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AN L_p INEQUALITY FOR POLYNOMIALS

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Abstract. In this paper an L^p-inequality for the sth derivative of a polynomial not vanishing in $|z| < k$ where $k \geq 1$ is obtained, which refines and generalizes some known polynomial inequalities.

1. Introduction

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree at most n. For $P \in P_n$, define

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p \right\}^{1/p}, \ 1 \leq p < \infty,$$
$$\|P\|_\infty := \underset{|z|=1}{Max} |P(z)| \ \ and \ \ m := \underset{|z|=k}{Min} |P(z)| .$$

If $P \in P_n$, then

$$\left\|P'\right\|_\infty \leq n \|P\|_\infty \tag{1.1}$$

and

$$\|P'\|_p \leq n\|P\|_p. \tag{1.2}$$

Inequality (1.1) is a well-known result of S. Bernstein (see [12] or [15]) whereas inequality (1.2) is due to Zygmund [16]. Arestov [1] proved that the inequality (1.2) remains true for $0 < p < 1$ as well. Equality in (1.1) and (1.2) holds for $P(z) = az^n$, $a \neq 0$. If we let $p \to \infty$ in (1.2), we get inequality (1.1)

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If we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then both the inequalities (1.1) and (1.2) can be improved. In fact, if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then (1.1) and (1.2) can be, respectively, replaced by

$$\left\|P'\right\|_{\infty} \leq \frac{n}{2}\left\|P\right\|_{\infty} \tag{1.3}$$

and

$$\left\|P'\right\|_{p} \leq \frac{n}{\left\|1+z\right\|_{p}}\left\|P\right\|_{p}, \ \ p \geq 1. \tag{1.4}$$

Inequality (1.3) was conjectured by P. Erdös and later verified by P. D. Lax [10] whereas the inequality (1.4) was found out by De Bruijn [6]. Rahman and Schmeisser [13] proved the inequality (1.4) remains true for $0 < p < 1$ as well. Both the estimates are sharp and equality in (1.3) and (1.4) holds for $P(z) = az^n + b$, $|a| = |b|$.

As an improvement of inequality (1.3), Aziz and Dawood [2] proved that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then

$$\left\|P'\right\|_{\infty} \leq \frac{n}{2}\left\{\left\|P\right\|_{\infty} - \underset{|z|=1}{Min}\left|P(z)\right|\right\}. \tag{1.5}$$

Malik [11] generalized inequality (1.3) by proving that if $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then

$$\left\|P'\right\|_{\infty} \leq \frac{n}{1+k}\left\|P\right\|_{\infty}. \tag{1.6}$$

Govil and Rahman [8] extended inequality (1.6) to L_p-norm by proving that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k$ where $k \geq 1$, then

$$\left\|P'\right\|_{p} \leq \frac{n}{\left\|k+z\right\|_{p}}\left\|P\right\|_{p}, \ \ p \geq 1. \tag{1.7}$$

It was shown by Gardner and Weems [7] and independently by Rather [14] that the inequality (1.7) remains true for $0 < p < 1$ as well. As a refinement of inequality (1.6), Govil et al.[9] proved that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then

$$\left\|P'\right\|_{\infty} \leq n\left\{\frac{n|a_0| + k^2|a_1|}{n|a_0|(1+k^2) + 2k^2|a_1|}\right\}\left\|P\right\|_{\infty}. \tag{1.8}$$

Inequality (1.6) was further generalized by Govil and Rahman [8] to the sth derivative of $P(z)$ by showing that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then

$$\left\|P^{(s)}\right\|_{\infty} \leq \frac{n(n-1)\cdots(n-s+1)}{1+k^s}\left\|P\right\|_{\infty}. \tag{1.9}$$

Aziz and Shah [5] extended inequality (1.9) to L^p mean of $|P(z)|$ by proving that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then for each $p > 0$,

$$\left\| P^{(s)} \right\|_p \leq \frac{n(n-1)\cdots(n-s+1)}{\|k^s + z\|_p} \|P\|_p . \tag{1.10}$$

