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SOME NOTES ON TVS-CONE 2-METRIC AND 2-METRIC

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Abstract. In this article, we show that the concept of a TVS-cone 2-metric space by a nonlinear scalarization function and our theorems extend some results in Du [3]. Then we prove the equivalence of cone 2-metric and 2-metric, and the main idear is to investigate the relationship between TVS-cone 2-metric and 2-metric.

1. Introduction

In 2007, Huang-Zhang [6] generalized the concept of a metric space, replacing the set of real numbers by an ordered Bananch space and introduced the cone metric space. Later, many authors generalized some theroems on cone 2-metric space. Since a cone metric space is a special case of a TVS-cone metric space. Du [3] has proved that the Banach contraction principle in general metric spaces and in TVS-cone metric space are equivalent. First, we introduct some concepts which are used, and we expand the concept of TVS-cone metric space and prove some theorems by the same idea used in [2] and [3]. At last, we provide an example of the conclusion.

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2. Preliminaries

Definition 2.1. ([3]) Let Y be always a topological vector space. A subset P of Y is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) if $a, b \in R$ with $a, b \geq 0$ and $x, y \in P$, then $ax + by \in P$;
(iii) $P \bigcap (-P) = \{0\}$.

For a given cone $P \subset Y$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, where $x \prec\prec y$ means that $y - x \in intP$ (the interior of P). In the following, we assume that Y is a locally convex Hausdorff t.v.s, P is a proper, closed and convex pointed cone in Y with $intP \neq \varnothing$ and $e \in intP$. The nonlinear scalarization function $\xi_e : Y \to R$ is defined by $\xi_e(y) = \inf\{r \in R : y \in re - P\}$ for all $y \in Y$.

Lemma 2.2. ([3]) *For each $r \in R$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \Leftrightarrow y \in re - P$;
(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - P$;
(iii) $\xi_e(y) \geq r \Leftrightarrow y \notin re - intP$;
(iv) $\xi_e(y) < r \Leftrightarrow y \in re - intP$;
(v) $\xi_e(.)$ *is positively homogeneous and continuous on* Y;
(vi) $y_1 \in y_2 + P \Rightarrow \xi_e(y_2) \leq \xi_e(y_1)$;
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1, y_2 \in Y$.

Definition 2.3. ([7]) Let X be a nonempty set. For all $x, y, z \in X$, a valued function $\rho : X \times X \times X \to R$ is said to be a 2-metric, if the following conditions hold:

(i) For all $x, y \in X$ and $x \neq y$, there is a point $z \in X$, such that $\rho(x, y, z) \neq \theta$;
(ii) $\rho(x, y, z) = \theta$ iff $x = y$ or $y = z$ or $x = z$;
(iii) $\rho(x, y, z) = \rho(y, z, x) = \rho(x, z, y)$;
(iv) $\rho(x, y, z) \preceq \rho(x, y, w) + \rho(x, w, z) + \rho(w, y, z)$, $\forall w \in X$.

Since we can't give a metric directly in TVS-cone 2-metic space, we need introduce the definition of the cone 2-metic, then we have the metric in TVS-cone 2-metric space by the nonlinear scalarization function in a cone 2-metric.

Definition 2.4. ([4]) Let X be a nonempty set, E be a real Banach space, suppose the mapping $D : X \times X \times X \to E$ satisfies:

(i) For $\forall x, y \in X$ there is a point $z \in X$, such that $D(x, y, z) \neq \theta$;
(ii) $D(x, y, z) = \theta$ iff $x = y$ or $y = z$ or $x = z$;
(iii) $D(x, y, z) = D(x, z, y) = D(y, z, x)$ for all $x, y, z \in X$;
(iv) $D(x, y, z) \preceq D(x, y, w) + D(x, w, z) + D(w, y, z)$ for all $x, y, z, w \in X$.

Then D is called a cone 2-metric on X and (X, D) is called a cone 2-metric space.

Now we introduct the definition of TVS-cone 2-metric space and its completeness.

