilon)$ be nondecreasing, continuous and nonnegative on $\alpha \leq \upsilon < \chi$ such that $g(\upsilon) \leq \Omega^*$ for some Ω. Let $\xi(\upsilon)$ be locally integrable on $\alpha \leq \upsilon < \chi$ with*

$$|\xi(\upsilon)| \leq a(\upsilon) + g(\upsilon)\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\xi(\sigma)d\sigma, \quad \upsilon \in J$$

with some $\wp > 0$. Then

$$|\xi(\upsilon)| \leq a(\upsilon) + \int_\alpha^\upsilon\left[\sum_{n=1}^{\infty}\frac{(g(\upsilon)\Gamma(\wp))^n}{\Gamma(n\wp)}\Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{n\wp-1}\right]\xi(\sigma)d\sigma, \alpha \leq \upsilon < \chi.$$

Proof. Theorem 1 in [28] is identical to the proof in this case. □

Lemma 2.13. ([28]) *Assume that* $\gamma = \wp + \beta - \wp\beta$, *where* $0 \leq \beta \leq 1$ *and* $0 < \wp < 1$. *If* $\varpi : J \times \mathbb{R}^3 \to \mathbb{R}$ *and* $\varpi(\cdot, \xi(\cdot)) \in \zeta_{1-\gamma}[\alpha, \chi]$ *for all* $\xi \in \zeta_{1-\gamma}[\alpha, \chi]$. *A function* $\xi \in \zeta^{\gamma}_{1-\gamma}[\alpha, \chi]$ *is a solution of* (1.1) *if and only if* ξ *satisfies*

$$\xi(\upsilon) = \frac{\xi_\alpha}{\Gamma(\gamma)}(\Im(\upsilon) - \Im(\alpha))^{\gamma-1} + \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon (\Im(\upsilon) - \Im(\sigma))^{\wp-1}\varpi(\sigma, \xi(\sigma), G\xi(\sigma), K\xi(\sigma))d\sigma. \tag{2.7}$$

3. Existence results

To back up our primary findings, we propose the following hypothesis.

(H1) Assume that $\varpi : J \times \mathbb{R} \to \mathbb{R}$ is a function and $\varpi(\cdot, \xi(\cdot)) \in \zeta_{1-\gamma,\Im}[J, \mathbb{R}]$ for any $\xi \in \zeta_{1-\gamma}[J, \mathbb{R}]$ for all $\xi, y \in \mathbb{R}$ and there exist $L > 0$ with

$$|\varpi(\upsilon, \xi) - \varpi(\upsilon, y)| \leq L|\xi - y|.$$

(H2) Assume that $g, k : \Delta \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist $G^*, K^* > 0$, and

$$\int_\alpha^\upsilon |g(\upsilon, \sigma, \xi) - g(\upsilon, \sigma, y)|ds \leq G^*|\xi - y|$$

and

$$\int_\alpha^\chi |k(\upsilon, \sigma, \xi) - k(\upsilon, \sigma, y)|ds \leq K^*|\xi - y|.$$

(H3) Assume that $\varpi : J \times \mathbb{R} \to \mathbb{R}$ is a function and there exists M, N with

$$|\varpi(\upsilon, \xi)| \leq M|\xi| + N, \ \ \forall \upsilon \in J, \ \xi \in \mathbb{R}.$$

Theorem 3.1. *Assume that both* (H1) *and* (H2) *are met. Then, there is at least one solution to the Eq.* (1.1).

Proof. Incorporate the map $N : \zeta_{1-\gamma,\Im}[\alpha, \chi] \to \zeta_{1-\gamma,\Im}[\alpha, \chi]$. The operator-form representation of the Eq. (2.7)

$$\xi(\upsilon) = N\xi(\upsilon),$$

where

$$(\Lambda\xi)(\upsilon) = \frac{\xi_\alpha}{\Gamma(\gamma)}(\Im(\upsilon) - \Im(\alpha))^{\gamma-1} + \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}\varpi(s, \xi(\sigma), G\xi(\sigma), K\xi(\sigma))d\sigma. \tag{3.1}$$

