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A NEW RELAXED TSENG METHOD FOR FINDING A COMMON SOLUTION OF FIXED POINT AND SPLIT MONOTONE INCLUSION PROBLEMS

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Abstract. In this paper, we study the problem of finding a common solution to a fixed point problem involving a finite family of ρ-demimetric operators and a split monotone inclusion problem with monotone and Lipschitz continuous operator in real Hilbert spaces. Motivated by the inertial technique and the Tseng method, a new and efficient iterative method for solving the aforementioned problem is introduced and studied. Also, we establish a strong convergence result of the proposed method under standard and mild conditions.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, C be a nonempty, closed and convex subset of H. The notion of split feasibility problem (SFP) was introduced and studied by Censor and Elfving [7] in the framework of finite dimensional Hilbert spaces. The SFP has found applications in many real-life problems such as image recovery, signal processing, control theory, data compression, computer tomography and so on (see [7, 8] and the references therein). The SFP is to find

$$x^* \in C \ \text{ such that } \ Tx^* \in Q, \tag{1.1}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and T is a bounded linear operator from H_1 onto H_2.

Censor and Elfving [7] proposed an iterative method of the form of a simultaneous multi-projections for approximating the solution of SFP (1.1) in a finite dimensional Euclidean space $\mathbb{R}^n$. The classical variational inequality problem (VIP), formulated as: Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \ \forall \ y \in C, \tag{1.2}$$

where A is a nonlinear operator. The solution set of the VIP (1.2) is denoted by $VI(C, A)$. For the purpose of modeling mechanics-related problems and solving Signorini problems, Stampacchia and Fichera [14, 15, 35] independently introduced and studied the concept of VIP. Numerous issues in economics, the mathematical sciences, and mathematical physics can be expressed as VIP (1.2) formulations. Due to its fruitful applications to real life problems, many authors have studied and generalized the notion of VIP (1.2) (see [17, 25, 27, 31, 32] and the references therein).

Censor et al. in [10] extended the concept of VIP (1.2) to the following split variational inequality problem (SVIP): Find

$$x^* \in C \text{ that solves } \langle Ax^*, x - x^* \rangle \geq 0, \ \ \forall \ x \in C \tag{1.3}$$

such that $y^* = Tx^* \in Q$ solves

$$\langle By^*, y - y^* \rangle \geq 0, \ \ \forall \ y \in Q, \tag{1.4}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_1, \ B : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear operator. It is easy to see that SVIP (1.3)-(1.4) is a generalization of the SFP (1.1). Letting $A = B = 0$, then the SVIP (1.3)-(1.4) reduces to the SFP (1.1).

Another interesting optimization problem is the monotone inclusion problem (MIP). This problem is defined as:

$$\text{Find } x \in H \text{ such that } \ 0 \in (A + A_1)(x), \tag{1.5}$$

where $A : H \to H$ and $A_1 : H \to 2^H$ are monotone operators. It is well known that if $A_1 = N_C$ is the normal cone of some nonempty, closed, and convex subset C of H, then problem (1.5) becomes the classical VIP (1.2). The problem (1.5) can be used to describe a variety of real-world problems. It is crucial in a variety of mathematical optimization problems, such as variational inequalities problems, minimization problems, linear inverse problems, saddle point problems, fixed point problems, split feasibility problems, Nash equilibrium problems in non-cooperative games, and many other problems (see [4, 7, 8, 12, 13, 28] and the references therein). Numerous authors have introduced and researched various iterative techniques to solve problem (1.5) as a result of its successful applications. The well-known forward-backward splitting technique is one of the simplest iterative techniques for solving problem (1.5). The iterative method is defined as follows:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_\lambda^{A_1}(x_n - \lambda A x_n), \end{cases} \tag{1.6}$$

where $\lambda > 0$ and $J_\lambda^{A_1} := (I + \lambda A)^{-1}$. In the iterative technique (1.6), the individual steps within each iteration involve forward evaluations in which the value of the single-valued operator is computed and the backward evaluations in which the set-valued operator is computed rather than their sum directly. The iterative method converges weakly to a solution of problem (1.5) provided that A is α-inverse strongly monotone. The method may fail to converge if A is monotone and Lipschits continuous. To overcome this setback, Tseng in [34], introduced and studied a modified forward-backward splitting technique. The method is defined as follows:

$$\begin{cases} x_0 \in H, \\ y_n = J_{\lambda_n}^{A_1}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n (A y_n - A x_n), \end{cases} \tag{1.7}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$. It is well known that (1.7) converges weakly when A is monotone and Lipschitz.

Moudafi [22] presented and explored an intriguing generalization of the MIP in [22], namely, the split monotone inclusion problem (SMIP). The problem is defined as follows:

$$\text{Find } \ x \in H_1 \ \text{ such that } \ 0 \in (A + A_1)(x), \tag{1.8}$$

such that

$$y = Tx \in H_2 \quad \text{such that} \;\; 0 \in (B + B_1)(y), \tag{1.9}$$

where $A : H_1 \to H_1, B : H_2 \to H_2$ are single valued operators, $A_1 : H_1 \to 2^{H_1}, B_1 : H_2 \to 2^{H_2}$ are multivalued operators and $T : H_1 \to H_2$ is a bounded linear operator. It is well known that if $A_1 = N_C$ and $B_1 = N_Q$, in problem (1.8)-(1.9), where N_C and N_Q are the normal cones associated with C and Q, respectively, then the SMIP becomes the SVIP (1.3)-(1.4) (see [9, 10]). In addition, we get the split common null point problem (see [6]) as a special case in problem (1.8)-(1.9) if we put $A = 0 = B$. It follows that problem (1.8)-(1.9) is highly generic in nature and naturally comprise a variety of significant optimization problems, such as split saddle point problems, split equilibrium problems, split minimization problems, and split common fixed point problems.

Moudafi [22], gave the following iterative method:

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = J_\mu^{A_1}(I^{H_1} - \mu A)(x_n + \gamma T^*(J_\mu^{B_1}(I^{H_2} - \mu B) - I^{H_2})Tx_n), \end{cases} \tag{1.10}$$

where $\gamma \in (0, \frac{2}{\|T\|}), I^{H_1}, I^{H_2}$ are the identity operators on H_1 and H_2, respectively, and $J_\mu^{A_1}$ and $J_\mu^{B_1}$ are the resolvents of A_1 and B_1, respectively. He established that the iterative sequence $\{x_n\}$ generated by Algorithm (1.10) converges weakly to a solution of (1.8)-(1.9) provided the solution set of problem (1.8)-(1.9) is nonempty, A_1, B_1 are maximal monotone, and A and B are inverse-strongly monotone. Since the introduction of the SMIP, many authors have proposed and studied different iterative techniques to solve the SMIP (see [16, 20, 27, 36] and the references therein). However, all of these authors use the assumption that the operators A and B are inverse-strongly monotone, which may rule out some of the potential applications of these techniques.

To avoid such disadvantage, Izuchukwu et al. [19], proposed the following iterative method:

Algorithm 1.1. Initialization Step: Choose $x_0, x_1 \in H$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.
Step 1: Compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n), \\ z_n &= Tw_n - \zeta\eta_n d_n, \end{aligned}$$

where $d_n := Tw_n - y_n - \lambda_n(ATw_n - Ay_n)$, $\eta_n = \frac{\langle Tw_n - y_n, d_n\rangle}{\|d_n\|^2}$ if $d_n \neq 0$, otherwise, $\eta_n = 0$ and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu_1\|Tw_n - y_n\|}{\|ATw_n - Ay_n\|}, \lambda_n\right\}, & \text{if } ATw_n \neq Ay_n, \\ \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{1.11}$$

Step 2: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{1.12}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 3: Compute

$$u_n = J_{\nu_n}^{B_1}(v_n - \nu_n Bv_n),$$
$$t_n = v_n - \phi\omega_n b_n,$$

where $b_n = v_n - u_n - \nu_n(Bv_n - Bu_n)$, $\omega_n = \frac{\langle v_n - u_n, b_n\rangle}{\|b_n\|^2}$ if $b_n \neq 0$, otherwise, $\omega_n = 0$ and

$$\nu_{n+1} = \begin{cases} \min\left\{\frac{\mu_2\|v_n - u_n\|}{\|Bv_n - Bu_n\|}, \nu_n\right\}, & \text{if } Bv_n \neq Bu_n, \\ \\ \nu_n, & \text{otherwise.} \end{cases} \tag{1.13}$$

Step 4: Compute

$$x_{n+1} = (1 - \alpha_n)w_n + \alpha_n t_n, \tag{1.14}$$

where $J_{\lambda_n}^{A_1}$ and $J_{\nu}^{B_1}$ are the resolvents of A_1 and B_1, respectively.

They established that the iterative sequence $\{x_n\}$ generated by Algorithm 1.1 converges weakly to a solution of (1.8)-(1.9) in as much the solution set of problem (1.8)-(1.9) is nonempty, A_1 and B_1 are maximal monotone, A and B are monotone and Lipschitz continuous with Lipschitz constant L_1 and L_2, respectively. Izuchukwu et al., [19], offered a positive response to the point in Remark 1 above by outlining the subsequent iterative method for resolving the problem (1.8)-(1.9). However, we also observed that

Remark 1.2. The authors could only establish weak convergence. It is well known that strong convergence is more desirable in this area of research. Thus, we are back to the point of asking, if an iterative algorithm can be developed to generate a strong convergence. In addition, can we further generalize the problem (1.8)-(1.9)?

The concept of inertial extrapolation was first suggested in 1964 by Polyak in [24] as a technique for accelerating the process of solving the smooth convex minimization problem. Since then, researchers have used this method to enhance their iterative processes. The first two starting terms of the iterative technique must be used for the inertial technique, and the prior two iterates must be used to determine the subsequent iterate. Numerous authors have improved, expanded, and generalized the inertial extrapolation method since its beginnings, see [1, 2, 3, 27, 37] and the references therein. Among many others, the relaxing technique has proven to increase the rate of convergence of iterative processes.

Motivated by the works of Moudafi [22], Izuchukwu et al. [19], Tseng [34] and the recent interest in this direction of research, our purpose in this study is to introduce and study a generalization of problem (1.8)-(1.9) in the framework of real Hilbert spaces. The problem is defined as follows:

$$\begin{aligned}&\text{Find } x \in H_1 \text{ such that } x \in (A + A_1)^{-1}(0) \cap_{i=1}^{N} F(S_i)\\ &\qquad \text{and } y = Tx \in H_2 \text{ that solves } 0 \in (B + B_1)^{-1}(y),\end{aligned} \tag{1.15}$$

where A, A_1, B, B_1 are as defined above and S is a nonlinear operation and $F(S)$ is the set of fixed point of S. It is easy to see that if $S = I$ (identity mapping), the problem (1.15) becomes (1.8)-(1.9).