Recently Aziz and Rather [4] refined inequality (1.10) by showing that if $P \in P_n$ and $P(z) \neq 0$ for $|z| < k, k \geq 1$, then for each $p > 0$ and $1 \leq s < n$,

$$\left\| P^{(s)} \right\|_p \leq \frac{n(n-1)\cdots(n-s+1)}{\|\delta_{k,s} + z\|_p} \|P\|_p , \tag{1.11}$$

where $\delta_{k,s}$ is defined by

$$\delta_{k,s} = \left\{ \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{C(n,s)|a_0| + |a_s|k^{s+1}} \right\}. \tag{1.12}$$

In this paper we present following result which includes not only a refinement of inequality (1.10) as a special case but also leads to some striking conclusion giving refinements and generalizations of other well known results.

Theorem 1.1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for every real or complex number β with $|\beta| \leq 1$, $p > 0$, $0 \leq t \leq 1$ and $1 \leq s < n$,*

$$\begin{aligned} &\left\| P^{(s)} + \beta \frac{tn(n-1)\cdots(n-s+1)m}{1+\phi(k,s,t)} \right\|_p \\ &\qquad \leq \frac{n(n-1)\cdots(n-s+1)}{\|\phi(k,s,t) + z\|_p} \|P\|_p \end{aligned} \tag{1.13}$$

where $\phi(k,s,t)$ is defined by

$$\phi(k,s,t) = k^s \left\{ \frac{k + \lambda(k,s,t)}{1 + k\lambda(k,s,t)} \right\} \tag{1.14}$$

and

$$\lambda(k,s,t) = \frac{|a_s|k^s}{C(n,s)(|a_0| - tm)} \leq 1. \tag{1.15}$$

Remark 1.2. For $t = 0$, inequality (1.13) reduces to inequality (1.11).

Making use of inequalities (1.14) and (1.15) and the fact that

$$\|\phi(k,s,t) + z\|_p \geq \|k^s + z\|_p , \quad p > 0,$$

we immediately obtain the following refinement as well as generalization of inequality (1.10).

Corollary 1.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for every real or complex number β with $|\beta| \leq 1$, $p > 0$, $0 \leq t \leq 1$ and $1 \leq s < n$,*

$$\begin{aligned} &\left\| P^{(s)} + \beta \frac{tn(n-1)\cdots(n-s+1)m}{1+\phi(k,s,t)} \right\|_p \\ &\qquad \leq \frac{n(n-1)\cdots(n-s+1)}{\|k^s + z\|_p} \|P\|_p . \end{aligned} \tag{1.16}$$

Letting $p \to \infty$ in (1.13) and choosing argument of β, with $|\beta| = 1$, suitably, we obtain the following interesting refinement of inequality (1.9).

Corollary 1.4. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $0 \leq t \leq 1$ and $1 \leq s < n$,*

$$\left\| P^{(s)} \right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{1+\phi(k,s,t)} \left\{ \|P\|_\infty - tm(p,k) \right\} . \tag{1.17}$$

Taking $t = 0$ in Corollary 2, we immediately get the following refinement of inequality (1.9).

Corollary 1.5. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$, then for $1 \leq s < n$,*

$$\left\| P^{(s)} \right\|_\infty \leq \frac{n(n-1)\cdots(n-s+1)}{1+\phi(k,s,0)} \|P\|_\infty . \tag{1.18}$$

2. Lemmas

For the proof of the theorem, we need the following lemmas. The first lemma is due to Aziz and Rather [4].