Consider the ball

$$\mathsf{C}_r = \{\xi \in \zeta_{1-\gamma,\Im}[\alpha, \chi] : \|\xi\| \leq r\}.$$

The operator Λ is obviously well defined. It is obvious that the problem's solutions are the fixed points of the operator Λ. Then, for all $\xi \in \zeta_{1-\gamma,\Im}[\alpha,\chi]$ and $\upsilon \in J$ we get,

$$\begin{aligned}
&\left|(\Lambda\xi)(\upsilon)(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}\right| \\
&\quad = \Big|\frac{\xi_\alpha}{\Gamma(\gamma)} + \frac{1}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{1-\gamma} \\
&\qquad \times \int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\xi(\sigma),G\xi(\sigma),K\xi(\sigma))d\sigma\Big| \\
&\quad \le \frac{\xi_\alpha}{\Gamma(\gamma)} + \frac{1}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{1-\gamma} \\
&\qquad \times \int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}|\varpi(\sigma,\xi(\sigma))|d\sigma \\
&\quad \le \frac{\xi_\alpha}{\Gamma(\gamma)} + \frac{1}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{1-\gamma} \\
&\qquad \times \int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\alpha))^{\wp-1}(M|\xi(\sigma)|+\Lambda)d\sigma \\
&\quad \le \frac{\xi_\alpha}{\Gamma(\gamma)} + \frac{M}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}(\Im(\upsilon)-\Im(\alpha))^{\wp+\gamma-1}B(\gamma,\wp)\|\xi\|_{\zeta_{1-\gamma,\Im}} \\
&\qquad + \frac{\Lambda}{\Gamma(\wp+1)}(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}(\Im(\upsilon)-\Im(\alpha))^{\wp} \\
&\quad \le \frac{\xi_\alpha}{\Gamma(\gamma)} + \frac{1}{\Gamma(\wp)}(\Im(\chi)-\Im(\alpha))^{\wp}B(\gamma,\wp)\|\xi\|_{\zeta_{1-\gamma,\Im}} \\
&\qquad + \frac{\Lambda}{\Gamma(\wp+1)}(\Im(\chi)-\Im(\alpha))^{\wp+1-\gamma}.
\end{aligned}$$

This demonstrates how the $\mathtt{C}_r = \left\{\xi \in \zeta_{1-\gamma,\Im}[\alpha,\chi] : \|\xi\|_{\zeta_{1-\gamma,\Im}} \le r\right\}$ is transformed by Λ into itself. The proof is broken down into different steps:

Step 1: Λ is continuous.

Let ξ_n and $\xi_n \to \xi$ in $\zeta_{1-\gamma,\Im}[\alpha,\chi]$. Then for all $\upsilon \in J$,

$$\begin{aligned}
&\Big|\left((\Lambda\xi_n)(\upsilon) - (\Lambda\xi)(\upsilon)\right)(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}\Big| \\
&\quad \le \Big|\frac{(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}}{\Gamma(\wp)} \\
&\qquad \times \int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi\left(\sigma,\xi_n(\sigma),G\xi_n(\sigma),K\xi_n(\sigma)\right)d\sigma \\
&\qquad - \frac{(\Im(\upsilon)-\Im(\alpha))}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\xi(\sigma),G\xi(\sigma),K\xi(\sigma))d\sigma\Big|
\end{aligned}$$

$$\leq \frac{(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}}{\Gamma(\wp)} \int_{\alpha}^{\upsilon} \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}$$
$$\times \Big|\varpi\left(\sigma, \xi_n(\sigma), G\xi_n(\sigma), K\xi_n(\sigma)\right)-\varpi(\sigma, \xi(\sigma), G\xi(\sigma), K\xi(\sigma))\Big| d\sigma$$
$$\leq \frac{(\Im(\upsilon)-\Im(\alpha))^{1-\gamma}}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{\wp+\gamma-1} B(\gamma, \wp)$$
$$\times \Big\|\varpi\left(\cdot, \xi_n(\cdot), G\xi_n(\cdot), H\xi_n(\cdot)\right)-\varpi(\cdot, \xi(\cdot), G\xi(\cdot), K\xi(\cdot))\Big\|_{\zeta_{1-\gamma, \Im}}$$

which implies

$$\|\Lambda\xi_n-\Lambda\xi\|_{\zeta_{1-\gamma, \Im}} \leq B(\gamma, \wp) \frac{(\Im(\chi)-\Im(\alpha))^{\wp}}{\Gamma(\wp)}$$
$$\times \Big\|\varpi\left(\cdot, \xi_n(\cdot), G\xi_n(\cdot), K\xi_n(\cdot)\right)-\varpi(\cdot, \xi(\cdot), G\xi(\cdot), K\xi(\cdot))\Big\|_{\zeta_{1-\gamma, \Im}}.$$

This means that Λ is continuous.