In addition, we propose a new relaxed inertial extrapolation viscosity Tseng method with self-adaptive step size for solving (1.15) when the underlying operators B and A are monotone and Lipschitz continuous with Lipschitz constant, which is a weaker assumption when compared to the cost operators that are inverse strongly monotone used in [16, 18, 20, 22] and the references therein.

Furthermore, we prove that the proposed method converges strongly to the solution of (1.15) in real Hilbert spaces. Finally, we present some examples and numerical experiments to show the efficiency and applicability of our method in comparison with some existing ones in the literature in the framework of infinite dimensional Hilbert spaces.

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant to our study. In Section 3, we present our proposed method and highlight some of its useful features. In Section 4, we establish strong convergence of our method and in Section 5, we gave some special cases of our proposed iterative algorithms. In Section 6, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Finally, in Section 7, we gave the conclusion of the paper.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denotes strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well known that

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \tag{2.1}$$

$$\|x \pm y\|^2 \leq \|x\|^2 + 2\langle y, x \pm y\rangle, \tag{2.2}$$

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \tag{2.3}$$

Definition 2.1. Let $T : H \to H$ be an operator. Then the operator T is called

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in H$. If $L = 1$, then T is called nonexpansive;

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H;$$

(c) inversely strongly monotone if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \ \ \forall\, x, y \in H;$$

(d) ρ-demimetric, where $\rho \in (-\infty, 1)$ if $F(T) \neq \emptyset$ such that

$$\langle x - y, (I - T)x\rangle \geq \frac{1}{2}(1 - \rho)\|x - Sx\|^2$$

for all $x \in H$ and $y \in F(T)$. Equivalently,

$$\|Sx - y\|^2 \leq \|x - y\|^2 + \rho\|x - Sx\|^2.$$

If B is a multivalued operator, that is $B : H \to 2^H$, then B is said to be monotone, if

$$\langle x - y, u - v\rangle \geq 0, \ \forall x, y \in H, \ u \in B(x), \ v \in B(y)$$

and B is maximal monotone, if the graph $G(B)$ of B defined by

$$G(B) := \{(x, y) \in H \times H : y \in B(x)\}$$

is not properly contained in the graph of any other monotone operator. It is generally known that B is maximal monotone if and only if for $(x, u) \in$

$H \times H$, $\langle x - y, u - v \rangle \geq 0$ for all $(y, v) \in G(B)$ implies that $u \in B(x)$. Then the resolvent operator $J_\lambda^B : H \to H$ associated with B is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x), \ \forall \, x \in H,$$

where $\lambda > 0$ and I is the identity operator on H.

Lemma 2.2. ([29, 30]) *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to H$ be a ρ-demimetric operator with $\rho \in (-\infty, 1)$ and $F(T) \neq \emptyset$. Let ψ be a real number with $0 < \psi < 1 - \rho$ and let $K = (1 - \psi)I + \psi T$. Then K is a quasinonexpansive operator.*

Lemma 2.3. ([5]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $F : C \to C$ be a nonexpansive mapping. For each $x \in C$ and the Cesaro mean $T_n x = \frac{1}{N}\sum_{i=0}^{N-1} T_i x$, then $\limsup_{n\to\infty} \|T_n x - F(T_n x)\| = 0$.*

Lemma 2.4. ([26]) *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \quad n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim\limits_{k\to\infty} a_n = 0$.

Lemma 2.5. ([21]) *Let H be a real Hilbert space, $A : H \to H$ be a monotone and Lipschitz continuous operator and $A_1 : H \to 2^H$ be maximal monotone operator, then $(A + A_1) : H \to 2^H$ is a maximal operator.*

3. Proposed algorithm

In this section, we present our proposed method for solving a problem (1.15) and highlight some of its important features.

Assumption 3.1. Suppose that
Condition A.

(1) H_1 and H_2 are two real Hilbert spaces.
(2) $B_1 : H_1 \to 2^{H_1}$ and $A_1 : H_2 \to 2^{H_2}$ are maximal monotone mappings and $T : H_1 \to H_2$ is a bounded linear operator with the adjoint operator T^*.

(3) $B : H_1 \to H_1$ and $A : H_2 \to H_2$ are monotone and Lipschitz continuous with Lipschitz constant L_1 and L_2, respectively.
(4) For all $i \in \{1, 2, \cdots, N\}$, $S_i : H_1 \to H_1$ be a finite family of ρ-demimetric operators with $\rho \in (-\infty, 1)$ and such that $I - S_i$ is demiclosed at zero.
(5) $\{S_n\}$ is a sequence of nonexpansive mapping.
(6) $f : H_1 \to H_1$ is a contraction mapping with $k \in [0, 1)$.
(7) The solution set of problem (1.15), denoted by $\Gamma \neq \emptyset$ is nonempty.

Condition B. Suppose that $\{\alpha_n\}$ is a real sequence such that

(1) $\{\beta_n\}, \{\zeta_n\}, \{\eta_n\}$ and $\{\psi_n\}$ are all real sequences in $(0, 1)$ such that $\eta_n \in (\alpha, 1 - \alpha_n)$ for some $\alpha > 0$.
(2) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(3) $\{\epsilon_n\}$ is a positive integer such that $\circ(\alpha_n) = \epsilon_n$ means that

$$\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

We present the following iterative algorithm.

Algorithm 3.2. Initialization Step: Given $\lambda_1, \nu_1 > 0, \mu, \delta \in (0, 1)$ and $\theta \in [0, 1)$, $0 \leq \theta_n \leq \overline{\theta}_n$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.

$$\overline{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 1: Compute

$$\begin{aligned} w_n &= (1 - \alpha_n)x_n + (1 - \alpha_n)\theta_n(S_n x_n - S_n x_{n-1}), \\ y_n &= J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n), \\ z_n &= (1 - \beta_n)Tw_n + \beta_n y_n + \beta_n \lambda_n (ATw_n - Ay_n), \end{aligned}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\|}\right\}, & \text{if } ATw_n \neq Ay_n, \\ \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{3.3}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 3: Compute

$$u_n = J_{\nu_n}^{B_1}(v_n - \nu_n B v_n),$$
$$t_n = (1-\zeta_n)v_n + \zeta_n u_n + \zeta_n \nu_n (Bv_n - Bu_n),$$

where

$$\nu_{n+1} = \begin{cases} \min\left\{\nu_n, \frac{\delta\|v_n - u_n\|}{\|Bv_n - Bu_n\|}\right\}, & \text{if } Bu_n \neq Bv_n, \\ \\ \nu_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n - \eta_n)x_n + \eta_n T_n t_n, \tag{3.5}$$

where $T_n = \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_n)I + \psi_n S_i)$.

Remark 3.3. (1) If $\{\beta_n\} = \{\zeta_n\} = 1$, we obtain a viscosity Tseng iterative method for solving (1.15).

(2) We note that $T_n = \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_n)I + \psi_n S_i)$ is quasi-nonexpansive mapping. To see this, let $p \in \Gamma$ and using Lemma 3.9, we get

$$\begin{aligned} \|T_n x - p\| &= \|\frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_n)I + \psi_n S_i)x - p\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|((1-\psi_n)I + \psi_n S_i)x - p\| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\|x - p\| \\ &= \|x - p\|. \end{aligned} \tag{3.6}$$

Thus, T_n is quasi-nonexpansive.

(3) The choice of the step size $\{\gamma_n\}$ in Algorithm 3.2 do not require the prior knowledge of the operator norm $\|T\|$. In addition, the step size is well defined. To see this, let $p \in \Gamma$, then $0 \in (A_1 + A)Tp$ and since $y_n = J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n)$, we obtain

$$Ay_n + \frac{1}{\lambda_n}(Tw_n - \lambda_n ATw_n - y_n) \in (A_1 + A)y_n. \tag{3.7}$$

Using Lemma 2.5, we have $\langle y_n - Tp, Ay_n + \frac{1}{\lambda_n}(Tw_n - \lambda_n ATw_n - y_n)\rangle \geq 0$, we thus get

$$\langle y_n - Tp, Tw_n - y_n - \lambda_n(ATw_n - Ay_n)\rangle \geq 0,$$

which is equivalent to

$$2\langle y_n - Tp, Tw_n - y_n\rangle - 2\lambda_n\langle y_n - Tp, ATw_n - Ay_n\rangle \geq 0.$$

Now, observe that

$$\begin{aligned}2\langle y_n - Tp, Tw_n - y_n\rangle = \|Tw_n - Tp\|^2 - \|y_n - Tp\|^2 \\ - \|Tw_n - y_n\|^2.\end{aligned} \tag{3.8}$$

Consequently, we have

$$\begin{aligned}\|y_n - Tp\|^2 \leq \|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 \\ - 2\lambda_n\langle y_n - Tp, ATw_n - Ay_n\rangle.\end{aligned} \tag{3.9}$$