Lemma 2.1. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for $1 \leq s < n$,*

$$\delta_{k,s} |P^{(s)}(z)| \leq |Q^{(s)}(z)| \ \ for \ \ |z| = 1, \tag{2.1}$$

where $\delta_{k,s}$ is defined by

$$\delta_{k,s} = \left\{ \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{C(n,s)|a_0| + |a_s|k^{s+1}} \right\} . \tag{2.2}$$

Lemma 2.2. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq \rho$ where $\rho \leq 1$, then for $1 \leq s < n$,*

$$|P^{(s)}(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{\rho^n} m(P,\rho) \ \ for \ \ |z| = 1. \tag{2.3}$$

Proof. By hypothesis all the zeros of $P(z)$ lie in $|z| \leq \rho$ where $\rho \leq 1$ and $m(P,\rho) = \underset{|z|=\rho}{Min}|P(z)|$, therefore, all the zeros of $F(z) = P(\rho z)$ lie in $|z| \leq 1$ and

$$\underset{|z|=1}{Min}|F(z)| = \underset{|z|=1}{Min}|P(\rho z)| = \underset{|z|=\rho}{Min}|P(z)| = m(P,\rho).$$

This implies $m(P,\rho)|z|^n = m(P,\rho) \leq |F(z)|$ for $|z| = 1$. We first show that the polynomial $G(z) = F(z) + \alpha m(P,\rho)z^n$ has all its zeros in $|z| \leq 1$ for every real or complex number α with $|\alpha| < 1$. This is obvious if $m(P,\rho) = 0$, that is, if $F(z)$ has a zero on $|z| = 1$. We now assume that all the zeros of $F(z)$ lie in $|z| < 1$, so that $m(P,\rho) > 0$. This gives for $|z| = 1$ and $|\alpha| < 1$.

$$|m(p,\rho)\alpha z^n| = m(P,\rho)|\alpha||z|^n = m(P,\rho)|\alpha| < m(P,\rho) \leq |F(z)|.$$

A direct application of Rouche's theorem shows that the polynomial $G(z) = F(z)+\alpha m(P,\rho)|z|^n$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| < 1$. By repeated application of Gauss-Lucas theorem, it follows that all the zeros of $G^{(s)}(z) = F^{(s)}(z) + n(n-1)\cdots(n-s+1)\alpha m(P,\rho)z^{n-s}$ lie in $|z| < 1$ for every real or complex number α with $|\alpha| < 1$. This implies

$$|F^{(s)}(z)| \geq n(n-1)\cdots(n-s+1)m(P,\rho)|z|^{n-s} \; for \; |z| \geq 1. \tag{2.4}$$

If the inequality (2.4) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|F^{(s)}(z_0)| < n(n-1)\cdots(n-s+1)m(P,\rho)|z_0|^{n-s}.$$

We choose

$$\alpha = -\frac{F^{(s)}(z_0)}{n(n-1)\cdots(n-s+1)m(P,\rho)z_0^{n-s}},$$

then clearly $|\alpha| < 1$ and with this choice of α, we get $G^{(s)}(z_0) = 0$ where $|z_0| \geq 1$. This is a contradiction to the fact that all the zeros of $G^{(s)}(z)$ lie in $|z| < 1$. Hence the inequality (2.4) is established. Replacing $F(z)$ by $P(\rho z)$ in (2.4), we obtain

$$\rho^s|P^{(s)}(\rho z)| \geq n(n-1)\cdots(n-s+1)m(P,\rho)|z|^{n-s} \; for \; |z| \geq 1.$$

Taking, in particular, $z = e^{i\theta}/\rho$ where $\rho \leq 1$ so that $|z| = (1/\rho) \geq 1$, we get

$$\rho^s|P^{(s)}(e^{i\theta})| \geq \frac{n(n-1)\cdots(n-s+1)}{\rho^{n-s}}m(P,\rho), \;\; 0 \leq \theta < 2\pi,$$

or equivalently,

$$|P^{(s)}(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{\rho^n}m(P,\rho) \;\; for \;\; |z| = 1.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \geq 1$ and $Q(z) = z^n\overline{P(1/\bar{z})}$, then for $0 \leq t \leq 1$ and $1 \leq s < n$,*

$$\phi(k,s,t)|P^{(s)}(z)| \leq |Q^{(s)}(z)| - tn(n-1)\cdots(n-s+1)m(P,k) \tag{2.5}$$

