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ACCELERATED STRONGLY CONVERGENT EXTRAGRADIENT ALGORITHMS TO SOLVE VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN REAL HILBERT SPACES

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Abstract. Two inertial extragradient-type algorithms are introduced for solving convex pseudomonotone variational inequalities with fixed point problems, where the associated mapping for the fixed point is a ρ-demicontractive mapping. The algorithm employs variable step sizes that are updated at each iteration, based on certain previous iterates. One notable advantage of these algorithms is their ability to operate without prior knowledge of Lipschitz-type constants and without necessitating any line search procedures. The iterative sequence constructed demonstrates strong convergence to the common solution of the variational inequality and fixed point problem under standard assumptions. In-depth numerical applications are conducted to illustrate theoretical findings and to compare the proposed algorithms with existing approaches.

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1. Introduction

The investigation of common solution problems is driven by their potential applicability within mathematical models characterized by fixed point constraints, particularly in practical domains such as signal processing, network resource allocation, and image recovery. This aspect holds considerable significance across various fields, encompassing signal analysis, composite reduction, optimization methodologies, and image recovery challenges, as evidenced by relevant literature (e.g., Maing et al., [20, 21], Iiduka et al., [10], Qin et al., [27], An et al., [4]). This study scrutinizes two focal issues within this realm.

Let $\mathcal{K}$ denote a nonempty, closed and convex subset of a real Hilbert space $\mathcal{X}$ endowed with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding induced norm $\| \cdot \|$. The study makes a significant contribution by exploring the convergence analysis of iterative algorithms designed to solve variational inequality problems and fixed point problems in real Hilbert spaces.

Considering an operator $\mathcal{N} : \mathcal{K} \to \mathcal{X}$, the variational inequality problem (VIP) (Stampacchia [30], 1964) is formulated as follows:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } \big\langle \mathcal{N}(u^*), y - u^* \big\rangle \geq 0, \text{ for all } y \in \mathcal{K}. \tag{VIP}$$

Let $VI(\mathcal{K}, \mathcal{N})$ denote the solution set corresponding to problem (VIP). Variational inequalities find applications across diverse domains, including but not limited to partial differential equations, optimization, engineering, applied mathematics, and economics (see, for instance, [1, 8, 11, 12, 15, 16, 19, 25, 31]). The variational inequality problem holds significant importance in the applied sciences. Numerous researchers have dedicated considerable effort not only to exploring the existence and stability of solutions but also to devising iterative algorithms for their resolution. Projection algorithms, in particular, play a pivotal role in ascertaining the numerical solutions to variational inequalities. [6, 7, 14, 17, 26, 29] along with others documented in works [2, 3, 9, 13, 23, 24, 35, 34].

The majority of algorithms employed for problem-solving rely on the projection technique computed within the feasible set $\mathcal{K}$. Among these methods, the extragradient algorithm, originating from the works of Korpelevich [17] and Antipin [5], stands out prominently. Consider $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ as a mapping. The associated fixed-point problem with $\mathcal{Q}$ is expressed as:

$$\mathcal{Q}(u^*) = u^*. \tag{FP}$$

The solution set of the fixed point problem (FP) is denoted as $Fix(\mathcal{Q})$. A considerable portion of algorithms aimed at solving (FP) is derived from the foundational Mann iteration scheme. Specifically, commencing with $u_1 \in \mathcal{X}$, this scheme generates the sequence $\{u_{k+1}\}$ for every $k \geq 1$ according to the recurrence relation:

$$u_{k+1} = \alpha_k u_k + (1 - \alpha_k)\mathcal{Q}(u_k). \tag{1.1}$$

For constructing weak convergence necessitates adherence to specific criteria by the variable sequence $\{\alpha_k\}$. The Halpern iteration introduces an alternative structured iterative algorithm, demonstrating heightened efficacy in achieving strong convergence within infinite-dimensional Hilbert spaces. The iterative sequence is delineated as follows:

$$u_{k+1} = \alpha_k u_1 + (1 - \alpha_k)\mathcal{Q}u_k, \tag{1.2}$$

where $u_1 \in \mathcal{X}$, and the sequence $\alpha_k \subset (0, 1)$ is non-summable and showcases gradual decline, conforming to the conditions:

$$\alpha_k \to 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \alpha_k = +\infty.$$

Additionally, the viscosity algorithm, as introduced by Moudafi [22], which merges the cost mapping $\mathcal{Q}$ with a contraction mapping iteratively, presents a generic variant of the Halpern iteration. Alongside the Halpern iteration, a generalized form known as the viscosity algorithm [22] exists, wherein the cost mapping $\mathcal{Q}$ is integrated with a contraction mapping within the iterative process. Lastly, the hybrid steepest descent approach, proposed in [33], represents another methodology offering substantial convergence benefits.