It also follows that

$$\begin{aligned}
&\|z_n - Tp\|^2 \\
&\quad = \|(1-\beta_n)Tw_n + \beta_n y_n + \beta_n\lambda_n(ATw_n - Ay_n) - Tp\|^2 \\
&\quad = \|(1-\beta_n)(Tw_n - Tp) + \beta_n(y_n - Tp) + \beta_n\lambda_n(ATw_n - Ay_n)\|^2 \\
&\quad = (1-\beta_n)^2\|Tw_n - Tp\|^2 + \beta_n^2\|y_n - Tp\|^2 + \beta_n^2\lambda_n^2\|ATw_n - Ay_n)\|^2 \\
&\qquad + 2\beta_n(1-\beta_n)\langle Tw_n - Tp, y_n - Tp\rangle \\
&\qquad + 2\lambda_n\beta_n(1-\beta_n)\langle Tw_n - Tp, ATw_n - Ay_n\rangle \\
&\qquad + 2\lambda_n\beta_n^2\langle y_n - Tp, ATw_n - Ay_n\rangle \\
&\quad = (1-\beta_n)^2\|Tw_n - Tp\|^2 + \beta_n^2\|y_n - Tp\|^2 + \beta_n^2\lambda_n^2\|ATw_n - Ay_n)\|^2 \\
&\qquad + \beta_n(1-\beta_n)[\|Tw_n - Tp\|^2 + \|y_n - Tp\|^2 - \|Tw_n - y_n\|^2] \\
&\qquad + 2\lambda_n\beta_n(1-\beta_n)\langle Tw_n - Tp, ATw_n - Ay_n\rangle \\
&\qquad + 2\lambda_n\beta_n^2\langle y_n - Tp, ATw_n - Ay_n\rangle \\
&\quad = (1-\beta_n)\|Tw_n - Tp\|^2 + \beta_n\|y_n - Tp\|^2 + \beta_n^2\lambda_n^2\|ATw_n - Ay_n)\|^2 \\
&\qquad - \beta_n(1-\beta_n)\|Tw_n - y_n\|^2 + 2\lambda_n\beta_n(1-\beta_n)\langle Tw_n - Tp, ATw_n - Ay_n\rangle \\
&\qquad + 2\lambda_n\beta_n^2\langle y_n - Tp, ATw_n - Ay_n\rangle \\
&\quad \leq (1-\beta_n)\|Tw_n - Tp\|^2 + \beta_n[\|Tw_n - Tp\|^2 - \|Tw_n - y_n\|^2 \\
&\qquad - 2\lambda_n\langle y_n - Tp, ATw_n - Ay_n\rangle] \\
&\qquad - \beta_n(1-\beta_n)\|Tw_n - y_n\|^2 + \beta_n^2\lambda_n^2\|ATw_n - Ay_n\|^2 \\
&\qquad + 2\lambda_n\beta_n(1-\beta_n)\langle Tw_n - Tp, ATw_n - Ay_n\rangle \\
&\qquad + 2\lambda_n\beta_n^2\langle y_n - Tp, ATw_n - Ay_n\rangle \\
&\quad = \|Tw_n - Tp\|^2 - \beta_n(2-\beta_n)\|Tw_n - y_n\|^2 + \beta_n^2\lambda_n^2\|ATw_n - Ay_n\|^2 \\
&\qquad + 2\lambda_n\beta_n(1-\beta_n)\langle ATw_n - Ay_n, Tw_n - y_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|Tw_n - Tp\|^2 - \beta_n(2-\beta_n)\|Tw_n - y_n\|^2 + \beta_n^2 \frac{\mu\lambda_n^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2 \\
&\quad + 2\lambda_n\beta_n(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}\|Tw_n - y_n\|^2 \\
&\leq \|Tw_n - Tp\|^2 - \beta_n(2-\beta_n)\|Tw_n - y_n\|^2 + \beta_n \frac{\mu\lambda_n^2}{\lambda_{n+1}^2}\|Tw_n - y_n\|^2 \\
&\quad + 2\lambda_n\beta_n(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}\|Tw_n - y_n\|^2 \\
&= \|Tw_n - Tp\|^2 - \beta_n\left[2 - \beta_n - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}\right]\|Tw_n - y_n\|^2. \quad (3.10)
\end{aligned}
$$

Since $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\lambda_{n+1}$, and say $\lim_{n\to\infty}\beta_n = \beta \in (0,1)$ we have

$$
\beta_n\left[2 - \beta_n - \mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2} - 2(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}\right] = \beta[2 - \beta - \mu^2 - 2(1-\beta)\mu] > 0.
$$

Thus, we get

$$
\|z_n - Tp\|^2 \leq \|Tw_n - Tp\|^2, \quad (3.11)
$$

which implies

$$
\|z_n - Tp\| \leq \|Tw_n - Tp\|. \quad (3.12)
$$

Using the Cauchy-Schwarz inequality and (3.11), we obtain

$$
\begin{aligned}
\|T^*(Tw_n - z_n)\|\|w_n - p\| &\geq \langle T^*(Tw_n - z_n), w_n - p\rangle \\
&= \langle Tw_n - z_n, Tw_n - Tp\rangle \\
&= \frac{1}{2}[\|Tw_n - z_n\|^2 + \|Tw_n - Tp\|^2 - \|z_n - Tp\|^2] \\
&\geq \frac{1}{2}\|Tw_n - z_n\|^2. \quad (3.13)
\end{aligned}
$$

Since $z_n \neq Tw_n$, we have $\|Tw_n - z_n\| \geq 0$, thus, we obtain that $\|T^*(Tw_n - z_n)\|\|w_n - p\| > 0$. Hence, we have

$$
\|T^*(Tw_n - z_n)\| \neq 0
$$

and so γ_n is well defined.

(4) The proposed algorithm can be viewed as a modified relaxed inertial Tseng method with self-adaptive step size that is generated at each iteration by some simple computations. Thus, the implementation and applicability of our method do not depend on the Lipschitz constant of the cost operator. This iterative technique uses both inertial and relaxation techniques, the scheme is constructed from an explicit discretization of a dynamical system in time.

4. Convergence analysis

In section, we establish strong convergence of our method.

Lemma 4.1. *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.2*, under Assumption* 3.1*. Then we obtain that* $\{x_n\}$ *is bounded.*

Proof. Let $p \in \Gamma$. By using the definition of w_n in Algorithm 3.2, we obtain

$$\begin{aligned}\|w_n - p\| &= \|(1-\alpha_n)x_n + (1-\alpha_n)\theta_n(S_nx_n - S_nx_{n-1}) - p\| \\ &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(S_nx_n - S_nx_{n-1}) - \alpha_n p\| \\ &\le (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\theta_n\|S_nx_n - S_nx_{n-1}\| + \alpha_n\|p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\right]. \end{aligned} \tag{4.1}$$

Using (5.15), we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le \frac{\epsilon_n}{\alpha_n} \to 0$. Hence, we have

$$\lim_{n\to\infty}\left[(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|\right] = \|p\|,$$

hence, there exists $M > 0$ such that

$$(1-\alpha_n)\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\| \le M.$$

Thus, (4.1) becomes

$$\begin{aligned}\|w_n - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n M \\ &\le \|x_n - p\| + \alpha_n M. \end{aligned} \tag{4.2}$$

Furthermore, using Algorithm 3.2 and the step size, we have

$$\begin{aligned}\|v_n - p\|^2 &= \|w_n + \gamma_n T^*(z_n - Tw_n) - p\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle w_n - p, T^*(z_n - Tw_n)\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + 2\gamma_n\langle Tw_n - Tp, z_n - Tw_n\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|z_n - Tp\|^2 \\ &\quad - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\ &\le \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 + \gamma_n\|Tw_n - Tp\|^2 \\ &\quad - \gamma_n\|Tw_n - Tp\|^2 - \gamma_n\|z_n - Tw_n\|^2 \\ &\le \|w_n - p\|^2 + \gamma_n^2\|T^*(z_n - Tw_n)\|^2 - \gamma_n(\gamma_n + \epsilon)\|T^*(z_n - Tw_n)\|^2 \\ &= \|w_n - p\|^2 - \gamma_n\epsilon\|T^*(z_n - Tw_n)\|^2 \\ &\le \|w_n - p\|^2, \end{aligned} \tag{4.3}$$

which implies that

$$\|v_n - p\| \le \|w_n - p\|. \tag{4.4}$$

Using a similar technique as above, we obtain

$$\begin{aligned}\|t_n - p\|^2 \le & \|v_n - p\|^2 \\ & - \zeta_n \left[2 - \zeta_n - \delta^2 \frac{\nu_n^2}{\nu_{n+1}^2} - 2(1-\beta_n)\frac{\delta \nu_n}{\nu_{n+1}}\right] \|v_n - u_n\|^2 \\ \le & \|v_n - p\|^2, \end{aligned} \tag{4.5}$$

which implies that

$$\|t_n - p\| \le \|v_n - p\|. \tag{4.6}$$

Finally, it is easy to see that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n-\eta_n)x_n + \eta_n T_n t_n - p\| \\ &\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| \\ &\quad + (1-\alpha_n-\eta_n)\|x_n - p\| + \eta_n \|T_n t_n - p\| \\ &\le \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\quad + (1-\alpha_n-\eta_n)\|x_n - p\| + \eta_n \|t_n - p\| \\ &\le (1 - \alpha_n(1-k) - \eta_n)\|x_n - p\| \\ &\quad + \alpha_n \|f(p) - p\| + \eta_n \|v_n - p\| \\ &\le (1 - \alpha_n(1-k) - \eta_n)\|x_n - p\| \\ &\quad + \alpha_n \|f(p) - p\| + \eta_n \|w_n - p\| \\ &\le (1 - \alpha_n(1-k) - \eta_n)\|x_n - p\| \\ &\quad + \alpha_n \|f(p) - p\| + \eta_n \|x_n - p\| + \alpha_n M \\ &= (1 - \alpha_n(1-k))\|x_n - p\| + \alpha_n(1-k)\left[\frac{\|f(p) - p\| + M}{(1-k)}\right] \\ &\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\| + M}{(1-k)}\right\}. \end{aligned} \tag{4.7}$$

It follows by induction that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\| + M}{(1-k)}\right\}. \tag{4.8}$$

Hence, $\{x_n\}$ is bounded. □

Theorem 4.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then under the Assumption 3.1, $\{x_n\}$ converges strongly to $p \in \Gamma$, where $p = P_\Gamma \circ f(p)$.*

Proof. Let $p \in \Gamma$, and Algorithm 3.2, we get

$$\begin{aligned}
\|w_n - p\|^2 &= \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(S_n x_n - S_n x_{n-1}) - \alpha_n p\|^2 \\
&\le \|(1-\alpha_n)(x_n - p) + (1-\alpha_n)\theta_n(S_n x_n - S_n x_{n-1})\|^2 \\
&\quad - 2\alpha_n\langle p, w_n - p\rangle \\
&= (1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2\theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
&\quad + 2\theta_n(1-\alpha_n)\|x_n - p\|\|S_n x_n - S_n x_{n-1}\| \\
&\quad - 2\alpha_n[\langle p, w_n - x_{n+1}\rangle + \langle p, x_{n+1} - p\rangle] \\
&\le (1-\alpha_n)^2\|x_n - p\|^2 + (1-\alpha_n)\theta_n^2\|x_n - x_{n-1}\|^2 \\
&\quad + 2\theta_n(1-\alpha_n)\|x_n - p\|\|x_n - x_{n-1}\| \\
&\quad - 2\alpha_n[\langle p, w_n - x_{n+1}\rangle + \langle p, x_{n+1} - p\rangle] \\
&\le (1-\alpha_n)^2\|x_n - p\|^2 \\
&\quad + (1-\alpha_n)\theta_n\|x_n - x_{n-1}\|[\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|] \\
&\quad + 2\alpha_n\|p\|\|w_n - x_{n+1}\| + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\quad + 2\alpha_n\|p\|\|x_{n+1} - w_n\|, \qquad (4.9)
\end{aligned}$$

where $N := \sup_{n\in\mathbb{N}}\{\theta_n\|x_n - x_{n-1}\|, 2\|x_n - p\|\}$.