where $\phi(k,s,t)$ is defined by

$$\phi(k,s,t) = k^s\left\{\frac{k+\lambda(k,s,t)}{1+k\lambda(k,s,t)}\right\} \tag{2.6}$$

and

$$\lambda(k,s,t) = \frac{|a_s|k^s}{C(n,s)(|a_0|-tm)} \leq 1. \tag{2.7}$$

Proof. By hypothesis, $P \in P_n$ has all its zeros in $|z| \geq k \geq 1$ and $m(P,k) = \underset{|z|=k}{Min}|P(z)|$, therefore,

$$m(P,k) \leq |P(z)| \quad for \quad |z| = k. \tag{2.8}$$

We first show that for every real or complex number α with $|\alpha| \leq 1$, the polynomial $f(z) = P(z) - \alpha m(P,k)$ has all its zeros in $|z| \geq k \geq 1$. This is clear if $P(z)$ has a zero on $|z| = k$, for then $m(P,k) = 0$ and $f(z) = P(z)$. In case $P(z)$ has no zero on $|z| = k$, the clearly $m(P,k) > 0$. since $m(P,k)/P(z)$ is not a constant, by the Maximum Modulus Principle, it follows from (2.8) that

$$m(P,k) < |P(z)| \quad for \quad |z| < k. \tag{2.9}$$

Now if $f(z)$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then $P(z_0) - \alpha m(P,k) = f(z_0) = 0$. This implies

$$|P(z_0)| = |\alpha m(P,k)| \leq m(P,k) \quad where \quad |z_0| < k,$$

which contradicts (2.9) and hence in any case $f(z) = P(z) - \alpha m(P,k)$ has all its zeros in $|z| \geq k \geq 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Lemma 2.1 to the polynomial

$$f(z) = P(z) - \alpha m(P,k) = (a_0 - \alpha m(P,k)) + \sum_{j=1}^{n} a_j z^j,$$

we get for every real or complex number α with $|\alpha| \leq 1$,

$$\begin{aligned} &\phi_1(k,s)|P^{(s)}(z)| \\ &\leq |Q^{(s)}(z) - \bar{\alpha}n(n-1)\cdots(n-s+1)m(P,k)z^{n-s}| \quad for \quad |z| = 1. \end{aligned} \tag{2.10}$$

where $\phi_1(k,s)$ is defined by

$$\phi_1(k,s) = k^s\left\{\frac{k+\lambda_1(k,s)}{1+k\lambda_1(k,s)}\right\} \tag{2.11}$$

and

$$\lambda_1(k,s) = \frac{|a_s|k^s}{C(n,s)|a_0 - \alpha m|} \leq 1 \ \ with \ \ m = m(P,k). \tag{2.12}$$

Using $|a_0| - |\alpha|m \leq |a_0 - \alpha m|$ and $k \geq 1$, it can be easily verified that

$$\begin{aligned} \phi(k,s,|\alpha|) &= k^s \left\{ \frac{k + \lambda(k,s,|\alpha|)}{1 + k\lambda(k,s,|\alpha|)} \right\} \\ &= k^s \left\{ \frac{k + \frac{|a_s|k^s}{C(n,s)(|a_0| - |\alpha|m)}}{1 + k\frac{|a_s|k^s}{C(n,s)(|a_0| - |\alpha|m)}} \right\} \\ &\leq k^s \left\{ \frac{k + \frac{|a_s|k^s}{C(n,s)|a_0 - \alpha m|}}{1 + k\frac{|a_s|k^s}{C(n,s)|a_0 - \alpha m|}} \right\} \\ &= k^s \left\{ \frac{k + \lambda_1(k,s)}{1 + k\lambda_1(k,s)} \right\} = \phi_1(k,s). \end{aligned} \tag{2.13}$$