Tan et al. [32] introduced a new numerical algorithm called the extragradient viscosity algorithm. It's designed to solve variational inequalities that involve a fixed-point problem with a specific kind of mapping called a ρ-demicontractive mapping. This algorithm combines ideas from two other algorithms: the extragradient algorithm and the Mann-type technique. The authors showed that all these algorithms have strong convergence if the operator is both monotonic and satisfies the Lipschitz condition. These methods offer an advantage because they can be estimated numerically using optimization tools, as demonstrated in [32]. However, one drawback of these algorithms is that they heavily rely on viscosity and Mann-type techniques to achieve strong convergence. Strong convergence is crucial for iterative sequences, especially in situations with infinite dimensions. Only a few algorithms achieve strong convergence using inertial schemes. Using Mann and viscosity procedures can be challenging from an algorithmic perspective, potentially slowing down convergence and limiting practical usefulness. These algorithms also require more numerical and computational steps, making the system more complex.

This brings us to a fundamental question:

Can we make self-adjusting, strongly convergent inertial extragradient algorithms that don't need Mann and Viscosity-type methods to solve variational inequalities and fixed-point problems?

In response, we have developed two strong convergence extragradient-type algorithms. These are designed to solve monotone variational inequalities and the ρ-demicontractive fixed point problem in real Hilbert spaces. Our inspiration comes from the research discussed in [32]. Additionally, we have intentionally avoided using any hybrid techniques, like the Mann-type scheme or the viscosity scheme, to ensure the strong convergence of these algorithms.

The paper is organized into several sections. In Section 2, we present some fundamental findings. Section 3 introduces four distinct algorithms and verifies their convergence analysis. Lastly, Section 4 offers numerical data to showcase the practical application of the presented algorithms.

2. Preliminaries

Let $\mathcal{K}$ denote a nonempty, closed and convex subset of a real Hilbert space $\mathcal{X}$. For any $u, y \in \mathcal{X}$, we establish the following properties:

(i) $\|u+y\|^2 = \|u\|^2 + 2\langle u, y\rangle + \|y\|^2$;

(ii) $\|u+y\|^2 \leq \|u\|^2 + 2\langle y, u+y\rangle$;

(iii) $\|bu+(1-b)y\|^2 = b\|u\|^2 + (1-b)\|y\|^2 - b(1-b)\|u-y\|^2$.

A metric projection $P_{\mathcal{K}}(u)$ of $u \in \mathcal{X}$ is defined as follows:

$$P_{\mathcal{K}}(u) = \arg\min\{\|u-y\| : y \in \mathcal{K}\}.$$

It is well established that $P_{\mathcal{K}}$ is nonexpansive and possesses the following significant properties:

(1) $\big\langle u - P_{\mathcal{K}}(u), y - P_{\mathcal{K}}(u)\big\rangle \leq 0, \ \forall\, y \in \mathcal{K}$;

(2) $\|P_{\mathcal{K}}(u) - P_{\mathcal{K}}(y)\|^2 \leq \big\langle P_{\mathcal{K}}(u) - P_{\mathcal{K}}(y), u-y\big\rangle, \ \forall\, y \in \mathcal{K}$.

Definition 2.1. Let $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ be a nonlinear mapping with $Fix(\mathcal{Q}) \neq \emptyset$. Then, $I - \mathcal{Q}$ is said to be demiclosed at zero if for any sequence $\{u_k\}$ in $\mathcal{X}$, the following statement holds:

$$u_k \rightharpoonup u \quad \text{and} \quad (I-\mathcal{Q})u_k \to 0 \ \ \Rightarrow \ \ u \in Fix(\mathcal{Q}),$$

where $Fix(\mathcal{Q}) = \{x \in \mathcal{X} : \mathcal{Q}x = x\}$ is the set of fixed points of $\mathcal{Q}$.