Furthermore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n-\eta_n)x_n + \eta_n T_n t_n - p\|^2 \\
&= \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)((1-\eta_n)x_n + \eta_n T_n t_n - p) \\
&\quad - \alpha_n(x_n - ((1-\eta_n)x_n + \eta_n T_n t_n)) - \alpha_n(p - f(p))\|^2 \\
&\le \|(1-\alpha_n)[(1-\eta_n)x_n + \eta_n T_n t_n - p] + \alpha_n(f(x_n) - f(p))\|^2 \\
&\quad - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_n T_n t_n) + p - f(p), x_{n+1} - p\rangle \\
&\le (1-\alpha_n)\|(1-\eta_n)x_n + \eta_n T_n t_n - p\|^2 + \alpha_n k\|x_n - p\|^2 \\
&\quad - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_n T_n t_n) + p - f(p), x_{n+1} - p\rangle \\
&= (1-\alpha_n)[(1-\eta_n)\|x_n - p\|^2 + \eta_n\|T_n t_n - p\|^2 \\
&\quad - \eta_n(1-\eta_n)\|T_n t_n - x_n\|^2] + \alpha_n k\|x_n - p\|^2 \\
&\quad - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_n T_n t_n) + p - f(p), x_{n+1} - p\rangle \\
&\le (1-\alpha_n)[(1-\eta_n)\|x_n - p\|^2 + \eta_n\|T_n t_n - p\|^2 \\
&\quad - \eta_n(1-\eta_n)\|T_n t_n - x_n\|^2] + \alpha_n k\|x_n - p\|^2 \\
&\quad - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_n T_n t_n) + p - f(p), x_{n+1} - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n\|t_n-p\|^2-\eta_n(1-\eta_n)\|T_nt_n-x_n\|^2] \\
&\quad +\alpha_nk\|x_n-p\|^2-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n)+p-f(p),x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n\|v_n-p\|^2-\eta_n(1-\eta_n)\|T_nt_n-x_n\|^2] \\
&\quad +\alpha_nk\|x_n-p\|^2-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n)+p-f(p),x_{n+1}-p\rangle \\
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n\|w_n-p\|^2-\eta_n(1-\eta_n)\|T_nt_n-x_n\|] \\
&\quad +\alpha_nk\|x_n-p\|^2-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n)+p-f(p),x_{n+1}-p\rangle \\
&\leq (1-\alpha_n(1-k))\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N+2\eta_n(1-\alpha_n)\alpha_n\langle p,p-x_{n+1}\rangle \\
&\quad +2\alpha_n\|p\|\|x_{n+1}-w_n\|-(1-\alpha_n)\eta_n(1-\eta_n)\|T_nt_n-x_n\| \\
&\quad +2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n),x_{n+1}-p\rangle+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle \\
&= (1-\alpha_n(1-k))\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N+2\eta_n(1-\alpha_n)\alpha_n\langle p,p-x_{n+1}\rangle \\
&\quad +2\alpha_n\|p\|\|x_{n+1}-w_n\|-(1-\alpha_n)\eta_n(1-\eta_n)\|T_nt_n-x_n\| \\
&\quad +2\alpha_n\|x_n-((1-\eta_n)x_n+\eta_nT_nt_n)\|\|x_{n+1}-p\|+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle \\
&\leq (1-\alpha_n(1-k))\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N+2\eta_n(1-\alpha_n)\alpha_n\langle p,p-x_{n+1}\rangle \\
&\quad +2\alpha_n\|p\|\|x_{n+1}-w_n\|+2\alpha_n\eta_n\|x_n-T_nt_n\|\|x_{n+1}-p\| \\
&\quad +2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle \\
&= (1-\alpha_n(1-k))\|x_n-p\|^2+\alpha_n(1-k)\Big[\frac{\theta_n}{\alpha_n(1-k)}\|x_n-x_{n-1}\|N \\
&\quad +\frac{2\eta_n(1-\alpha_n)}{(1-k)}\langle p,p-x_{n+1}\rangle+\frac{2}{(1-k)}\|p\|\|x_{n+1}-w_n\|] \\
&\quad +\frac{2\eta_n}{(1-k)}\|x_n-T_nt_n\|\|x_{n+1}-p\|+\frac{2}{(1-k)}\langle f(p)-p,x_{n+1}-p\rangle\Big] \\
&= (1-\alpha_n(1-k))\|x_n-p\|^2+\alpha_n(1-k)\Psi_n \qquad (4.10)
\end{aligned}
$$

where,

$$
\begin{aligned}
\Psi_n = &\frac{\theta_n}{\alpha_n(1-k)}\|x_n-x_{n-1}\|N+\frac{2\eta_n(1-\alpha_n)}{(1-k)}\langle p,p-x_{n+1}\rangle \\
&+\frac{2}{(1-k)}\|p\|\|x_{n+1}-w_n\|+\frac{2\eta_n}{(1-k)}\|x_n-T_nt_n\|\|x_{n+1}-p\| \\
&+\frac{2}{(1-k)}\langle f(p)-p,x_{n+1}-p\rangle.
\end{aligned}
$$

According to Lemma 2.4, to conclude our proof, it is sufficient to establish that

$$\limsup_{k\to\infty}\Psi_{n_k}\leq 0$$

for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \tag{4.11}$$

From (4.10), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n(1-k))\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N \\ &\quad - 2\eta_n(1-\alpha_n)\alpha_n\langle p, p - x_{n+1}\rangle \\ &\quad + 2\alpha_n\|p\|\|x_{n+1} - w_n\| - (1-\alpha_n)\eta_n(1-\eta_n)\|T_nt_n - x_n\| \\ &\quad + 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_nT_nt_n), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle \\ &\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N - 2\eta_n(1-\alpha_n)\alpha_n\langle p, p - x_{n+1}\rangle \\ &\quad + 2\alpha_n\|p\|\|x_{n+1} - w_n\| - (1-\alpha_n)\eta_n(1-\eta_n)\|T_nt_n - x_n\| \\ &\quad + 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_nT_nt_n), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle f(p) - p, x_{n+1} - p\rangle, \end{aligned} \tag{4.12}$$

which implies that

$$\begin{aligned}&\limsup_{k\to\infty}\left((1-\alpha_{n_k})\eta_{n_k}(1-\eta_{n_k})\|T_{n_k}t_{n_k} - x_{n_k}\|^2\right) \\ &\quad\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k} - p\|^2 + \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|N \\ &\qquad - 2\alpha_{n_k}(1-\alpha_{n_k})\eta_{n_k}\langle p, p - x_{n_k+1}\rangle + 2\alpha_{n_k}\|p\|\|x_{n_k+1} - w_{n_k}\| \\ &\qquad + 2\alpha_{n_k}\langle x_{n_k} - ((1-\eta_{n_k})x_{n_k} + \eta_{n_k}T_{n_k}t_{n_k}), x_{n_k+1} - p\rangle \\ &\qquad + 2\alpha_{n_k}\langle f(p) - p, x_{n_k+1} - p\rangle - \|x_{n_k+1} - p\|^2\Bigg] \\ &\quad\leq -\liminf_{k\to\infty}[\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2] \leq 0.\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|T_{n_k}t_{n_k} - x_{n_k}\| = 0. \tag{4.13}$$

In addition, it is easy to see from (4.10) and (4.3) that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad\leq (1-\alpha_n)[(1-\eta_n)\|x_n - p\|^2 + \eta_n\|v_n - p\|^2 - \eta_n(1-\eta_n)\|T_nt_n - x_n\|^2] \\ &\qquad + \alpha_nk\|x_n - p\|^2 - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_nT_nt_n) + p - f(p), x_{n+1} - p\rangle \\ &\quad\leq (1-\alpha_n)[(1-\eta_n)\|x_n - p\|^2 + \eta_n\|w_n - p\|^2 - \eta_n\epsilon^2\|T^*(z_n - Tw_n)\|^2] \\ &\qquad + \alpha_nk\|x_n - p\|^2 - 2\alpha_n\langle x_n - ((1-\eta_n)x_n + \eta_nT_nt_n) + p - f(p), x_{n+1} - p\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N \\
&\quad -2\alpha_n\langle p,p-x_{n+1}\rangle+2\alpha_n\|p\|\|x_{n+1}-w_n\|]-\eta_n\epsilon^2\|T^*(z_n-Tw_n)\|^2] \\
&\quad +\alpha_n k\|x_n-p\|^2-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n)+p-f(p),x_{n+1}-p\rangle \\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N-2\alpha_n\eta_n(1-\alpha_n)\langle p,p-x_{n+1}\rangle \\
&\quad +2\alpha_n\|p\|\|x_{n+1}-w_n\|-\eta_n(1-\alpha_n)\epsilon^2\|T^*(z_n-Tw_n)\|^2 \\
&\quad -2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_nT_nt_n)+p-f(p),x_{n+1}-p\rangle, \qquad (4.14)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(\eta_n(1-\alpha_n)\epsilon^2\|T^*(z_n-Tw_n)\|^2\right) \\
&\quad\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N \\
&\qquad -2\alpha_{n_k}(1-\alpha_{n_k})\eta_{n_k}\langle p,p-x_{n_k+1}\rangle+2\alpha_{n_k}\|p\|\|x_{n_k+1}-w_{n_k}\| \\
&\qquad +2\alpha_{n_k}\langle x_{n_k}-((1-\eta_{n_k})x_{n_k}+\eta_{n_k}T_{n_k}t_{n_k}),x_{n_k+1}-p\rangle \\
&\qquad +2\alpha_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Bigg] \\
&\quad\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2] \\
&\quad\leq 0.
\end{aligned}
$$