Step 2: $\Lambda\left(\mathtt{C}_r\right)$ is uniformly bounded.

It is obvious that $\Lambda\left(\mathtt{C}_r\right) \subset \mathtt{C}_r$ is bounded

Step 3: $\Lambda\left(\mathtt{C}_r\right)$ is relatively compact.

It makes sense when $\Lambda\left(\mathtt{C}_r\right) \subset \mathtt{C}_r$ that $\Lambda\left(\mathtt{C}_r\right)$ is uniformly bounded, and to demonstrate that Λ is an equicontinuous operator. Assume that $\upsilon_1, \upsilon_2 \in J$, $\upsilon_1 < \upsilon_2$, $\mathtt{C}_r$ is bounded in $\zeta_{1-\gamma,\rho}[\alpha, \chi]$. Then,

$$|((\Lambda\xi)(\upsilon_1)-(\Lambda\xi)(\upsilon_2))|$$
$$\leq \frac{\xi_\alpha}{\Gamma(\gamma)}\left|(\Im(\upsilon_1)-\Im(\alpha))^{\gamma-1}-(\Im(\upsilon_2)-\Im(\alpha))^{\gamma-1}\right|$$
$$+\frac{B(\gamma, \wp)}{\Gamma(\wp)}\left((\Im(\upsilon_1)-\Im(\alpha))^{\wp+\gamma-1}-(\Im(\upsilon_2)-\Im(\alpha))^{\wp+\gamma-1}\right)\|\varpi\|_{\zeta_{1-\gamma, \Im}}.$$

As $\upsilon_1 \rightarrow \upsilon_2$, The aforementioned inequality's right side goes to zero. We may infer that $\Lambda : \zeta_{1-\gamma,\Im}[\alpha, \chi] \rightarrow \zeta_{1-\gamma,\Im}[\alpha, \chi]$ is continuous and totally continuous as a result of Steps 1 through 3 and the Arzela-Ascoli theorem. □

4. Generalized stability

Let $\epsilon > 0$ and $\varphi : I \rightarrow \mathbb{R}^+$ be continuous. We consider the following inequalities:

$$\left|D_{\alpha+}^{\wp,\beta;\Im}\eta(\upsilon)-\varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon))\right| \leq \epsilon, \quad \upsilon \in J, \tag{4.1}$$

$$\left|D_{\alpha+}^{\wp,\beta;\Im}\eta(\upsilon)-\varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon))\right| \leq \epsilon\varphi(\upsilon), \quad \upsilon \in J, \tag{4.2}$$

$$\left|D_{\alpha+}^{\wp,\beta;\Im}\eta(\upsilon)-\varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon))\right| \leq \varphi(\upsilon), \quad \upsilon \in J. \tag{4.3}$$

Definition 4.1. ([9]) The problem (1.1) is said to be Hyers-Ulam stable if for all $\epsilon > 0$, there exist $\zeta_{\varpi} > 0$ and for each solution $\eta \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ of the (4.1) there exists a solution $\xi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ of the problem (1.1) such that

$$|\eta(\upsilon) - \xi(\upsilon)| \leq \zeta_{\varpi}\epsilon, \quad \upsilon \in J.$$

Definition 4.2. ([10]) The problem (1.1) is said to be generalized Hyers-Ulam stable if there exists $\varphi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ with $\varphi_{\varpi}(0) = 0$ such that for all $\eta \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ solution of (4.1) there exists a solution $\xi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ of (1.1) with

$$|\eta(\upsilon) - \xi(\upsilon)| \leq \varphi_{\varpi}\epsilon, \quad \upsilon \in J.$$

