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FIXED POINT THEOREMS FOR SIX MAPS IN G-METRIC SPACES

Shaban Sedghi[1] and Nabi Shobkolaei[2]

[1]Department of Mathematics, Qaemshahr Branch
Islamic Azad University, Qaemshahr, Iran

e-mail: sedghi.gh@qaemshahriau.ac.ir

[2]Department of Mathematics, Babol Branch
Islamic Azad University, Babol, Iran

e-mail: nabi_shobe@yahoo.com

Abstract. In this paper we obtain a unique common fixed point theorem for six mappings in G-metric spaces.

1. Introduction

In 1992 Dhage introduced a new class of generalized metric spaces called D-metric spaces [1-4] as a generalization of ordinary metric function (X, d) and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [11] and Naidu et al. [7-9] demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect, alternatively, Mustafa and Sims [12] introduce more appropriate notion of generalized metric space which called G-metric spaces, and obtained some topological properties. Later several authors like [5, 6, 10, 13-17] obtained some fixed point theorems for a single map in G-metric spaces.

In this paper, we obtain a unique common fixed point theorem for six mappings in G-metric spaces and obtain some theorems of [10] and [18] as corollaries to our theorem.

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First, we present some known definitions and propositions in G-metric spaces.

Definition 1.1. ([12]) Let X be a nonempty set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties :
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G_3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.
Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. ([12]) Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be G-convergent to x.

Definition 1.3. ([12]) Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G-Cauchy if $\lim_{n,m,l\to\infty} G(x_l, x_n, x_m) = 0$. And (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Proposition 1.4. ([12]) *In a G-metric space (X, G), the following are equivalent.*

(1) *The sequence $\{x_n\}$ is G-Cauchy.*
(2) *For every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Proposition 1.5. ([12]) *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.6. ([12]) *Let (X, G) be a G-metric space. Then for any x, y, z, $a \in X$, it follows that*

(i) *if $G(x, y, z) = 0$ then $x = y = z$,*
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(x, x, y)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

Proposition 1.7. ([12]) *Let (X, G) be a G-metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent*

(i) $\{x_n\}$ *is G-convergent to* x,
(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Definition 1.8. ([12]) Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function. Then f is said to be *G-continuous at a point* $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.9. ([12]) *Let* (X, G), *and* (X', G') *be two G-metric spaces. Then a function* $f : X \longrightarrow X'$ *is G-continuous at a point* $x \in X$ *if and only if it is G-sequentially continuous at* x*; that is, whenever* $\{x_n\}$ *is G-convergent to* x *we have* $\{f(x_n)\}$ *is G-convergent to* $f(x)$.

We recall that two maps f and g are said to be weak compatible if they commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.

The main aim of this paper is to present a generalization of Theorem 2.1 in [18].

2. Main Result

Let Φ denote the class of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is non decreasing, continuous and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$. It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$, for all $t > 0$.

Theorem 2.1. *Let* (X, G) *be a G-Complete metric space and* $A, B, C, S, T, R : X \to X$ *be satisfying:*
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq R(X)$, $C(X) \subseteq S(X)$ *and either one of* $T(X)$, $R(X)$ *or* $S(X)$ *is a closed subset of* X,
(ii) *the pair* (A, S), (B, T) *and* (C, R) *are weakly compatible,*
(iii)

$$G(Ax, By, Cz) \leq \phi\left(\max\left\{\begin{array}{c} G(Sx, Ty, Rz), \frac{1}{2}G(Sx, Ax, By), \\ \frac{1}{2}G(Ty, By, Cz), \frac{1}{2}G(Rz, Cz, Ax) \end{array}\right\}\right) \tag{2.1}$$

for all $x, y, z \in X$, *where* $\phi \in \Phi$. *Then the maps* A, B, C, T, R *and* S *have a unique common fixed point, say* $p \in X$ *and if* T, R *and* S *are G-continuous at* p *then* A, B *and* C *are G-continuous at* p.