Also, since all the zeros $Q(z) = z^n\overline{P(1/\bar{z})}$ lie in $|z| \leq (1/k) \leq 1$ and

$$\underset{|z|=\frac{1}{k}}{Min}|Q(z)| = \underset{|z|=\frac{1}{k}}{Min}|z^n\overline{P(1/\bar{z})}| = \underset{|z|=1}{Min}|\frac{z^n}{k^n}\overline{P(k/\bar{z})}| = \frac{1}{k^n}\underset{|z|=k}{Min}|P(z)| = \frac{m(P,k)}{k^n},$$

we conclude by Lemma 2.2 (with $P(z)$ replaced by $Q(z)$ and ρ by $1/k$) that

$$\begin{aligned} |Q^{(s)}(z)| &\geq n(n-1)\cdots(n-s+1)k^n m(Q,\frac{1}{k}) \\ &= n(n-1)\cdots(n-s+1)m(P,k) \ \ for \ \ |z|=1. \end{aligned} \tag{2.14}$$

Choosing argument of $\alpha, |\alpha| \leq 1$, on the right hand side of (2.10) such that for $|z| = 1$,

$$\begin{aligned} &|Q^{(s)}(z) - \bar{\alpha}n(n-1)\cdots(n-s+1)m(P,k)z^{n-s}| \\ &= |Q^{(s)}(z)| - |\alpha|n(n-1)\cdots(n-s+1)m(P,k) \end{aligned}$$

(which is possible by (2.14)), it follows from (2.10) by using (2.13) that for $|\beta| \leq 1$ and $|z| = 1$,

$$\phi(k,s,|\alpha|)|P^{(s)}(z)| \leq |Q^{(s)}(z)| - |\alpha|n(n-1)\cdots(n-s+1)m(P,k) \tag{2.15}$$

where

$$\phi(k,s,|\alpha|) = k^s \left\{ \frac{k + \lambda(k,s,|\alpha|)}{1 + k\lambda(k,s,|\alpha|)} \right\} \tag{2.16}$$

and

$$\lambda(k,s,|\alpha|) = \frac{|a_s|k^s}{C(n,s)(|a_0| - |\alpha|m|)}.$$

Since (2.11) is true for all α with $|\alpha| \leq 1$ and by (2.9), $m = m(p,k) < |P(0)| = |a_0|$, we can choose argument of α in (2.12) such that $|a_0 - \alpha m| = |a_0| - |\alpha| m$. For this choice of the argument of α, we get from (2.12) that

$$\lambda(k, s, |\alpha|) = \lambda_1(k, s) \leq 1. \tag{2.17}$$

The inequalities (2.15),(2.16) and (2.17) are equivalent to (2.5),(2.6) and (2.7) respectively with $t = |\alpha|$ where $0 \leq t \leq 1$. This completes the proof of lemma 2.3. □

Lemma 2.4. *If $P \in P_n$ and $Q(z) = z^n \overline{P(1/\bar{z})}$, then for each α, $0 \leq \alpha < 2\pi$ and $p > 0$,*

$$\int_0^{2\pi} \int_0^{2\pi} \left| P'(e^{i\theta}) + e^{i\alpha} Q'(e^{i\theta}) \right|^p d\theta d\alpha \leq n^p \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta. \tag{2.18}$$

Lemma 4 is due to Aziz and Rather [4].

Lemma 2.5. *If A, B, C are non-negative real numbers such that $B + C \leq A$, then for every real α,*

$$\left| (A - C)e^{i\alpha} + (B + C) \right| \leq \left| Ae^{i\alpha} + B \right|.$$

Lemma 5 is also due to Aziz and Rather [3].

3. Proof of the theorem 1.1

Proof. By hypothesis $P \in P_n$ and $P(z)$ does not vanish in $|z| < k$, $k \geq 1$, therefore, by Lemma 2.3, for each θ, $0 \leq \theta < 2\pi$, we have

$$\phi(k, s, t) |P^{(s)}(z)| \leq |Q^{(s)}(z)| - tn(n-1) \cdots (n-s+1) m(P, k)$$

where $\phi(k, s, t)$ is defined by (2.6). This implies

$$\phi(k, s, t) \left\{ |P^{(s)}(z)| + \frac{tn(n-1) \cdots (n-s+1) m(P,k)}{1 + \phi(k,s,t)} \right\}$$

$$\leq \left\{ |Q^{(s)}(z)| - \frac{tn(n-1) \cdots (n-s+1) m(P,k)}{1 + \phi(k,s,t)} \right\}. \tag{3.1}$$