Definition 2.5. Let X be a non-empty set and Y be a topological vector space. Suppose that the mapping $d : X \times X \times X \to Y$ satisfies:

(i) $d(x, y, z) \succeq \theta$ for all $x, y, z \in X$;
(ii) $d(x, y, z) = \theta$ iff $x = y$ or $x = z$ or $y = z$;
(iii) $d(x, y, z) = d(x, z, y) = d(z, x, y)$ for all $x, y, z \in X$;
(iv) $d(x, y, z) \preceq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z \in X$.

Then d is called a TVS-valued cone 2-metric on X and (X, d) is called a TVS-cone 2-metric space.

Definition 2.6. Let (X, d) be a TVS-cone 2-metric space. Let $\{x_n\}$ be a sequence in X and $\forall c \in P$ with $\theta \preceq c$ and $a \in X$.

(i) The sequence $\{x_n\}$ is called Cauchy if for $\forall c \in P$, $c \succeq \theta$, there exsits $N > 0$, when $n, m > N$ and $\forall a \in X$, such that $d(x_n, x_m, a) \prec\prec c$;
(ii) The sequence $\{x_n\}$ is said to be convergent if there exist a positive integer $N > 0$ and $x \in X$ such that $d(x_n, x, a) \prec\prec c$ for all $n > N$ and $\forall a \in X$;
(iii) A TVS-cone 2 metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X.

3. Main Results

Lemma 3.1. *Let (X, D) be a cone 2-metric space. Then*

$$\rho(x, y, z) = \inf\{\|u\| | D(x, y, z) \preceq u, u \in P\}$$

is a 2-metric for all $x, y, z \in X$, where D is a cone 2-metric.

Proof. (i) By Definition 2.4, for all $x, y \in X$, there is a point $z \in X$, such that we have $D(x, y, z) \neq 0$. Let $D(x, y, z) = a$. Then we have

$$\rho(x, y, z) = \inf\{\|u\| | a \preceq u, u \in P\} = \|a\| \neq 0.$$

(ii) If $\rho(x, y, z) = 0$, that is $\inf\{\|u\| | D(x, y, z) \preceq u, u \in P\} = 0$, there exists $u_n \in P$, let $\|u_n\| < \frac{1}{n}$, such that $D(x, y, z) \preceq u_n$. Since $D(x, y, z) \preceq u_n$ and $u_n \to 0 (n \to \infty)$, *i.e.*, $u_n - D(x, y, z) \in P \Rightarrow -D(x, y, z) \in P$ and $D(x, y, z) \in P$, then we have $D(x, y, z) = 0$, and so $x = y$.

If $x = y$, by Definition 2.4, $D(x, y, z) = 0$, that is

$$\rho(x, y, z) = \inf\{\|u\| | 0 \preceq u, u \in P\} = 0.$$

The same way we prove that $y = z$ and $x = z$.

(iii) $D(x, y, z) = D(x, z, y) = D(y, z, x)$ implies $\rho(x, y, z) = \rho(x, z, y) = \rho(y, z, x)$ for all $x, y, z \in X$.

(iv) Now we prove

$$\rho(x, y, z) \leq \rho(x, y, w) + \rho(x, w, z) + \rho(w, y, z)$$

for all $x, y, z, w \in X$. In fact, since

$$\begin{aligned} \rho(x, y, z) &= \inf\{\|u_1\| | D(x, y, z) \preceq u_1, u_1 \in P\}, \\ \rho(x, y, w) &= \inf\{\|u_2\| | D(x, y, w) \preceq u_2, u_2 \in P\}, \\ \rho(x, w, z) &= \inf\{\|u_3\| | D(x, w, z) \preceq u_3, u_3 \in P\}, \\ \rho(w, y, z) &= \inf\{\|u_4\| | D(w, y, z) \preceq u_4, u_4 \in P\}, \end{aligned}$$

for $u_2, u_3, u_4 \in P$, $D(x, y, w) \preceq u_2, D(x, w, z) \preceq u_3, D(w, y, z) \preceq u_4$, then

$$D(x, y, z) \preceq D(x, y, w) + D(x, w, z) + D(w, y, z) \preceq u_2 + u_3 + u_4.$$