Definition 2.2. Let $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ be an operator. It is said to be:

(1) *monotone* if
$$\langle \mathcal{N}(u_1) - \mathcal{N}(u_2), u_1 - u_2 \rangle \geq 0, \quad \forall\, u_1, u_2 \in \mathcal{K};$$
(2) *Lipschitz-continuous* with constant $L > 0$ such that
$$\|\mathcal{N}(u_1) - \mathcal{N}(u_2)\| \leq L\|u_1 - u_2\|, \quad \forall\, u_1, u_2 \in \mathcal{K};$$
(3) *sequentially weakly continuous* if a sequence $\{\mathcal{N}(u_k)\}$ converges weakly to $\mathcal{N}(u)$ for any sequence $\{u_k\}$ converging weakly to u.

Definition 2.3. Let $\mathcal{Q} : \mathcal{K} \to \mathcal{K}$ be a mapping such that $Fix(\mathcal{Q}) \neq \emptyset$. $\mathcal{Q}$ is said to be ρ-demicontractive if for any fixed number $0 \leq \rho < 1$, the following holds:
$$\|\mathcal{Q}(u_1) - u_2\|^2 \leq \|u_1 - u_2\|^2 + \rho\|(I - \mathcal{Q})(u_1)\|^2, \quad \forall\, u_2 \in Fix(\mathcal{Q}), u_1 \in \mathcal{X}.$$

Lemma 2.4. ([18]) *Let $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ be an operator that is L-Lipschitz continuous and monotone on $\mathcal{K}$. Consider $\mathcal{Q} = P_{\mathcal{K}}(I - \eta\mathcal{N})$, where $\eta > 0$. If $\{u_k\}$ is a sequence in $\mathcal{X}$ such that $u_k \rightharpoonup q$ and $u_k - \mathcal{N}(u_k) \to 0$, then $q \in VI(\mathcal{K}, \mathcal{N}) = Fix(\mathcal{Q})$.*

Lemma 2.5. ([28]) *Suppose that $\{c_k\} \subset [0, +\infty)$, $\{d_k\} \subset (0, 1)$ and $\{e_k\} \subset \mathbb{R}$ are sequences satisfying the following conditions:*
$$c_{k+1} \leq (1 - d_k)c_k + d_k e_k, \quad \forall\, k \in \mathbb{N} \quad \textit{and} \quad \sum_{k=1}^{+\infty} d_k = +\infty.$$
If $\limsup_{j\to+\infty} y_{k_j} \leq 0$ for any subsequence $\{c_{k_j}\}$ of $\{c_j\}$ such that
$$\liminf_{j\to+\infty}(c_{k_j+1} - c_{k_j}) \geq 0.$$
Then $\lim_{k\to+\infty} c_k = 0$.

3. Main results

In this section, we delve into the convergence analysis of two new inertial extragradient algorithms designed for solving fixed point and variational inequality problems. Initially, we scrutinize the proposed algorithms. To establish strong convergence, it is presupposed that the following conditions hold:

- ($\mathcal{N}$1) The common solution set, denoted by $Fix(\mathcal{Q}) \cap VI(\mathcal{K}, \mathcal{N})$, is nonempty.
- ($\mathcal{N}$2) The operator $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is monotone.
- ($\mathcal{N}$3) The operator $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is Lipschitz continuous.

- ($\mathcal{N}4$) The mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ is ρ-demicontractive for $0 \le \rho < 1$ and demiclosed at zero.
- ($\mathcal{N}5$) The operator $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ is sequentially weakly continuous.