Taking

$$A = |Q^{(s)}(e^{i\theta})|, \; B = |P^{(s)}(e^{i\theta})| \; and \; C = \frac{tn(n-1) \cdots (n-s+1) m(P,k)}{1 + \phi(k,s,t)},$$

in Lamma 2.5 and noting by (3.1) that for $\phi(k, s, t) \geq 1$,

$$B + C \leq \phi(k, s, t)(B + C) \leq (A - C) \leq A,$$

we get for every real α,

$$\begin{aligned}&\Bigg|\left\{|Q^{(s)}(e^{i\theta})| - \frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}\right\}e^{i\alpha}\\&\quad+\left\{|P^{(s)}(e^{i\theta})| + \frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}\right\}\Bigg|\\&\leq \left||Q^{(s)}(e^{i\theta})|e^{i\alpha} + |P^{(s)}(e^{i\theta})|\right|.\end{aligned}$$

This implies for each $p > 0$,

$$\int_0^{2\pi} |G(\theta) + e^{i\alpha}F(\theta)|^p d\theta \leq \int_0^{2\pi} \left||P^{(s)}(e^{i\theta})| + e^{i\alpha}|Q^{(s)}(e^{i\theta})|\right|^p d\theta, \tag{3.2}$$

where

$$F(\theta) = |Q^{(s)}(e^{i\theta})| - \frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}|$$

and

$$G(\theta) = |P^{(s)}(e^{i\theta})| + \frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}|.$$

Integrating both sides of (3.2) with respect to α on $[0,2\pi]$, we get for each $p > 0$ and α real,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi} |G(\theta) + e^{i\alpha}F(\theta)|^p d\theta d\alpha\\&\leq \int_0^{2\pi}\int_0^{2\pi} \left||P^{(s)}(e^{i\theta})| + e^{i\alpha}|Q^{(s)}(e^{i\theta})|\right|^p d\theta d\alpha.\end{aligned} \tag{3.3}$$

Now for points $e^{i\theta}$, for which $P^{(s)}(e^{i\theta}) \neq 0$, we have

$$\begin{aligned}\int_0^{2\pi} \left|P^{(s)}(e^{i\theta}) + e^{i\alpha}Q^{(s)}(e^{i\theta})\right|^p d\alpha &= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi} \left|\frac{Q^{(s)}(e^{i\theta})}{P^{(s)}(e^{i\theta})}e^{i\alpha} + 1\right|^p d\alpha\\&= |P^{(s)}(e^{i\theta})|^p \int_0^{2\pi} \left|\left|\frac{Q^{(s)}(e^{i\theta})}{P^{(s)}(e^{i\theta})}\right|e^{i\alpha}+1\right|^p d\alpha\\&= \int_0^{2\pi} \left||P^{(s)}(e^{i\theta})| + e^{i\alpha}|Q^{(s)}(e^{i\theta})|\right|^p d\alpha.\end{aligned}$$

Since this inequality is trivially true for points $e^{i\theta}$ for which $P^{(s)}(e^{i\theta}) = 0$, it follows that

$$\int_0^{2\pi} \left||P^{(s)}(e^{i\theta})| + e^{i\alpha}|Q^{(s)}(e^{i\theta})|\right|^p d\alpha = \int_0^{2\pi} \left|P^{(s)}(e^{i\theta}) + e^{i\alpha}Q^{(s)}(e^{i\theta})\right|^p d\alpha. \tag{3.4}$$