Definition 4.3. ([10]) The problem (1.1) is said to be Hyers-Ulam-Rassias stable in relation to $\varphi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ if there exists a real number $\zeta_{\varpi,\varphi} > 0$ and for all $\epsilon > 0$ and $\eta \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ of the inequality (4.2) there exists a solution of (1.1), $\xi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ with

$$|\eta(\upsilon) - \xi(\upsilon)| \leq \zeta_{\varpi,\varphi}\varphi(\upsilon), \quad \upsilon \in J.$$

Definition 4.4. ([9]) The problem (1.1) is said to be generalized Hyers-Ulam-Rassias stable in relation to $\varphi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ if there exists $\zeta_{\varpi,\varphi} > 0$ such that for each solution $\eta \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ of (4.3) there exists $\xi \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ is a solution of (1.1) with

$$|\eta(\upsilon) - \xi(\upsilon)| \leq \zeta_{\varpi,\varphi}\varphi(\upsilon), \quad \upsilon \in J.$$

Remark 4.5. Clearly, we have the following implication:

(1) The Hyers-Ulam stability is the generalized Hyers-Ulam stability.
(2) The Hyers-Ulam-Rassias stability is generalized Hyers-Ulam-Rassias stability.
(3) If the problem (1.1) is generalized Hyers-Ulam-Rassias stable with $\varphi(\upsilon) = 1$ then it is Hyers-Ulam stable.

Remark 4.6. Let a function $\eta \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$. Then η is a solution of the inequality (4.1),

$$\left| D_{\alpha+}^{\wp,\beta;\Im}\eta(\upsilon) - \varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon)) \right| \leq \epsilon, \quad \upsilon \in J$$

if and only if there exists $g \in \zeta_{1-\gamma,\Im}[\alpha, \chi]$ such that
(i) $|g(\upsilon)| \leq \epsilon$, $\upsilon \in J$
(ii) $D_{\alpha+}^{\wp,\beta;\Im}\eta(\upsilon) = \varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon)) + g(\upsilon), \;\; \upsilon \in J.$

Remark 4.7. If the inequality (4.1) has a solution of η, then the following inequality has a solution of η.

$$\Big|\eta(\upsilon) - \frac{\eta_\alpha}{\Gamma(\gamma)}(\Im(\upsilon) - \Im(\alpha))^{\gamma-1} - \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}$$
$$\times(\varpi(\sigma, \eta(\sigma), G\eta(\sigma), K\eta(\upsilon))d\sigma\Big| \leq \epsilon\frac{(\Im(\upsilon) - \Im(\alpha))^{\wp}}{\Gamma(\wp+1)}.$$

Indeed, by Remark 4.6 we have that

$$D_{\alpha_+}^{\wp,\beta;\Im}\eta(\upsilon) = \varpi(\upsilon, \eta(\upsilon), G\eta(\upsilon), K\eta(\upsilon)) + g(\upsilon), \quad \upsilon \in J.$$

Then

$$\eta(\upsilon) = \frac{\eta_\alpha}{\Gamma(\gamma)}(\Im(\upsilon) - \Im(\alpha))^{\gamma-1}$$
$$+ \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}(\varpi(\sigma, \eta(\sigma), G\eta(\sigma), K\eta(\upsilon)) + g(\sigma))d\sigma.$$

From this it follows that

$$\Big|\eta(\upsilon) - \frac{\eta_\alpha}{\Gamma(\gamma)}(\Im(\upsilon) - \Im(\alpha))^{\gamma-1}$$
$$- \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}(\varpi(\sigma, \eta(\sigma), G\eta(\sigma), K\eta(\sigma))d\sigma\Big|$$
$$= \left|\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}g(\sigma)d\sigma\right|$$
$$\leq \frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon) - \Im(\sigma))^{\wp-1}|g(\sigma)|d\sigma$$
$$\leq \epsilon\frac{(\Im(\upsilon) - \Im(\alpha))^{\wp}}{\Gamma(\wp+1)}.$$

We make similar observations about inequality (4.2) and inequality (4.3).

The major findings of generalized Hyers-Ulam-Rassias stability are now presented in this section.