Thus, we obtain

$$
\lim_{k\to\infty}\|T^*(z_{n_k}-Tw_{n_k})\|=0. \qquad (4.15)
$$

Using (3.13), we have

$$
\|Tw_{n_k}-z_{n_k}\|\leq 2\|T^*(z_{n_k}-Tw_{n_k})\|\|w_{n_k}-p\|, \qquad (4.16)
$$

thus, using (4.23), we get

$$
\lim_{k\to\infty}\|Tw_{n_k}-z_{n_k}\|=0. \qquad (4.17)
$$

Furthermore, observe that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad=\|\alpha_n(f(p)-p)+(1-\alpha_n-\eta_n)(x_n-p)+\eta_n(T_nt_n-p)\|^2 \\
&\quad=\alpha_n\|f(p)-p\|^2+(1-\alpha_n-\eta_n)\|x_n-p\|^2+\eta_n\|T_nt_n-p\|^2 \\
&\qquad -\alpha_n\|f(x_n)-x_n\|^2 \\
&\qquad -\eta_n(1-\alpha_n-\eta_n)\|x_n-T_nt_n\|^2-\alpha_n\eta_n\|f(x_n)-T_nt_n\|^2 \\
&\quad\leq \alpha_n\|f(p)-p\|^2+(1-\alpha_n-\eta_n)\|x_n-p\|^2+\eta_n\|t_n-p\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n\|f(p)-p\|^2 + (1-\alpha_n-\eta_n)\|x_n-p\|^2 \\
&\quad + \eta_n\left[\|v_n-p\|^2 - \zeta_n\left[2-\zeta_n-\delta^2\frac{\nu_n^2}{\nu_{n+1}^2} - 2(1-\beta_n)\frac{\delta\nu_n}{\nu_{n+1}}\right]\|v_n-u_n\|^2\right] \\
&\le \alpha_n\|f(p)-p\|^2 + (1-\alpha_n-\eta_n)\|x_n-p\|^2 \\
&\quad + \eta_n\left[\|w_n-p\|^2 - \zeta_n\left[2-\zeta_n-\delta^2\frac{\nu_n^2}{\nu_{n+1}^2} - 2(1-\beta_n)\frac{\delta\nu_n}{\nu_{n+1}}\right]\|v_n-u_n\|^2\right] \\
&\le \alpha_n\|f(p)-p\|^2 + (1-\alpha_n-\eta_n)\|x_n-p\|^2 + \eta_n[\|x_n-p\|^2 \\
&\quad + \theta_n\|x_n-x_{n-1}\|N - 2\alpha_n\langle p, p-x_{n+1}\rangle + 2\alpha_n\|p\|\|x_{n+1}-w_n\|] \\
&\quad - \eta_n\zeta_n\left[2-\zeta_n-\delta^2\frac{\nu_n^2}{\nu_{n+1}^2} - 2(1-\beta_n)\frac{\delta\nu_n}{\nu_{n+1}}\right]\|v_n-u_n\|^2 \\
&\le \alpha_n\|f(p)-p\|^2 + \|x_n-p\|^2 + \theta_n\|x_n-x_{n-1}\|N \\
&\quad - 2\eta_n\alpha_n\langle p, p-x_{n+1}\rangle + 2\eta_n\alpha_n\|p\|\|x_{n+1}-w_n\| \\
&\quad - \eta_n\zeta_n\left[2-\zeta_n-\delta^2\frac{\nu_n^2}{\nu_{n+1}^2} - 2(1-\beta_n)\frac{\delta\nu_n}{\nu_{n+1}}\right]\|v_n-u_n\|^2, \qquad (4.18)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(\eta_{n_k}\zeta_{n_k}[2-\zeta_{n_k}-\delta^2\frac{\nu_{n_k}^2}{\nu_{n_k+1}^2} - 2(1-\beta_{n_k})\frac{\delta\nu_{n_k}}{\nu_{n_k+1}}]\|v_{n_k}-u_{n_k}\|^2\right) \\
&\quad \le \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2 + \alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N - 2\alpha_{n_k}\eta_{n_k}\langle p, p-x_{n_k+1}\rangle \\
&\qquad + 2\alpha_{n_k}\|p\|\|x_{n_k+1}-w_{n_k}\| + \alpha_{n_k}\|f(p)-p\|^2 - \|x_{n_k+1}-p\|^2\Bigg] \\
&\quad \le -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2 - \|x_{n_k}-p\|^2] \le 0.
\end{aligned}
$$

Thus, we have

$$
\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\| = 0. \qquad (4.19)
$$

Hence

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2 + \eta_n\|v_n-p\|^2 - \eta_n(1-\eta_n)\|T_nt_n-x_n\|] \\
&\qquad + \alpha_n k\|x_n-p\|^2 - 2\alpha_n\langle x_n - ((1-\eta_n)x_n+\eta_nT_nt_n) + p - f(p), x_{n+1}-p\rangle \\
&\quad \le (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2 + \eta_n\|v_n-p\|^2] + \alpha_n k\|x_n-p\|^2 \\
&\qquad - 2\alpha_n\langle x_n - ((1-\eta_n)x_n+\eta_nT_nt_n) + p - f(p), x_{n+1}-p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n\|w_n-p\|^2+\eta_n\gamma_n^2\|T^*(z_n-Tw_n)\|^2\\
&\quad+\gamma_n\eta_n\|z_n-Tp\|^2-\gamma_n\eta_n\|Tw_n-Tp\|^2]+\alpha_n k\|x_n-p\|^2\\
&\quad-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_n T_n t_n)+p-f(p),x_{n+1}-p\rangle\\
&\leq (1-\alpha_n)[(1-\eta_n)\|x_n-p\|^2+\eta_n\|w_n-p\|^2+\eta_n\gamma_n^2\|T^*(z_n-Tw_n)\|^2\\
&\quad+\gamma_n\eta_n\|Tw_n-Tp\|^2-\beta_n\gamma_n\eta_n[2-\beta_n-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\\
&\quad-2(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}]\|Tw_n-y_n\|^2-\gamma_n\eta_n\|Tw_n-Tp\|^2]+\alpha_n k\|x_n-p\|^2\\
&\quad-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_n T_n t_n)+p-f(p),x_{n+1}-p\rangle\\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N-2\eta_n\alpha_n\langle p,p-x_{n+1}\rangle\\
&\quad+2\alpha_n\|p\|\|x_{n+1}-w_n\|+\epsilon^2\|T^*(z_n-Tw_n)\|^2-\beta_n\epsilon\eta_n[2-\beta_n-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\\
&\quad-2(1-\beta_n)\frac{\mu\lambda_n}{\lambda_{n+1}}]\|Tw_n-y_n\|^2\\
&\quad-2\alpha_n\langle x_n-((1-\eta_n)x_n+\eta_n T_n t_n)+p-f(p),x_{n+1}-p\rangle, \qquad (4.20)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(\beta_{n_k}\eta_{n_k}\epsilon[2-\beta_{n_k}-\mu^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}-2(1-\beta_{n_k})\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}]\|Tw_{n_k}-y_{n_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|N-2\alpha_{n_k}\eta_{n_k}\langle p,p-x_{n_k+1}\rangle\\
&\quad+2\alpha_{n_k}\|p\|\|x_{n_k+1}-w_{n_k}\|+\epsilon_n^2\|T^*(z_n-Tw_n)\|^2)\\
&\quad-2\alpha_{n_k}\langle x_{n_k}-((1-\eta_{n_k})x_{n_k}+\eta_{n_k}T_{n_k}t_{n_k})+p-f(p),x_{n_k+1}-p\rangle\\
&\quad-\|x_{n_k+1}-p\|^2\Bigg]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.
\end{aligned}
$$

Thus, we obtain

$$\lim_{k\to\infty}\|Tw_{n_k}-y_{n_k}\|=0. \qquad (4.21)$$

In addition, observe that

$$
\begin{aligned}
\|t_{n_k}-v_{n_k}\|&=\|(1-\zeta_{n_k})v_{n_k}+\zeta_{n_k}u_{n_k}+\zeta_{n_k}\nu_{n_k}(Bv_{n_k}-Bu_{n_k})-v_{n_k}\|\\
&\leq \zeta_{n_k})\|u_{n_k}-u_{n_k}\|+\zeta_{n_k}\nu_{n_k}\|Bv_{n_k}-Bu_{n_k}\|\\
&=\zeta_{n_k}\|u_{n_k}-u_{n_k}\|+\frac{\delta\zeta_{n_k}\nu_{n_k}}{\nu_{n_k+1}}\|v_{n_k}-u_{n_k}\|. \qquad (4.22)
\end{aligned}
$$

Thus, using (4.19), we have

$$\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|=0, \tag{4.23}$$

$$\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|\le\lim_{k\to\infty}\|z_{n_k}-Tw_{n_k}\|+\lim_{k\to\infty}\|Tw_{n_k}-y_{n_k}\|=0, \tag{4.24}$$

$$\lim_{k\to\infty}\|t_{n_k}-u_{n_k}\|\le\lim_{k\to\infty}\|t_{n_k}-v_{n_k}\|+\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|=0. \tag{4.25}$$

In addition, we have that

$$\begin{aligned}\|w_{n_k}-x_{n_k}\|&\le\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\alpha_{n_k}\|x_{n_k}\|\\&\quad+\alpha_{n_k}^2\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\\&\to 0\quad\text{as}\quad k\to\infty.\end{aligned}\tag{4.26}$$

Also, we have

$$\|v_{n_k}-x_{n_k}\|\le\|w_{n_k}-x_{n_k}\|+\gamma_{n_k}\|T^*(z_{n_k}-Tw_{n_k})\|\to 0\text{ as }k\to\infty,$$

$$\|t_{n_k}-x_{n_k}\|\le\|t_{n_k}-v_{n_k}\|+\|v_{n_k}-x_{n_k}\|\to 0\text{ as }k\to\infty, \tag{4.27}$$

$$\begin{aligned}\|t_{n_k}-w_{n_k}\|&\le\|t_{n_k}-x_{n_k}\|+\|x_{n_k}-w_{n_k}\|\to 0\text{ as }k\to\infty,\\\|y_{n_k}-x_{n_k}\|&\le\|y_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0\text{ as }k\to\infty,\\\|z_{n_k}-x_{n_k}\|&\le\|z_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\to 0\text{ as }k\to\infty.\end{aligned}$$

Using Lemma 2.3, we have

$$\begin{aligned}\|t_{n_k}-T_{n_k}t_{n_k}\|\le\|t_{n_k}-x_{n_k}\|+\|x_{n_k}-T_{n_k}t_{n_k}\|\to 0\quad\text{as}\quad k\to\infty,\\\limsup_{k\to\infty}\|T_{n_k}t_{n_k}-F(T_{n_k}t_{n_k})\|=0.\end{aligned}\tag{4.28}$$