Proof. Choose $x_0 \in X$. Define the sequence $y_{3n} = Ax_{3n} = Tx_{3n+1}$, $y_{3n+1} = Bx_{3n+1} = Rx_{3n+2}$, $y_{3n+2} = Cx_{3n+2} = Sx_{3n+3}$, $n = 0, 1, 2, \cdots$. Denote $d_{n+1} = G(y_n, y_{n+1}, y_{n+2})$. Then in general we have

$$\begin{aligned}
& d_{3n+1} = G(y_{3n}, y_{3n+1}, y_{3n+2}) \\
& = G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \\
& \leq \phi\left(\max\left\{\begin{array}{c} G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2}), \frac{1}{2}G(Sx_{3n}, Ax_{3n}, Bx_{3n+1}), \\ \frac{1}{2}G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \frac{1}{2}G(Rx_{3n+2}, Cx_{3n+2}, Ax_{3n}) \end{array}\right\}\right) \\
& = \phi\left(\max\left\{\begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}), \frac{1}{2}G(y_{3n-1}, y_{3n}, y_{3n+1}), \\ \frac{1}{2}G(y_{3n}, y_{3n+1}, y_{3n+2}), \frac{1}{2}G(y_{3n+1}, y_{3n+2}, y_{3n}) \end{array}\right\}\right) \\
& \leq \phi\left(\max\left\{d_{3n}, \tfrac{1}{2}d_{3n}, \tfrac{1}{2}d_{3n+1}, \tfrac{1}{2}d_{3n+1}\right\}\right).
\end{aligned}$$

Hence, we must have $d_{3n+1} \leq \phi(d_{3n})$. Similarly, we will have $d_{3n+2} \leq \phi(d_{3n+1})$ and $d_{3n+3} \leq \phi(d_{3n+2})$. Thus $d_{n+1} \leq \phi(d_n), n = 1, 2, 3, \cdots$, so,

$$\begin{aligned}
G(y_n, y_{n+1}, y_{n+2}) & \leq \phi(G(y_{n-1}, y_n, y_{n+1})) \\
& \leq \phi^2(G(y_{n-2}, y_{n-1}, y_n)) \\
& . \\
& . \\
& . \\
& \leq \phi^n(G(y_0, y_1, y_2)). \qquad (2.2)
\end{aligned}$$

From (G_3) and (2.2), we have

$$G(y_n, y_n, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+2}) \leq \phi^n(G(y_0, y_1, y_2)). \qquad (2.3)$$

For $m > n$, since $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, we have

$$\begin{aligned}
& G(y_n, y_n, y_m) \\
& \leq \sum_{i=n}^{m-1} G(y_i, y_i, y_{i+1}) \\
& \leq \phi^n(G(y_0, y_1, y_2)) + \phi^{n+1}(G(y_0, y_1, y_2)) + ... + \phi^{m-1}(G(y_0, y_1, y_2)) \\
& \leq \sum_{i=n}^{\infty} \phi^i(G(y_0, y_1, y_2)) \\
& \to 0 \ as \ n \to \infty.
\end{aligned}$$

Hence, $\{y_n\}$ is a G-Cauchy sequence from Proposition 1.4. Since X is G-complete, there exists $p \in X$ such that $y_n \to p$ $(n \to \infty)$, that is, $\lim_{n\to\infty} y_n = p$,

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} Ax_{3n} = \lim_{n\to\infty} Bx_{3n+1} = \lim_{n\to\infty} Cx_{3n+2} \\
&= \lim_{n\to\infty} Tx_{3n+1} = \lim_{n\to\infty} Rx_{3n+2} = \lim_{n\to\infty} Sx_{3n+3} = p.
\end{aligned}$$