(H4) There are rising functions ($\lambda_\varphi > 0$) such that for each $\upsilon \in J$, $\varphi \in \zeta_{1-\gamma,\rho}[\alpha, \chi]$,

$$I_{\alpha_+}^{\wp;\Im}\varphi(\upsilon) \leq \lambda_\varphi\varphi(\upsilon).$$

Theorem 4.8. *If the hypotheses* (H1) *and* (H4) *are true, then the issue* (1.1) *is Hyers-Ulam-Rassias stable in general.*

Proof. Assuming that η is the solution to (4.3). Then ξ is the only possible solution to the

$$\begin{cases} D_{\alpha_+}^{\wp,\beta;\Im}\xi(\upsilon) &= \varpi(\upsilon,\xi(\upsilon),G\eta(\upsilon),K\eta(\upsilon)), \quad \upsilon\in J,\\ I_{\alpha_+}^{1-\gamma;\Im}\xi(\alpha) &= I_{\alpha_+}^{1-\gamma;\Im}\eta(\alpha), \end{cases}$$

when

$$\frac{L(1+G^*+K^*)}{\Gamma(\wp)}B(\gamma\cdot\wp)(\Im(\chi)-\Im(\alpha))^{\wp}<1.$$

Then we have

$$\begin{aligned}\xi(\upsilon) =& \frac{\eta_\alpha}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{\gamma-1}\\ &+\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\xi(\sigma),G\xi(\sigma),K\xi(\sigma))d\sigma.\end{aligned}$$

By differentiating inequality (4.3), we have

$$\begin{aligned}&\Big|\eta(\upsilon)-\frac{\eta_\alpha}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{\gamma-1}\\ &\quad -\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\eta(\sigma),G\eta(\sigma),K\eta(\sigma))d\sigma\Big|\\ &\quad\le \lambda_\varphi\varphi(\upsilon).\end{aligned}$$

Hence it follows

$$\begin{aligned}&|\eta(\upsilon)-\xi(\upsilon)|\\ &\quad\le \Big|\eta(\upsilon)-\frac{\eta_\alpha}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{\gamma-1}\\ &\qquad -\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\xi(\sigma),G\xi(\sigma),K\xi(\sigma))d\sigma\Big|\\ &\quad\le \Big|\eta(\upsilon)-\frac{\eta_\alpha}{\Gamma(\wp)}(\Im(\upsilon)-\Im(\alpha))^{\gamma-1}\\ &\qquad -\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\eta(\sigma),G\eta(\sigma),K\eta(\sigma))d\sigma\Big|\\ &\qquad +\Big|\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\eta(\sigma),G\eta(\sigma),K\eta(\sigma))d\sigma\\ &\qquad -\frac{1}{\Gamma(\wp)}\int_\alpha^\upsilon \Im'(\sigma)(\Im(\upsilon)-\Im(\sigma))^{\wp-1}\varpi(\sigma,\xi(\sigma),G\xi(\sigma),K\xi(\sigma))d\sigma\Big|\\ &\quad\le \lambda_\varphi\varphi(\upsilon)+\frac{L(1+H)}{\Gamma(\wp)}\int_\alpha^\upsilon(\Im(\upsilon)-\Im(\sigma))^{\wp-1}|\eta(\sigma)-\xi(\sigma)|d\sigma.\end{aligned}$$

According to Lemma 2.5, there exists a constant $M^*>0$ that is not reliant on $\lambda_\varphi\varphi(\upsilon)$ such that

$$|\eta(\upsilon)-\xi(\upsilon)|\le M^*\lambda_\varphi\varphi(\upsilon):=\zeta_{\varpi,\varphi}\varphi(\upsilon).$$

As a result, the problem (1.1) is Ulam-Hyers-Rassias stable in its generalized form. □

Remark 4.9. Clearly, we have the followings:

(1) We take into consideration the inequality (4.2) as well as the issue (1.1) under the premise of Theorem 4.8. The Ulam-Hyers-Rassias stability of the issue (1.1) may be confirmed by repeating the same procedure.

(2) We take into consideration the inequality (4.1) and the issue (1.1) under the premise of Theorem 4.8. The same procedure may be used to confirm that the Ulam-Hyers stability of the issue (1.1) can be established.

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