Algorithm 1 (Accelerated Tseng's Extragradient Algorithm With Monotone Variable Step Size Rule)

STEP 0: Choose $u_0, u_1 \in \mathcal{K}$, $\theta \in (0,1)$, $\mu \in (0,1)$, $\eta_1 > 0$, and a sequence $\{\varsigma_k\} \subset (0,1)$ satisfying:

$$\lim_{k\to+\infty} \varsigma_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \varsigma_k = +\infty.$$

STEP 1: Compute:

$$q_k = u_k + \theta_k(u_k - u_{k-1}) - \varsigma_k\big[u_k + \theta_k(u_k - u_{k-1})\big],$$

where θ_k is defined as follows:

$$0 \le \theta_k \le \hat{\theta}_k \quad \text{and} \quad \hat{\theta}_k = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\chi_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \ne u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise.} \end{cases} \tag{3.1}$$

Additionally, a sequence $\chi_k = \circ(\varsigma_k)$ satisfies the condition $\lim_{k\to+\infty} \frac{\chi_k}{\varsigma_k} = 0$.

STEP 2: Compute:

$$y_k = P_{\mathcal{K}}(q_k - \eta_k \mathcal{N}(q_k)).$$

If $q_k = y_k$, then STOP; otherwise, proceed to **STEP 3**.

STEP 3: Compute:

$$p_k = y_k + \eta_k\big[\mathcal{N}(q_k) - \mathcal{N}(y_k)\big].$$

STEP 4: For any sequence $\alpha_k \subset (0, 1-\rho)$, compute:

$$u_{k+1} = (1-\alpha_k)p_k + \alpha_k \mathcal{Q}(p_k).$$

STEP 5: Compute:

$$\eta_{k+1} = \begin{cases} \min\left\{\eta_k, \ \frac{\mu\|q_k - y_k\|}{\|\mathcal{N}(q_k) - A(y_k)\|}\right\} & \text{if} \quad \mathcal{N}(q_k) \ne \mathcal{N}(y_k), \\ \eta_k, & \text{otherwise.} \end{cases} \tag{3.2}$$

Update $k := k+1$ and return to **STEP 1**.

Algorithm 2 (Accelerated Tseng's Extragradient Algorithm With Non-Monotone Variable Step Size Rule)

STEP 0: Choose initial points $u_0, u_1 \in \mathcal{K}$, parameters $\theta \in (0,1)$, $\mu \in (0,1)$, and $\eta_1 > 0$. Also, select a sequence $\{\daleth_k\}$ such that $\sum_{k=1}^{+\infty} \daleth_k < +\infty$ and a sequence $\{\varsigma_k\} \subset (0, 1-\rho)$ satisfying:

$$\lim_{k \to +\infty} \varsigma_k = 0 \quad \text{and} \quad \sum_{k=1}^{+\infty} \varsigma_k = +\infty.$$

STEP 1: Compute

$$q_k = u_k + \theta_k(u_k - u_{k-1}) - \varsigma_k\big[u_k + \theta_k(u_k - u_{k-1})\big],$$

where θ_k is defined as follows:

$$0 \le \theta_k \le \hat{\theta}_k \quad \text{and} \quad \hat{\theta}_k = \begin{cases} \min\left\{\frac{\theta}{2}, \frac{\chi_k}{\|u_k - u_{k-1}\|}\right\} & \text{if} \quad u_k \neq u_{k-1}, \\ \frac{\theta}{2} & \text{otherwise}, \end{cases} \tag{3.3}$$

with $\chi_k = \mathcal{O}(\varsigma_k)$ satisfying $\lim_{k \to +\infty} \frac{\chi_k}{\varsigma_k} = 0$.

STEP 2: Compute

$$y_k = P_{\mathcal{K}}(q_k - \eta_k \mathcal{N}(q_k)).$$

If $q_k = y_k$, then STOP; otherwise, proceed to **STEP 3**.

STEP 3: Compute

$$p_k = y_k + \eta_k\big[\mathcal{N}(q_k) - \mathcal{N}(y_k)\big].$$

STEP 4: Choose a sequence $\alpha_k \subset (0, 1-\rho)$. Compute

$$u_{k+1} = (1 - \alpha_k)p_k + \alpha_k \mathcal{Q}(p_k).$$

STEP 5: Compute

$$\begin{cases} \min\left\{\eta_k + \daleth_k, \ \frac{\mu\|q_k - y_k\|}{\|\mathcal{N}(q_k) - A(y_k)\|}\right\} & \text{if } \mathcal{N}(q_k) \neq \mathcal{N}(y_k), \\ \eta_k + \daleth_k, & \text{otherwise}, \end{cases} \tag{3.4}$$

where $A(y_k)$ denotes a specific operator. Then, increment k and return to **STEP 1**.