And, we have

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\|&=\|\alpha_{n_k}f(x_{n_k})+(1-\alpha_{n_k}-\eta_{n_k})x_{n_k}+\eta_{n_k}T_{n_k}t_{n_k}-x_{n_k}\|\\&=\|\alpha_{n_k}(f(x_{n_k})-x_{n_k})+\eta_{n_k}(T_{n_k}t_{n_k}-x_{n_k})\|\\&=\alpha_{n_k}\|f(x_{n_k})-x_{n_k}\|+\eta_{n_k}\|T_{n_k}t_{n_k}-x_{n_k})\|\\&\to 0\quad\text{as}\quad k\to\infty\end{aligned}\tag{4.29}$$

and

$$\|x_{n_k+1}-w_{n_k}\|\le\|x_{n_k+1}-x_{n_k}\|+\|x_{n_k}-w_{n_k}\|\to\ 0\quad\text{as}\quad n\to\infty. \tag{4.30}$$

Now, since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H_1$. In addition, using (4.27) and the boundedness of $\{t_{n_k}\}$, there exists a subsequence $\{t_{n_{k_j}}\}$ of $\{t_{n_k}\}$ such that $\{t_{n_{k_j}}\}$ converges weakly to $x^*\in H_1$ and since T_n is demiclosed with (4.28), we have that $x^* \in F(T_n) = F(S_i)$. Let $(v,u) \in G(A+A_1)$. Then $u - Av \in A_1 v$. Also it follow from (3.7), $\frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - \lambda_{n_{k_j}} ATw_{n_{k_j}} - y_{n_{k_j}}) \in Py_{n_{k_j}}$. Thus, using the monotonicity of A_1, we have

$$\left\langle v - y_{n_{k_j}}, u - Av - \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - \lambda_{n_{k_j}} ATw_{n_{k_j}} - y_{n_{k_j}}) \right\rangle \geq 0. \tag{4.31}$$

Using (4.31) and the monotonicity of A, we have

$$\begin{aligned} \langle v - y_{n_{k_j}}, u \rangle &\geq \left\langle v - y_{n_{k_j}}, Av + \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}}) - ATw_{n_{k_j}} \right\rangle \\ &= \langle v - y_{n_{k_j}}, Av - Ay_{n_{k_j}} \rangle + \langle v - y_{n_{k_j}}, Ay_{n_{k_j}} - ATw_{n_{k_j}} \rangle \\ &\quad + \left\langle v - y_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}}) \right\rangle \\ &\geq \langle v - y_{n_{k_j}}, Ay_{n_{k_j}} - ATw_{n_{k_j}} \rangle \\ &\quad + \left\langle v - y_{n_{k_j}}, \frac{1}{\lambda_{n_{k_j}}}(Tw_{n_{k_j}} - y_{n_{k_j}}) \right\rangle. \end{aligned} \tag{4.32}$$

From (4.26), we can choose a subsequence $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$ such that $\{w_{n_{k_j}}\}$ converges weakly to x^*. Also, since T is a bounded linear operator, we have that $\{Tw_{n_{k_j}}\}$ converges weakly to Tx^*. Using the step size (3.4) and (4.21), we have

$$\lim_{j\to\infty} \|ATw_{n_{k_j}} - Ay_{n_{k_j}}\| \leq \lim_{j\to\infty} \frac{\mu}{\lambda_{n_j+1}} \|Tw_{n_{k_j}} - y_{n_{k_j}}\| = 0. \tag{4.33}$$

Thus, we have that $\langle v - Tx^*, u \rangle \geq 0$ as $j \to \infty$. Thus, by the maximal monoticity of $A + A_1$, we have $Tx^* \in (A + A_1)^{-1}(0)$. Also, using similar approach as above, (4.33) and using (4.19), we have

$$\lim_{j\to\infty} \|Bv_{n_{k_j}} - Bu_{n_{k_j}}\| = 0$$

and

$$\lim_{j\to\infty} \|v_n - w_n\| = \lim_{j\to\infty} \gamma_{n_{k_j}} \|T^*(z_{n_{k_j}} - Tw_{n_{k_j}})\| = 0.$$

Since $\{w_{n_{k_j}}\}, \{v_{n_{k_j}}\}$ and $\{u_{n_{k_j}}\}$ converges weakly to x^* and $\lim_{j\to\infty} \|Bv_{n_{k_j}} - Bu_{n_{k_j}}\| = 0$, we have $x^* \in (B + B_1)^{-1}(0)$. Thus, $x^* \in \Gamma$. Furthermore, since $\{x_{n_{k_j}}\}$ converges weakly to x^*, we obtain

$$\limsup_{k\to\infty} \langle f(p) - p, x_{n_k} - p \rangle = \lim_{j\to\infty} \langle f(p) - p, x_{n_{k_j}} - p \rangle = \langle f(p) - p, x^* - p \rangle. \tag{4.34}$$

It follows that

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, x^*-p\rangle \le 0 \tag{4.35}$$

and we obtain from (4.34) and (4.35)

$$\limsup_{k\to\infty}\langle f(p)-p, x_{n_k+1}-p\rangle \le 0. \tag{4.36}$$

Lastly, we also obtain

$$\|x_{n_k+1}-w_{n_k}\| \le \|x_{n_k+1}-x_{n_k}\| + \|x_{n_k}-w_{n_k}\| \to 0 \quad \text{as} \quad n\to\infty. \tag{4.37}$$

In addition, we have

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle = \limsup_{j\to\infty}\langle p, p-x_{n_{k_j}}\rangle = \langle p, p-x^*\rangle. \tag{4.38}$$

Hence, we have obtain from the above equality that

$$\limsup_{k\to\infty}\langle p, p-x_{n_k}\rangle = \langle p, p-x^*\rangle \le 0, \tag{4.39}$$

which implies that

$$\limsup_{k\to\infty}\langle p, p-x_{n_k+1}\rangle \le 0. \tag{4.40}$$

Using our assumption and the above inequality, we have that

$$\begin{aligned}\limsup_{k\to\infty}\Psi_{n_k} &= \frac{\theta_{n_k}}{\alpha_{n_k}(1-k)}\|x_{n_k}-x_{n_k-1}\|N + \frac{2\eta_{n_k}(1-\alpha_{n_k})}{(1-k)}\langle p, p-x_{n_k+1}\rangle \\ &\quad + \frac{2}{(1-k)}\|p\|\|x_{n_k+1}-w_{n_k}\| + \frac{2\eta_{n_k}}{(1-k)}\|x_{n_k}-T_{n_k}t_{n_k}\|\|x_{n_k+1}-p\| \\ &\quad + \frac{2}{(1-k)}\langle f(p)-p, x_{n_k+1}-p\rangle \\ &\le 0.\end{aligned}$$

Thus, by Lemma 2.4, we have $\lim_{n\to\infty}\|x_n-p\| = 0$. □

5. Special cases of our proposed algorithm

5.1. **Split Convex Minimization Problems.** The concept of convex minimization problem is defined as:

$$\text{Find } x\in H \quad \text{such that } h(x)+g(x) = \min_{x^*\in H}\{h(x^*)+g(x^*)\}, \tag{5.1}$$

where $h : H \to \mathbb{R}$ is a convex differentiable function and $g : H \to \mathbb{R}$ is a convex lower semicontinuous function. It is well known that the problem (5.1) is equivalent to the following problem.

$$\text{Find } x\in H \quad \text{such that } 0 \in \nabla h(x) + \partial g(x), \tag{5.2}$$

where the subdifferential ∂g is the maximal monotone operator and the gradient operator ∇h is L-Lipschitz continuous. Using this concept, we further extend this to a split convex minimization problem (SCMP), which is defined as

$$\text{Find } \ x \in H_1 \ \text{ such that } \ 0 \in \nabla h_1(x) + \partial g_1(x) \tag{5.3}$$

such that

$$\text{Find } \ y = Tx \in H_2 \ \text{ such that } \ 0 \in \nabla h_2(y) + \partial g_2(y). \tag{5.4}$$

Our proposed iterative Algorithm 3.2, takes the form:

Algorithm 5.1. Initialization Step: Given $\lambda_1, \nu_1 > 0$, $\mu, \delta \in (0,1)$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|\}}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, \quad \text{otherwise.} \end{cases} \tag{5.5}$$

Step 1: Compute

$$\begin{aligned} w_n &= (1-\alpha_n)x_n + (1-\alpha_n)\theta_n(S_n x_n - S_n x_{n-1}), \\ y_n &= J_{\lambda_n}^{\partial g_2}(Tw_n - \lambda_n \nabla h_2 Tw_n), \\ z_n &= (1-\beta_n)Tw_n + \beta_n y_n + \beta_n \lambda_n(\nabla h_2 Tw_n - \nabla h_2 y_n), \end{aligned}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|Tw_n - y_n\|}{\|\nabla h_2 Tw_n - \nabla h_2 y_n\|\}}\right\}, & \text{if } \nabla h_2 Tw_n \neq \nabla h_2 y_n, \\ \lambda_n, \quad \text{otherwise.} \end{cases} \tag{5.6}$$

Step 2: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{5.7}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.
Step 3: Compute

$$\begin{aligned} u_n &= J_{\nu_n}^{\partial g_1}(v_n - \nu_n \nabla h_1 v_n), \\ t_n &= (1-\zeta_n)v_n + \zeta_n u_n + \zeta_n \nu_n(\nabla h_1 v_n - \nabla h_1 u_n), \end{aligned}$$

where

$$\nu_{n+1} = \begin{cases} \min\left\{\nu_n, \frac{\delta\|v_n - u_n\|}{\|\nabla h_1 v_n - \nabla h_1 u_n\|\}}\right\}, & \text{if } \nabla h_1 u_n \neq \nabla h_1 v_n, \\ \nu_n, \quad \text{otherwise.} \end{cases} \tag{5.8}$$

Step 4: Compute

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\eta_n)x_n + \eta_n T_n t_n, \tag{5.9}$$

where $T_n = \frac{1}{N}\sum_{i=0}^{N-1}((1-\psi_n)I + \psi_n S_i)$.