Let $T(X)$ be a closed subset of X. Then there exists $v \in X$ such that $Tv = p$. We prove that $Bv = p$. For

$$\begin{aligned}
&G(Ax_{3n}, Bv, Cx_{3n+2}) \\
&\leq \phi\left(\max\left\{\begin{array}{c} G(Sx_{3n}, Tv, Rx_{3n+2}), \frac{1}{2}G(Sx_{3n}, Ax_{3n}, Bv), \\ \frac{1}{2}G(Tv, Bv, Cx_{3n+2}), \frac{1}{2}G(Rx_{3n+2}, Cx_{3n+2}, Ax_{3n}) \end{array}\right\}\right) \\
&= \phi\left(\max\left\{\begin{array}{c} G(y_{3n-1}, p, y_{3n+1}), \frac{1}{2}G(y_{3n-1}, y_{3n}, Bv), \\ \frac{1}{2}G(p, Bv, y_{3n+2}), \frac{1}{2}G(y_{3n+1}, y_{3n+2}, y_{3n}) \end{array}\right\}\right).
\end{aligned}$$

Letting $n \to \infty$, we get

$$G(p, Bv, p) \leq \phi\left(\max\left\{0, \frac{1}{2}G(p, p, Bv), \frac{1}{2}G(p, Bv, p), 0\right\}\right),$$

and we have $Bv = Tv = p$, since $\phi(t) < t$ for all $t > 0$. Since $p = Bv \in B(X) \subseteq R(X)$, there exists $z \in X$ such that $Rz = p$. Similarly, putting $x = x_{3n}, y = x_{3n+1}, z = p$ in (1) and letting $n \to \infty$, we get $Cz = p$. Since $p = Cz \in C(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = p$.

Similarly, putting $x = p, y = x_{3n+1}, z = x_{3n+2}$ in (1) and letting $n \to \infty$, we get $Au = p$. By weak compatible the pair (A, S), we have $ASu = SAu$, and $Ap = Sp$. Also, $Bp = Tp$ and $Rp = Cp$. We prove that $Ap = p$. If $Ap \neq p$, then

$$\begin{aligned}
&G(Ap, Bx_{3n+1}, Cx_{3n+2}) \\
&\leq \phi\left(\max\left\{\begin{array}{c} G(Sp, Tx_{3n+1}, Rx_{3n+2}), \frac{1}{2}G(Sp, Ap, Bx_{3n+1}), \\ \frac{1}{2}G(Tx_{3n+1}, Bx_{3n+1}, Cx_{3n+2}), \frac{1}{2}G(Rx_{3n+2}, Cx_{3n+2}, Ap) \end{array}\right\}\right) \\
&= \phi\left(\max\left\{\begin{array}{c} G(Ap, y_{3n}, y_{3n+1}), \frac{1}{2}G(Ap, Ap, y_{3n+1}), \\ \frac{1}{2}G(y_{3n}, y_{3n+1}, y_{3n+1}), \frac{1}{2}G(y_{3n+1}, y_{3n+2}, Ap) \end{array}\right\}\right).
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
G(Ap, p, p) &\leq \phi\left(\max\left\{G(Ap, p, p), \frac{1}{2}G(Ap, Ap, p), 0, \frac{1}{2}G(p, p, Ap)\right\}\right) \\
&\leq \phi\left(\max\left\{G(Ap, p, p), G(Ap, p, p), 0, \frac{1}{2}G(p, p, Ap)\right\}\right)
\end{aligned}$$

and we have $Ap = p$, since $\phi(t) < t$ for all $t > 0$. Similarly, putting $x = x_{3n}, y = p, z = x_{3n+2}$ and $x = x_{3n}, y = x_{3n+1}, z = p$ in (1) and letting $n \to \infty$, we get $Bp = p$ and $Cp = p$, respectively. Thus p is a common fixed point of A, B, C, T, R and S.

Suppose $p\,'$ is another common fixed point of A, B, C, T, R and S. Then from (1), we have

$$G(Ap, Bp, Cp\,') \leq \phi\left(\max\left\{G(p, p, p\,'), 0, \frac{1}{2}G(p, p, p\,'), \frac{1}{2}G(p\,', p\,', p)\right\}\right).$$