Lemma 3.1. *Let $\{\eta_k\}$ be a sequence generated by the expression* (3.2)*. Then, $\{\eta_k\}$ is monotonically decreasing and bounded by* $\min\left\{\frac{\mu}{L}, \eta_1\right\} \le \eta \le \eta_1$.

Proof. Given that the mapping $\mathcal{N}$ is Lipschitz continuous, we have:

$$\begin{aligned} \frac{\mu\|q_k - y_k\|}{\|\mathcal{N}(q_k) - \mathcal{N}(y_k)\|} &\ge \frac{\mu\|q_k - y_k\|}{L\|q_k - y_k\|} \\ &\ge \frac{\mu}{L}. \end{aligned} \tag{3.5}$$

This implies that the sequence $\{\eta_k\}$ is monotonically decreasing and bounded. □

Lemma 3.2. *A sequence* $\{\eta_k\}$ *generated by the expression* (3.4) *is convergent to* η *and bounded by* $\min\left\{\frac{\mu}{L}, \eta_1\right\} \leq \eta \leq \eta_1 + P$*, where*

$$P = \sum_{k=1}^{+\infty} \daleth_k.$$

Proof. Given that the mapping $\mathcal{N}$ is Lipschitz continuous, we have

$$\begin{aligned} \frac{\mu\|q_k - y_k\|}{\|\mathcal{N}(q_k) - \mathcal{N}(y_k)\|} &\geq \frac{\mu\|q_k - y_k\|}{L\|q_k - y_k\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{3.6}$$

By the definition of η_{k+1}, we have

$$\min\left\{\frac{\mu}{L}, \eta_1\right\} \leq \eta_k \leq \eta_1 + P.$$

Let

$$[\eta_{k+1} - \eta_k]^+ = \max\{0, \eta_{k+1} - \eta_k\}$$

and

$$[\eta_{k+1} - \eta_k]^- = \max\{0, -(\eta_{k+1} - \eta_k)\}.$$

Using the definition of $\{\eta_k\}$, we have

$$\sum_{k=1}^{+\infty}(\eta_{k+1} - \eta_k)^+ = \sum_{k=1}^{+\infty} \max\{0, \eta_{k+1} - \eta_k\} \leq P < +\infty. \tag{3.7}$$

This implies that the series $\sum_{k=1}^{+\infty}(\eta_{k+1} - \eta_k)^+$ converges. Next, we need to demonstrate the convergence of

$$\sum_{k=1}^{+\infty}(\eta_{k+1} - \eta_k)^-.$$

Suppose $\sum_{k=1}^{+\infty}(\eta_{k+1} - \eta_k)^- = +\infty$. Then we obtain

$$\eta_{k+1} - \eta_k = (\eta_{k+1} - \eta_k)^+ - (\eta_{k+1} - \eta_k)^-.$$

Hence,

$$\eta_{k+1} - \eta_1 = \sum_{k=0}^{k}(\eta_{k+1} - \eta_k) = \sum_{k=0}^{k}(\eta_{k+1} - \eta_k)^+ - \sum_{k=0}^{k}(\eta_{k+1} - \eta_k)^-. \tag{3.8}$$

Letting $k \to +\infty$ in (3.8), we get $\eta_k \to -\infty$ as $k \to +\infty$, which is a logical contradiction. Hence, due to the convergence of the series $\sum_{k=0}^{k}(\eta_{k+1} - \eta_k)^+$ and $\sum_{k=0}^{k}(\eta_{k+1} - \eta_k)^-$, taking $k \to +\infty$ in (3.8) yields $\lim_{k\to+\infty} \eta_k = \eta$. This completes the proof. □

Lemma 3.3. *Let $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ be an operator satisfying conditions ($\mathcal{N}$1)–($\mathcal{N}$5). Suppose that $\{u_k\}$ is a sequence generated by Algorithms 1 and 2. For any $u^* \in VI(\mathcal{K}, \mathcal{N})$, we have*

$$\|p_k - u^*\|^2 \le \|q_k - u^*\|^2 - \left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right) \|q_k - y_k\|^2 .$$