So

$$\begin{aligned} & \{u_2 + u_3 + u_4 \in P | D(x, y, w) \preceq u_2, D(x, w, z) \preceq u_3, D(w, y, z) \preceq u_4\} \\ \subset & \{u_1 \in P | D(x, y, z) \preceq u_1\}, \end{aligned}$$

which implies

$$\begin{aligned} & \inf\{\|u_1\| | D(x, y, z) \preceq u_1\} \\ \leq & \inf\{\|u_2 + u_3 + u_4\| | D(x, y, w) \preceq u_2, D(x, w, z) \preceq u_3, D(w, y, z) \preceq u_4\} \end{aligned}$$

and we note that

$$\begin{aligned} & \inf\{\|u_2 + u_3 + u_4\| | D(x, y, w) \preceq u_2, D(x, w, z) \preceq u_3, D(w, y, z) \preceq u_4\} \\ \leq & \inf\{\|u_2\| + \|u_3\| + \|u_4\| | D(x, y, w) \preceq u_2, D(x, w, z) \preceq u_3, D(w, y, z) \preceq u_4\} \\ \leq & \inf\{\|u_2\| | D(x, y, w) \preceq u_2, u_2 \in P\} + \inf\{\|u_3\| | D(x, w, z) \preceq u_3, u_3 \in P\} \\ & + \inf\{\|u_4\| | D(w, y, z) \preceq u_4, u_4 \in P\}. \end{aligned}$$

Thus

$$\begin{aligned} & \inf\{\|u_1\|| D(x,y,z) \preceq u_1, u_1 \in P\} \\ \leq \ & \inf\{\|u_2\|| D(x,y,w) \preceq u_2, u_2 \in P\} + \inf\{\|u_3\|| D(x,w,z) \preceq u_3, u_3 \in P\} \\ & + \inf\{\|u_4\|| D(w,y,z) \preceq u_4, u_4 \in P\}, \end{aligned}$$

that is

$$\rho(x,y,z) \leq \rho(x,y,w) + \rho(x,w,z) + \rho(w,y,z).$$

From (i),(ii),(iii),(iv), we show that ρ is a 2-metric on X. □

Theorem 3.2. *Let (X,d) be a TVS-cone 2-metric space. Then $d_p : X \times X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ d$ is a 2-metric.*

Proof. (i) Since $\xi_e(y) = \inf\{r \in R : y \in re - P\}$, *i.e.*, $\xi_e(y) = inf\{r \in R : re \geq y\}$, then $\xi_e(d(x,y,z)) = \inf\{r \in R : re \geq d(x,y,z)\}$. Since $d(x,y,z) \succeq \theta$, we have $r \geq 0$, so $d_p(x,y,z) = \xi_e(d(x,y,z)) \geq 0$.
(ii) And $d(x,y,z) = d(x,z,y) = d(z,x,y)$, moreover

$$\begin{aligned} d_p(x,y,z) = \xi_e(d(x,y,z)), d_p(x,z,y) = \xi_e(d(x,z,y)), d_p(z,x,y) \\ = \xi_e(d(z,x,y)). \end{aligned}$$

Then by Definition 2.5, we have $d_p(x,y,z) = d_p(x,z,y) = d_p(z,x,y)$ for all $x,y,z \in X$.
(iii) If $x = y$, then $d_p(x,y,z) = \xi_e(\theta) = 0$. Conversely, if $d_p(x,y,z) = \xi_e(d(x,y,z)) = 0$, then $d(x,y,z) \in P \cap (-P) = \{\theta\}$ implies $x = y$ or $x = z$ or $y = z$.
(iv) Since $d(x,y,z) \preceq d(x,y,w)+d(x,w,z)+d(w,y,z)$, by Lemma 2.2-(vi),(vii),