Using this fact, $T_n = S$ (quasinonexpansive mapping) and taking $\alpha_n = 0$, the iterative Algorithm 3.2 reduces to

Algorithm 5.2. Initialization Step: Given $\lambda_1, \nu_1 > 0$, $\mu, \delta \in (0,1)$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{5.10}$$

Step 1: Compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J_{\lambda_n}^{A_1}(Tw_n - \lambda_n ATw_n), \\ z_n &= (1-\beta_n)Tw_n + \beta_n y_n + \beta_n\lambda_n(ATw_n - Ay_n), \end{aligned}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\|}\right\}, & \text{if } ATw_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.11}$$

Step 2: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{5.12}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$, if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 3: Compute

$$\begin{aligned} u_n &= J_{\nu_n}^{B_1}(v_n - \nu_n Bv_n), \\ t_n &= (1-\zeta_n)v_n + \zeta_n u_n + \zeta_n\nu_n(Bv_n - Bu_n), \end{aligned}$$

where

$$\nu_{n+1} = \begin{cases} \min\left\{\nu_n, \frac{\delta\|v_n - u_n\|}{\|Bv_n - Bu_n\|}\right\}, & \text{if } Bv_n \neq Bu_n, \\ \nu_n, & \text{otherwise.} \end{cases} \tag{5.13}$$

Step 4: Compute

$$x_{n+1} = (1-\eta_n)x_n + \eta_n St_n, \tag{5.14}$$

where S is a quasinonexpansive mapping.

Algorithm 5.3. Initialization Step: Given $\lambda_1, \nu_1 > 0$, $\mu, \delta \in (0,1)$. Choose $x_0, x_1 \in H_1$, given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$.

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{5.15}$$

Step 1: Compute

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= J^{A_1}_{\lambda_n}(Tw_n - \lambda_n ATw_n), \\ z_n &= y_n + \lambda_n(ATw_n - Ay_n), \end{aligned}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\mu\|Tw_n - y_n\|}{\|ATw_n - Ay_n\|}\right\}, & \text{if } ATw_n \neq Ay_n, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{5.16}$$

Step 2: Compute

$$v_n = w_n + \gamma_n T^*(z_n - Tw_n), \tag{5.17}$$

where γ_n is chosen such that for small enough $\epsilon > 0$, $\gamma_n \in \left[\epsilon, \frac{\|Tw_n - z_n\|^2}{\|T^*(Tw_n - z_n)\|^2} - \epsilon\right]$ if $Tw_n \neq z_n$, otherwise $\gamma_n = \gamma$.

Step 3: Compute

$$\begin{aligned} u_n &= J^{B_1}_{\nu_n}(v_n - \nu_n Bv_n), \\ t_n &= u_n + \nu_n(Bv_n - Bu_n), \end{aligned}$$

where

$$\nu_{n+1} = \begin{cases} \min\left\{\nu_n, \frac{\delta\|v_n - u_n\|}{\|Bv_n - Bu_n\|}\right\}, & \text{if } Bu_n \neq Bv_n, \\ \nu_n, & \text{otherwise.} \end{cases} \tag{5.18}$$

Step 4: Compute

$$x_{n+1} = (1 - \eta_n)x_n + \eta_n St_n, \tag{5.19}$$

where S is a quasinonexpansive mapping.

6. Numerical examples

In this section, we present several numerical examples to illustrate the main result. We compare our proposed algorithm (Algorithm 3.2) with Algorithm 3.1 in [11], Algorithm 2 in [23] and Algorithm 3.1 in [33].

Example 6.1. Let $H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Suppose the operators $T; S_i; A_1; A; B; B_1 : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by

$$\begin{aligned} A(x) &= (4x_1, 4x_2, 4x_3, \cdots, 4x_i, \cdots), \\ B(x) &= \left(\frac{x_1 + |x_1|}{3}, \frac{x_2 + |x_2|}{3}, \frac{x_3 + |x_3|}{3}, \cdots, \frac{x_i + |x_i|}{3}, \cdots\right), \end{aligned}$$

$$\begin{aligned}
A_1(x) &= (7x_1, 7x_2, 7x_3, \cdots, 7x_i, \cdots),\\
B_1(x) &= (3x_1, 3x_2, 3x_3, \cdots, 3x_i, \cdots),\\
T(x) &= \left(0, \frac{x_1}{2}, \frac{x_2}{4}, \frac{x_3}{3}, \cdots\right),\\
f(x) &= \left(0, \frac{x_1}{6}, \frac{x_2}{6}, \cdots\right),\\
T_n(x) &= (0, x_1, x_2, \cdots),\\
S_n(x) &= (x_1, x_2, \cdots).
\end{aligned}$$

It is easy to see that all the conditions in Assumption 3.1 regarding the operators are satisfied.

Example 6.2. Let $H_1 = H_2 = L_2([0,1])$ be equipped with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt,\ \forall\, x, y \in L_2([0,1]),\quad \|x\|^2 = \int_0^1 |x(t)|^2 dt,\ \forall\, x, y \in L_2([0,1]).$$

Suppose the operators $T; S_i; A_1; A; B; B_1; f : L_2([0,1]) \to L_2([0,1])$ are defined by

$$\begin{aligned}
Ax(t) &= \int_0^t x(s)ds,\ \forall\ x \in L_2([0,1]),\\
Bx(t) &= \int_0^1 \left(x(t) - (\frac{2tse^{t+s}}{e\sqrt{e^2-1}} \cos x(s))\right) ds + \frac{2te^t}{e\sqrt{e^2-1}},\ \forall\ x \in L_2([0,1]),\\
A_1x(t) &= \frac{1}{3}\max\{0, x(t)\},\ x \in L_2([0,1]),\\
B_1x(t) &= \max\{0, x(t)\},\ x \in L_2([0,1]),\\
T(x) &= \int_0^1 K(s,t)x(t)dt,\ \forall\ x \in L_2([0,1]),\\
fx(t) &= \int_0^t \frac{x(s)}{7} ds,\ \forall\ x \in L_2([0,1]),\\
T_nx(t) &= \int_0^t \frac{x(s)}{2} ds,\ \forall\ x \in L_2([0,1]),\\
S_nx(t) &= \int_0^t \frac{x(s)}{3} ds,\ \forall\ x \in L_2([0,1]).
\end{aligned}$$

Then, it is easy to see that all the conditions in Assumption 3.1 regarding the operators are satisfied.

Example 6.3. It is crucial in many practical problems to be able to find the problem's minimum-norm solution. Such a problem can be formulated as (see, [27, Example 3.4]):

$$\text{Find } x \in H \quad \text{such that } \|x\| = \min\{\|x^*\| : x^* \in H\}, \tag{6.1}$$

where H is a real Hilbert space. It is well known that (6.1) can be reformulated as the following variational inequality problem:

$$\text{Find } x \in H \quad \text{such that } \langle x, x - x^* \rangle \leq 0, \ \forall \ x^* \in H. \tag{6.2}$$

Suppose that $H_1 = H_2 = L([a,b])$, $C = \{x \in L_2([a,b]) : \langle a, x\rangle = b\}$ and $Q = \{x \in L_2([a,b]) : \langle a, x\rangle \geq b\}$ for some $b \in \mathbb{R}$ and $a \in L_2([a,b]) - \{0\}$. Then x minimizes $\|\cdot\| + \delta_C$ if and only if $0 \in \partial(\|\cdot\| + \delta_C)(x)$ and Tx minimizes $\|\cdot\| + \delta_Q$, where δ_C and δ_Q denotes the indicator functions of C and Q. Now suppose that $A_1 = B_1 = 0$ and $S_i = I$, thus problem (1.15) becomes

$$\text{Find an element } x \in C \text{ such that } x = argmin\{\|x^*\| : x^* \in C\},$$
$$\text{and such that } Tx \in Q \text{ solves } Tx = argmin\{\|y\| : y \in Q\}.$$

During the computation, we make use of the following:

- Algorithm 3.2: $\lambda_1 = 0.5$, $\nu_1 = 0.6$, $\alpha_n = \frac{1}{n+1}$, $\zeta_n = \beta_n = \frac{1}{5n+3}$, $\eta_n = \frac{2n^2}{10n^2+100}$, $\mu = \delta = 0.5$, $\epsilon_n = \frac{100}{(n+1)^2}$ and $\theta = 0.7$.
- CN (Alg. 3.1) in [11]: $\gamma_n = \frac{1}{2\|T\|^2}$ and $\nu_n = \frac{1}{2L_1}$ (where L_1 is the Lipschitz constant of B).
- OIM (Alg. 2) in [23]: $\lambda_1 = 0.5$, $\nu_1 = 0.6$, $\alpha_n = \frac{1}{n+1}$, $\eta_n = \frac{2n}{2n^2+100}$, $\mu = \delta = 0.5$, $\epsilon_n = \frac{100}{(n+1)^2}$ and $\theta = 0.7$.
- TJ (Alg. 3.1) in [33]: $\alpha_n = \frac{1}{n+1}, \gamma_n = \frac{1}{2\|T\|^2}$ and $\nu_n = \frac{1}{2L_1}$ (where L_1 is the Lipschitz constant of B).

We then use the stopping criterion:

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - J_\nu^{B1}(x_n - \nu Bx_n)||^2 + ||Tx_n - J_\lambda^{A1}(Tx_n - \lambda ATx_n)||^2\right) < \varepsilon,$$

where $\varepsilon = 10^{-5}$ is the predetermined error. Furthermore, we choose x_0 and x_1 as follows:

For Example 6.1: Case 1: $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$; Case 2: $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$, $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$; Case 3: $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$, $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$; Case 4: $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.
For Example 6.2 and 6.3: Case 1: $x_0(t) = 5 + t$, $x_1(t) = 1 + t^2$; Case 2: $x_0(t) = t^3 + t$, $x_1(t) = t^4 + t^3 + t^2 + t + 1$; Case 3: $x_0(t) = \cot(t)$, $x_1(t) = \sin(t)$; Case 4: $x_0(t) = e^t$, $x_1(t) = t^2 + t + 1$.

All the computations are performed using Matlab 2016 (b) which is running on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 6 and 2, "Iter" and "CPU" mean the CPU time in seconds and the number of iterations, respectively. Also, in the tables and figures, CN (Alg. 3.1), OIM (Alg. 2) and TJ (Alg. 3.1) represent Algorithm 3.1 in [11], Algorithm 2 in [23] and Algorithm 3.1 in [33], respectively.