Proof. Consider the following expression

$$\begin{aligned}
\|p_k - u^*\|^2 &= \|y_k + \eta_k[\mathcal{N}(u_k) - \mathcal{N}(y_k)] - u^*\|^2 \\
&= \|y_k - u^*\|^2 + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 \\
&\quad + 2\eta_k\langle y_k - u^*, \mathcal{N}(u_k) - \mathcal{N}(y_k)\rangle \\
&= \|y_k + u_k - u_k - u^*\|^2 + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 \\
&\quad + 2\eta_k\langle y_k - u^*, \mathcal{N}(u_k) - \mathcal{N}(y_k)\rangle \\
&= \|y_k - u_k\|^2 + \|u_k - u^*\|^2 + 2\langle y_k - u_k, u_k - u^*\rangle \\
&\quad + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 + 2\eta_k\langle y_k - u^*, \mathcal{N}(u_k) - \mathcal{N}(y_k)\rangle \\
&= \|u_k - u^*\|^2 + \|y_k - u_k\|^2 + 2\langle y_k - u_k, y_k - u^*\rangle \\
&\quad + 2\langle y_k - u_k, u_k - y_k\rangle + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 \\
&\quad + 2\eta_k\langle y_k - u^*, \mathcal{N}(u_k) - \mathcal{N}(y_k)\rangle.
\end{aligned} \tag{3.9}$$

Additionally, we can express

$$\langle u_k - \eta_k\mathcal{N}(u_k) - y_k, y - y_k\rangle \le 0, \ \forall\, y \in \mathcal{K}. \tag{3.10}$$

For a given $u^* \in VI(\mathcal{K}, \mathcal{N})$, we can state

$$\langle u_k - y_k, u^* - y_k\rangle \le \eta_k\langle \mathcal{N}(u_k), u^* - y_k\rangle. \tag{3.11}$$

By combining equations (3.9) and (3.11), we obtain

$$\begin{aligned}\|p_k - u^*\|^2 \le{}& \|u_k - u^*\|^2 + \|y_k - u_k\|^2 + 2\eta_k\langle \mathcal{N}(u_k), u^* - y_k\rangle \\ &- 2\langle u_k - y_k, u_k - y_k\rangle + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 \\ &- 2\eta_k\langle \mathcal{N}(u_k) - \mathcal{N}(y_k), u^* - y_k\rangle \\ ={}& \|u_k - u^*\|^2 - \|u_k - y_k\|^2 + \eta_k^2\|\mathcal{N}(u_k) - \mathcal{N}(y_k)\|^2 \\ &- 2\eta_k\langle \mathcal{N}(y_k), y_k - u^*\rangle. \end{aligned} \tag{3.12}$$

Utilizing the concept of a mapping $\mathcal{N}$ on $\mathcal{K}$, we derive

$$\langle \mathcal{N}(u^*), y - u^*\rangle - \langle \mathcal{N}(y), y - u^*\rangle \le 0, \quad \forall\, y \in \mathcal{K}.$$

Employing $u^* \in VI(\mathcal{K}, \mathcal{N})$, we obtain

$$\langle \mathcal{N}(y), y - u^*\rangle \ge 0, \quad \forall\, y \in \mathcal{K}.$$

Substituting $y = y_k \in \mathcal{K}$ yields

$$\langle \mathcal{N}(y_k), y_k - u^*\rangle \ge 0. \tag{3.13}$$

From equations (3.12) and (3.13), we deduce

$$\begin{aligned}\left\|p_k - u^*\right\|^2 &\le \left\|u_k - u^*\right\|^2 - \left\|u_k - y_k\right\|^2 + \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\left\|u_k - y_k\right\|^2 \\ &= \left\|u_k - u^*\right\|^2 - \left(1 - \mu^2\frac{\eta_k^2}{\eta_{k+1}^2}\right)\left\|u_k - y_k\right\|^2. \end{aligned} \tag{3.14}$$

□

Theorem 3.4. *Let $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ be an operator satisfying the conditions ($\mathcal{N}1$)–($\mathcal{N}5$). Then, the sequence $\{u_k\}$ generated by Algorithms 1 and 2 converges strongly to $u^* \in VI(\mathcal{K}, \mathcal{N}) \cap Fix(\mathcal{Q})$, where $u^* = P_{VI(\mathcal{K},\mathcal{N})\cap Fix(\mathcal{Q})}(0)$.*

Proof. **Claim 1:** $\{u_k\}$ is a bounded sequence.