$$\begin{aligned} \xi_e(d(x,y,z)) \leq \xi_e(d(x,y,w) + (d(x,w,z) + (d(w,y,z)) \leq \xi_e(d(x,y,w)) \\ + \xi_e(d(x,w,z)) + \xi_e(d(w,y,z)), \end{aligned}$$

we have

$$\xi_e(d(x,y,z)) \leq \xi_e(d(x,y,w)) + \xi_e(d(x,w,z)) + \xi_e(d(w,y,z)),$$

for all $x,y,z,w \in X$, that is

$$d_p(x,y,z) \leq d_p(x,y,w) + d_p(x,w,z) + d_p(w,y,z).$$

□

Corollary 3.3. *Let (X,D) be a cone 2-metric space in the sense of Huang-Zhang* [6] *and follow the idea in it. Then $d_p : X \times X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ D$ is a 2-metric space.*

Theorem 3.4. *Let (X,d) be a TVS-valued cone 2-metric space, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. For $\forall a \in X$. The following statements hold:*

(i) *If $\{x_n\}$ TVS-cone converges to x, then $d_p(x_n, x, a) \to 0$ as $n \to \infty$;*
(ii) *If $\{x_n\}$ is a TVS-cone Cauchy sequence in (X, d), then it is a Cauchy sequence in (X, d_p);*
(iii) *If (X, d) is TVS-cone complete, then (X, d_p) is a complete metric space.*

Proof. (i) Applying Theorem 3.2, d_p is a 2-metric on X. Let $\varepsilon > 0$, if $\{x_n\}$ TVS-cone converges to $x(n \to \infty)$, since $e \succ\succ \theta(e \in intP)$, $\exists n_0 \in N$ such that

$$\begin{aligned} d(x_n, x, a) \prec\prec \varepsilon e &\Leftrightarrow -d(x_n, x, a) + \varepsilon e \in P \\ &\Leftrightarrow -d(x_n, x, a) \in -\varepsilon e + P \\ &\Leftrightarrow d(x_n, x, a) \in \varepsilon e - P \text{ (Lemma 2.2-(i))} \\ &\Leftrightarrow d_p(x_n, x, a) = \xi_e \circ d(x_n, x, a) < \varepsilon \end{aligned}$$

for all $n \geq n_0$, so $d_p(x_n, x, a) \to 0$ as $n \to \infty$. Therefore, (i) holds.
(ii) Let $\{x_n\}$ be a TVS-cone Cauchy sequence in (X, d). Then there exists $n_1 \in N$ such that

$$\begin{aligned} d(x_n, x_m, a) \prec\prec \varepsilon e &\Leftrightarrow -d(x_n, x_m, a) + \varepsilon e \in P \\ &\Leftrightarrow -d(x_n, x_m, a) \in -\varepsilon e + P \\ &\Leftrightarrow d(x_n, x_m, a) \in \varepsilon e - P \text{ (Lemma 2.2-(i))} \\ &\Leftrightarrow d_p(x_n, x_m, a) = \xi_e \circ d(x_n, x_m, a) < \varepsilon \end{aligned}$$

for all $n, m \geq n_1$. So $\{x_n\}$ is a Cauchy sequence in (X, d_p), then (ii) holds.
(iii) We can get immediately from (i) and (ii). □

The following results are the main idea of this paper. First, by Lemma 3.1, we show that the cone 2-metric is equivalent to a 2-metric. Then we proof the 2-metric ρ in Lemma 3.1 is equivalent with d_p in Corollary 3.3. At last, we give an example which use the metric of Lemma 3.1.

Theorem 3.5. *For every cone 2-metric $D : X \times X \times X \to E$, there exists a 2-metric $\rho : X \times X \times X \to R^+$ which is equivalent to D on X.*