Thus, $G(p, p, p\,') \leq \phi(G(p, p, p\,')$ so that $p = p\,'$. Thus, p is the unique common fixed point of A, B, C, T, R and S. Let $\{y_n\}$ be any sequence in X

such that $\{y_n\}$ is G-convergent to p. Then we have

$$\begin{aligned} &G(Ay_n, Ap, Ap) = G(Ay_n, Bp, Cp) \\ &\leq \phi\left(\max\left\{\begin{array}{c} G(Sy_n, p, p), \frac{1}{2}G(Sy_n, Ay_n, p), \\ 0, \frac{1}{2}G(p, p, Ay_n) \end{array}\right\}\right) \\ &\leq \phi\left(\max\left\{\begin{array}{c} G(Sy_n, p, p), \frac{1}{2}G(Sy_n, p, p) + \frac{1}{2}G(p, Ay_n, p), \\ 0, \frac{1}{2}G(p, p, Ay_n) \end{array}\right\}\right). \end{aligned}$$

Letting $n \to \infty$, then since S is G-continuous at p we get, $\alpha \leq \phi(\frac{1}{2}\alpha)$, where

$$\alpha = \lim_{n\to\infty} G(Ay_n, Ap, Ap).$$

This implies that $\alpha = 0$. Then from Proposition 1.9, we deduce that A is G-continuous at p. Similarly, we can show that B and C are also G-continuous at p. □

Corollary 2.2. (Corollary 2.1 of [18]) *Let (X, G) be a G-Complete metric space and $A, B, C : X \to X$ be satisfying:*

$$G(Ax, By, Cz) \leq \phi\left(\max\left\{\begin{array}{c} G(x, y, z), \frac{1}{2}G(x, Ax, By), \\ \frac{1}{2}G(y, By, Cz), \frac{1}{2}G(z, Cz, Ax) \end{array}\right\}\right) \tag{2.4}$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Then the maps A, B, C have a unique common fixed point, say $p \in X$ and A, B and C are G-continuous at p.

Proof. Take $T = R = S = I$ identity map in Theorem 2.1. □

Corollary 2.3. *Let (X, G) be a G-Complete metric space and $A, B : X \to X$ be satisfying:*
(i) *$A(X) \subseteq B(X)$, $B(X)$ is a closed subset of X and the pair (A, B) is weakly compatible,*
(ii)

$$G(Ax, Ay, Az) \leq \phi\left(\max\left\{\begin{array}{c} G(Bx, By, Bz), \frac{1}{2}G(Bx, Ax, Ay), \\ \frac{1}{2}G(By, Ay, Az), \frac{1}{2}G(Bz, Az, Ax) \end{array}\right\}\right) \tag{2.5}$$

for all $x, y, z \in X$, where $\phi \in \Phi$. Then the maps A, B have a unique common fixed point, say $p \in X$ and if B is G-continuous at p then A is G-continuous at p.

Proof. Take $B = C = A$ and $T = R = S = B$ in Theorem 2.1. □

The following example illustrates Theorem 2.1 with $\phi(t) = \frac{t}{2}$.

Example 2.4. *Let $X = [0,1]$ and $G(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in X$. Define $A,B,C,T,R,S : X \to X$ as $Ax = Cx = 1$, $Bx = \frac{3+x}{4}$, $Tx = x$, $Sx = \frac{x^2+x}{2}$, $Rx = \frac{2+x^2}{3}$ for all $x \in X$.*

$$\begin{aligned}
&G(Ax,By,Cz)\\
&= \tfrac{1-y}{2}\\
&= \tfrac{1}{4}2(1-y)\\
&= \tfrac{1}{4}G(y,By,Cz) = \tfrac{1}{2}(\tfrac{1}{2}G(Ty,By,Cz))\\
&\le \tfrac{1}{2}\max\left\{G(Sx,Ty,Cz), \tfrac{1}{2}G(Sx,Ax,By), \tfrac{1}{2}G(Ty,By,Cz), \tfrac{1}{2}G(Rz,Cz,Ax)\right\}.
\end{aligned}$$

Also, it is easy to see that $A(X) \subseteq T(X)$, $B(X) \subseteq R(X)$, $C(X) \subseteq S(X)$ and $T(X) = X$ is a closed subset of X. Moreover, the pair (A,S), (B,T) and (C,R) are weakly compatible. Hence the all of conditions of Theorem 2.1 are hold and 1 is the unique common fixed point of A,B,C,T,R and S.

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