Let's consider that

$$u_{k+1} = (1 - \alpha_k)p_k + \alpha_k\mathcal{Q}(p_k).$$

By utilizing the definition of the sequence $\{u_{k+1}\}$, we have

$$\begin{aligned}\|u_{k+1} - u^*\|^2 &= \left\|(1 - \alpha_k)p_k + \alpha_k\mathcal{Q}(p_k) - u^*\right\|^2 \\ &= \|p_k - u^*\|^2 + 2\alpha_k\langle p_k - u^*, \mathcal{Q}(p_k) - p_k\rangle + \alpha_k^2\|\mathcal{Q}(p_k) - p_k\|^2 \\ &\le \|p_k - u^*\|^2 + \alpha_k(\rho - 1)\|\mathcal{Q}(p_k) - p_k\|^2 + \alpha_k^2\|\mathcal{Q}(p_k) - p_k\|^2 \\ &= \|p_k - u^*\|^2 - \alpha_k(1 - \rho - \alpha_k)\|\mathcal{Q}(p_k) - p_k\|^2. \end{aligned} \tag{3.15}$$

By using the value of $\{q_k\}$, we obtain

$$\begin{aligned}\|q_k - u^*\| &= \|u_k + \theta_k(u_k - u_{k-1}) - \varsigma_k u_k - \theta_k \varsigma_k (u_k - u_{k-1}) - u^*\| \\ &= \|(1-\varsigma_k)(u_k - u^*) + (1-\varsigma_k)\theta_k(u_k - u_{k-1}) - \varsigma_k u^*\| \\ &\le (1-\varsigma_k)\|u_k - u^*\| + (1-\varsigma_k)\theta_k\|u_k - u_{k-1}\| + \varsigma_k\|u^*\| \\ &\le (1-\varsigma_k)\|u_k - u^*\| + \varsigma_k M_1, \end{aligned} \tag{3.16}$$

for some fixed number M_1, we have

$$(1-\varsigma_k)\frac{\theta_k}{\varsigma_k}\|u_k - u_{k-1}\| + \|u^*\| \le M_1.$$

By using $\eta_k \to \eta$ such that $\chi \in (0, 1-\mu^2)$, we have

$$\lim_{k\to+\infty}\left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right) = 1 - \mu^2 > \chi > 0.$$

Thus, there exists some fixed $k_0 \in \mathbb{N}$ such that

$$\left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right) > \chi > 0, \ \forall\, k \ge k_0. \tag{3.17}$$

By using Lemma 3.3, we can rewrite

$$\|p_k - u^*\|^2 \le \|q_k - u^*\|^2, \ \forall\, k \ge k_0. \tag{3.18}$$

From expressions (3.15), (3.16), and (3.18), we infer that

$$\|u_{k+1} - u^*\| \le (1-\varsigma_k)\|u_k - u^*\| + \varsigma_k M_1 - \alpha_k(1-\rho-\alpha_k)\|\mathcal{Q}(p_k) - p_k\|^2. \tag{3.19}$$

Thus, for $\{\alpha_k\} \subset (0, 1-\rho)$, we obtain

$$\begin{aligned}\|u_{k+1} - u^*\| &\le (1-\varsigma_k)\|u_k - u^*\| + \varsigma_k M_1 \\ &\le \max\big\{\|u_k - u^*\|, M_1\big\} \\ &\ \ \vdots \\ &\le \max\big\{\|u_{k_0} - u^*\|, M_1\big\}. \end{aligned} \tag{3.20}$$

Consequently, we may infer that the sequence $\{u_k\}$ is a bounded sequence.