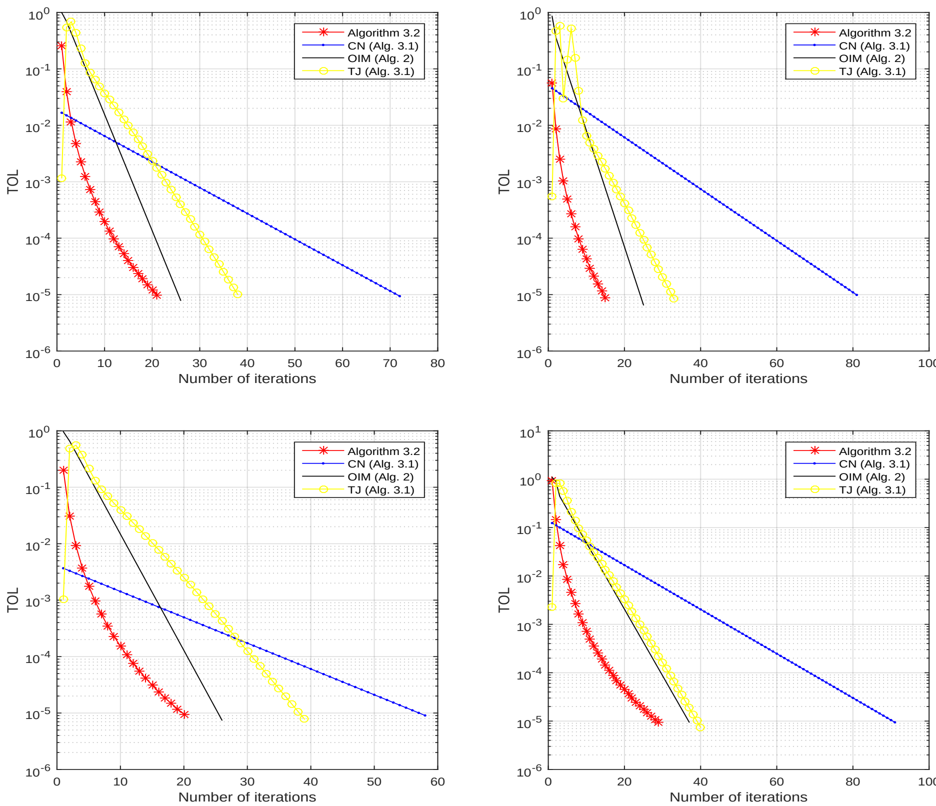

FIGURE 1. The behavior of TOL_n for Example 6.1 with $\varepsilon = 10^{-5}$: Top Left: Case 1; Top Right: Case 2; Bottom left: Case 3; Bottom Right: Case 4.

Algorithms	Case 1		Case 2		Case 3		Case 4	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Algorithm 3.2	0.0600	21	0.0549	15	0.0630	20	0.0684	29
CN (Alg. 3.1) in [11]	2.1208	72	2.1433	81	2.0962	58	2.1699	91
OIM (Alg. 2) in [23]	0.1230	26	0.1224	25	0.1214	26	0.2229	37
TJ (Alg. 3.1) in [33]	1.0127	38	1.0138	33	1.1138	39	1.0138	40

TABLE 1. Example 6.1: Comparison of algorithms with $\varepsilon = 10^{-5}$

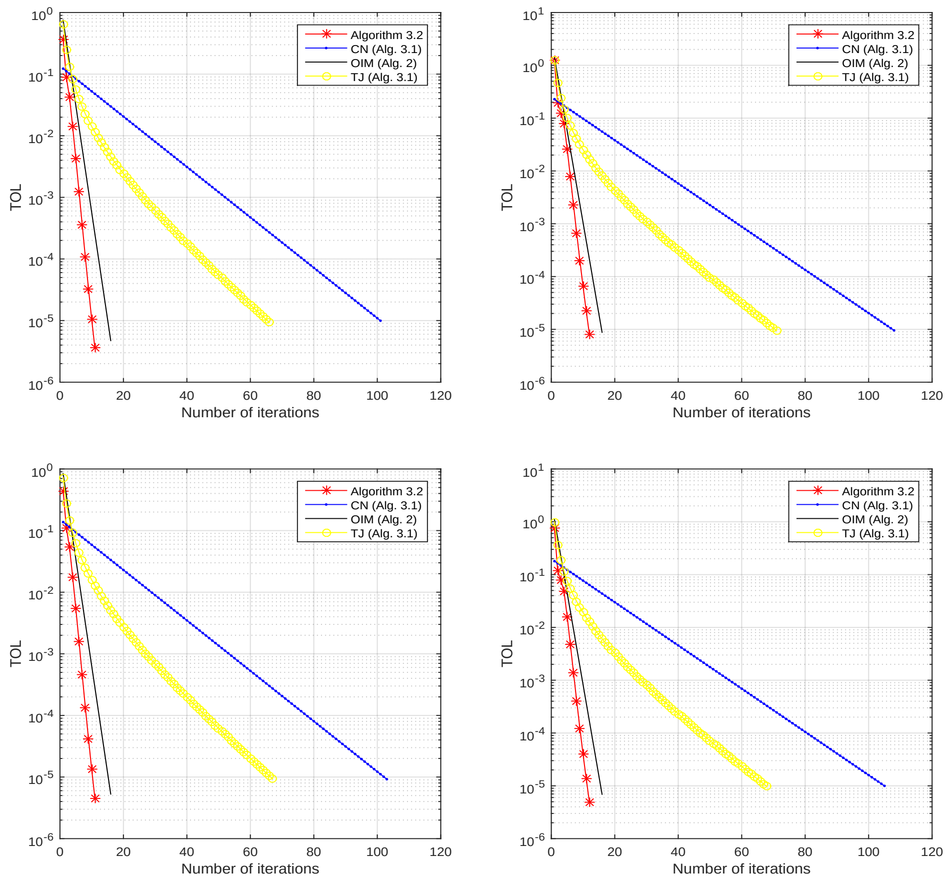

FIGURE 2. The behavior of TOL_n for Example 6.2 with $\varepsilon = 10^{-5}$: Top Left: Case 1; Top Right: Case 2; Bottom left: Case 3; Bottom Right: Case 4.

Algorithms	Case 1		Case 2		Case 3		Case 4	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Algorithm 3.2	5.7371	11	5.8280	12	5.3589	11	7.7976	12
CN (Alg. 3.1) in [11]	26.5351	101	29.4771	108	26.5171	103	31.1889	105
OIM (Alg. 2) in [23]	10.9454	16	12.1152	16	10.8555	16	12.0558	16
TJ (Alg. 3.1) in [33]	18.0298	66	22.5571	71	17.4289	67	26.6996	68

TABLE 2. Example 6.2: Comparison of algorithms with $\varepsilon = 10^{-5}$

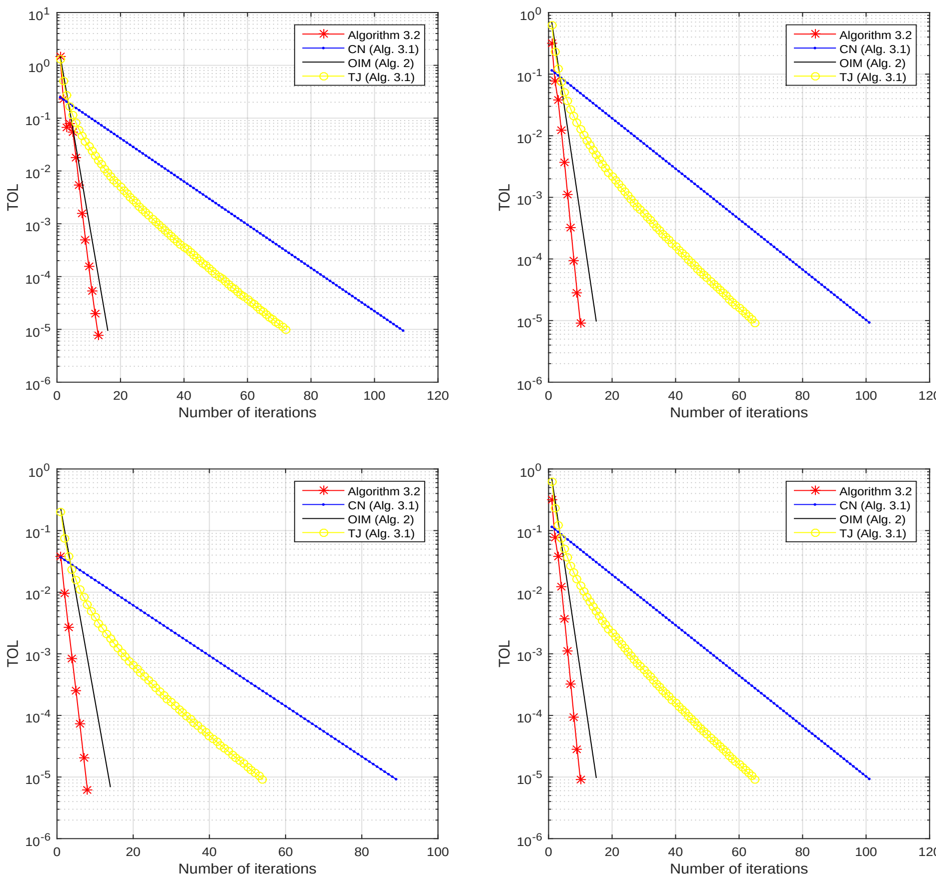

FIGURE 3. The behavior of TOL_n for Example 6.3 with $\varepsilon = 10^{-5}$: Top Left: Case 1; Top Right: Case 2; Bottom left: Case 3; Bottom Right: Case 4.

Algorithms	Case 1		Case 2		Case 3		Case 4	
	CPU	Iter	CPU	Iter	CPU	Iter	CPU	Iter
Algorithm 3.2	3.1945	13	3.8731	10	3.8571	8	4.1389	10
CN (Alg. 3.1) in [11]	25.8159	109	68.7095	101	60.6853	89	71.7865	101
OIM (Alg. 2) in [23]	10.6016	16	13.1424	15	13.2811	14	14.3358	15
TJ (Alg. 3.1) in [33]	15.6142	72	22.0141	65	18.0408	54	24.2571	65

TABLE 3. Example 6.3: Comparison of algorithms with $\varepsilon = 10^{-5}$

7. Conclusion

In this paper we have introduced and studied an iterative algorithm for finding a common solution to a fixed point problem for a finite family of demimetric mappings and a split monotone inclusion problem in the framework of real Hilbert spaces. We have obtained a strong convergence result without assuming that the single valued operators are inversely strongly monotone. We emphasize that the value of the Lipschitz constant is not required for the iterative technique to be implemented. Furthermore, we present several numerical experiments to show the efficiency and the applicability of our proposed iterative method.

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