Proof. (i) Define $\rho(x, y, z) = \inf\{\|u\| | D(x, y, z) \preceq u\}$. By Lemma 3.1 we have ρ is a 2-metric. Since $\rho(x_n, x, a) = \inf\{\|u_{mn}\| | D(x_n, x, a) \preceq u_{mn}\}$, we have $\forall n, m \in N$, $\exists u_{nm}$ such that

$$\|u_{nm}\| < \rho(x_n, x, a) + \frac{1}{m}, D(x_n, x, a) \preceq u_{nm}.$$

Put $v_n = u_{nn}$, then $\|v_n\| < \rho(x_n, x, a) + \frac{1}{n}, (n \to \infty)$ and $D(x_n, x, a) \preceq v_n$. If $x_n \to x$ in (X, ρ), then $\rho(x_n, x, a) \to 0$ and $v_n \to 0$. Therefore for all $c \succ\succ \theta$, there exsits $n \in N$ such that $v_n \preceq c$ for all $n \geq N$. That implies $D(x_n, x, a) \prec\prec c$. Namely $x_n \to x$ in (X, D).

(ii) Conversely, we have $x_n \to x$ $(n \to \infty)$ in (X, D). For $\varepsilon > 0$, choose $c \in E$, $\forall c \in E$ with $c \succ\succ \theta$ such that $\|c\| < \varepsilon$, there exsits $N > 0$, when $n > N$, we have $D(x_n, x, a) \prec\prec c$, $\forall a \in X$. So

$$\rho(x_n, x, a) = \inf\{\|c\| | D(x_n, x, a) \preceq c\} \leq \|c\| < \varepsilon.$$

Then we have $x_n \to x$ $(n \to \infty)$ in (X, ρ). □

Theorem 3.6. *If $\rho(x, y, z) = \inf\{\|u\| | D(x, y, z) \preceq u\}$ and $d_p = \xi_e(D(x, y, z))$ where D is a cone 2-metric on X, then ρ is equivalent with d_p.*

Proof. (i) By Corollary 3.3, d_p is a 2-metric, now we show that each sequence $\{x_n\} \subseteq X$ which converges to a point $x \in X$ in (X, ρ) also converges to x in the (X, d_p). Let $x_n \to x$. Then $\rho(x_n, x, a) \to 0$ $(n \to \infty)$. By Theorem 3.5, we have $D(x_n, x, a) \to 0$ $(n \to \infty)$. So $\forall \varepsilon > 0$, $\forall e \succ\succ 0$, $\exists N > 0$, for all $n \geq N$ such that

$$D(x_n, x, a) \prec\prec \varepsilon e \Rightarrow \varepsilon e - D(x_n, x, a) \in intP \Rightarrow D(x_n, x, a) \in \varepsilon e - intP.$$

By Lemma 2.2-(iv), we have $\xi_e(D(x_n, x, a)) < \varepsilon$, that is $d_p(x_n, x, a) < \varepsilon$, so $d_p(x_n, x, a) \to 0$ $(n \to \infty)$.

(ii) Conversely, we show that each sequence $\{x_n\} \subseteq X$ which converges to a point $x \in X$ in (X, d_p) also converges to x in the (X, ρ). Let $x_n \to x$. Then $d_p(x_n, x, a) \to 0$ $(n \to \infty)$, that is $\xi_e(D(x_n, x, a)) \to 0$, so $\forall \varepsilon > 0$, $\exists N > 0$, for all $n \geq N$, $\xi_e(D(x_n, x, a)) < \varepsilon$, for some $p \in intP$, by Lemma 2.2-(iv), such that

$$D(x_n, x, a) \in \varepsilon e - intP \Rightarrow D(x_n, x, a) \in \varepsilon e - p \Rightarrow D(x_n, x, a) \prec\prec \varepsilon e,$$

this implies that $x_n \to x$ $(n \to \infty)$ in (X, D). By Theorem 3.5, we have $x_n \to x$ $(n \to \infty)$ in (X, ρ). □

Example 3.7. *Let $0 \neq a \in R^n$ and $\|a\| = 1$, for every $x, y, z \in R^n$ define*

$$D(x, y, z) = \begin{cases} a, & x \neq y \neq z; \\ 0, & others. \end{cases}$$

Then D is a cone 2-metric on R^n and its equivalent 2-metric ρ is

$$\rho(x, y, z) = \begin{cases} 1, & x \neq y \neq z; \\ 0, & others. \end{cases}$$

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