Let $f(z)=P(z)+e^{i\alpha}Q(z)$ where $Q(z)=z^n\overline{P(1/\bar{z})}$, then $f\in P_n$ and we have

$$f^{(s)}(z)=P^{(s)}(z)+e^{i\alpha}Q^{(s)}(z),$$

which is clearly a polynomial of degree at most $n-s$, $1\leq s<n$. Integrating the two sides of (3.4) with respect to θ from 0 to 2π, we obtain by repeated application of inequality (1.2), for each $p>0$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}\left||P^{(s)}(e^{i\theta})|+e^{i\alpha}|Q^{(s)}(e^{i\theta})|\right|^p d\theta d\alpha\\
&=\int_0^{2\pi}\int_0^{2\pi}\left|P^{(s)}(e^{i\theta})+e^{i\alpha}Q^{(s)}(e^{i\theta})\right|^p d\theta d\alpha\\
&\leq 2\pi(n-s+1)^p\int_0^{2\pi}\int_0^{2\pi}\left|P^{(s-1)}(e^{i\theta})+e^{i\alpha}Q^{(s-1)}(e^{i\theta})\right|^p d\theta d\alpha\\
&\leq 2\pi(n-s+1)^p(n-s+2)^p\cdots(n-1)^p\\
&\qquad\times\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\alpha}Q'(e^{i\theta})\right|^p d\theta d\alpha\\
&\leq 2\pi(n-s+1)^p(n-s+2)^p\cdots(n-1)^p n^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \qquad (3.5)
\end{aligned}$$

Combining (3.3) and (3.5), it follows for each $p>0$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi}|F(\theta)+e^{i\alpha}G(\theta)|^p d\theta d\alpha\\
\leq\ & 2\pi(n-s+1)^p(n-s+2)^p\cdots(n-1)^p n^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta. \qquad (3.6)
\end{aligned}$$

Now for every real α and $R\geq r\geq 1$, we have

$$|R+e^{i\alpha}|\geq|r+e^{i\alpha}|,$$

which implies for each $p>0$,

$$\int_0^{2\pi}|R+e^{i\alpha}|^p d\alpha\geq\int_0^{2\pi}|r+e^{i\alpha}|^p d\alpha.$$

If $G(\theta)\neq 0$, we take

$$R=|F(\theta)|/|G(\theta)|\ \ \textit{and}\ \ r=\phi(k,s,t),$$

then by (3.1), $R\geq r\geq 1$ and we get

$$\begin{aligned}
\int_0^{2\pi}|G(\theta)+e^{i\alpha}F(\theta)|^p d\alpha &= |G(\theta)|^p\int_0^{2\pi}\left|\frac{F(\theta)}{G(\theta)}e^{i\alpha}+1\right|^p d\alpha\\
&= |G(\theta)|^p\int_0^{2\pi}\left|\left|\frac{F(\theta)}{G(\theta)}\right|e^{i\alpha}+1\right|^p d\alpha\\
&= |G(\theta)|^p\int_0^{2\pi}\left|\left|\frac{F(\theta)}{G(\theta)}\right|+e^{i\alpha}\right|^p d\alpha\\
&\geq |G(\theta)|^p\int_0^{2\pi}\left|\phi(k,s,t)+e^{i\alpha}\right|^p d\alpha.
\end{aligned}$$

For $G(\theta)=0$, this inequality is trivially true. Using this in (3.6), we conclude that for each $p>0$ and α real,

$$\int_0^{2\pi}\left|\phi(k,s,t)+e^{i\alpha}\right|^p d\alpha\int_0^{2\pi}\left\{|P^{(s)}(e^{i\theta})|+\frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}\right\}^p d\theta$$
$$\leq 2\pi n^p(n-1)^p\cdots(n-s+1)^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta.$$

This implies for every real or complex number β with $|\beta|\leq 1$, $p>0$ and α real,

$$\int_0^{2\pi}\left|\phi(k,s,t)+e^{i\alpha}\right|^p d\alpha\int_0^{2\pi}\left|P^{(s)}(e^{i\theta})+\beta\frac{tn(n-1)\cdots(n-s+1)m(P,k)}{1+\phi(k,s,t)}\right|^p d\theta$$
$$\leq 2\pi n^p(n-1)^p\cdots(n-s+1)^p\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta,$$

which immediately leads to (1.13) and this completes the proof of the Theorem 1.1. □

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