Claim 2:

$$\begin{aligned}&\left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right)\|q_k - y_k\|^2 + \alpha_k(1-\rho-\alpha_k)\|\mathcal{Q}(p_k) - p_k\|^2 \\ &\le \|u_k - u^*\|^2 - \|u_{k+1} - u^*\|^2 + \varsigma_k M_2, \end{aligned} \tag{3.21}$$

for some fixed $M_2 > 0$. Indeed, by using the definition of $\{u_{k+1}\}$, we have

$$\begin{aligned}
\|u_{k+1} - u^*\|^2 &= \|(1-\alpha_k)p_k + \alpha_k \mathcal{Q}(p_k) - u^*\|^2 \\
&= \|p_k - u^*\|^2 + 2\alpha_k \langle p_k - u^*, \mathcal{Q}(p_k) - p_k\rangle + \alpha_k^2 \|\mathcal{Q}(p_k) - p_k\|^2 \\
&\le \|p_k - u^*\|^2 + \alpha_k(\rho - 1)\|\mathcal{Q}(p_k) - p_k\|^2 + \alpha_k^2 \|\mathcal{Q}(p_k) - p_k\|^2 \\
&= \|p_k - u^*\|^2 - \alpha_k(1-\rho-\alpha_k)\|\mathcal{Q}(p_k) - p_k\|^2 . \qquad (3.22)
\end{aligned}$$

By using Lemma 3.3, we obtain

$$\|p_k - u^*\|^2 \le \|q_k - u^*\|^2 - \left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right)\|q_k - y_k\|^2 . \qquad (3.23)$$

By using expression (3.16), we can obtain

$$\begin{aligned}
\|q_k - u^*\|^2 &\le (1-\varsigma_k)^2\|u_k - u^*\|^2 + \varsigma_k^2 M_1^2 + 2M_1\varsigma_k(1-\varsigma_k)\|u_k - u^*\| \\
&\le \|u_k - u^*\|^2 + \varsigma_k\left[\varsigma_k M_1^2 + 2M_1(1-\varsigma_k)\|u_k - u^*\|\right] \\
&\le \|u_k - u^*\|^2 + \varsigma_k M_2, \qquad (3.24)
\end{aligned}$$

where M_2 is some fixed constant > 0. From expressions (3.22), (3.23) and (3.24), we obtain

$$\begin{aligned}
\|u_{k+1} - u^*\|^2 \le \|u_k - u^*\|^2 &+ \varsigma_k M_2 - \alpha_k(1-\rho-\alpha_k)\|\mathcal{Q}(p_k) - p_k\|^2 \\
&- \left(1 - \mu^2 \frac{\eta_k^2}{\eta_{k+1}^2}\right)\|q_k - y_k\|^2 . \qquad (3.25)
\end{aligned}$$

Claim 3:
Using the value of $\{q_k\}$, we can express as follows

$$\begin{aligned}
\big\|q_k - u^*\big\|^2 &= \big\|u_k + \theta_k(u_k - u_{k-1}) - \varsigma_k u_k - \theta_k\varsigma_k(u_k - u_{k-1}) - u^*\big\|^2 \\
&= \big\|(1-\varsigma_k)(u_k - u^*) + (1-\varsigma_k)\theta_k(u_k - u_{k-1}) - \varsigma_k u^*\big\|^2 \\
&\le \big\|(1-\varsigma_k)(u_k - u^*) + (1-\varsigma_k)\theta_k(u_k - u_{k-1})\big\|^2 + 2\varsigma_k\langle -u^*, q_k - u^*\rangle \\
&= (1-\varsigma_k)^2\big\|u_k - u^*\big\|^2 + (1-\varsigma_k)^2\theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\
&\quad + 2\theta_k(1-\varsigma_k)^2\big\|u_k - u^*\big\|\big\|u_k - u_{k-1}\big\| \\
&\quad + 2\varsigma_k\langle -u^*, q_k - u_{k+1}\rangle + 2\varsigma_k\langle -u^*, u_{k+1} - u^*\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\varsigma_k)\big\|u_k - u^*\big\|^2 + \theta_k^2\big\|u_k - u_{k-1}\big\|^2 \\
&\quad + 2\theta_k(1-\varsigma_k)\big\|u_k - u^*\big\|\big\|u_k - u_{k-1}\big\| \\
&\quad + 2\varsigma_k\big\|u^*\big\|\big\|q_k - u_{k+1}\big\| + 2\varsigma_k\langle -u^*, u_{k+1} - u^*\rangle \\
&= (1-\varsigma_k)\big\|u_k - u^*\big\|^2 + \varsigma_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| \\
&\quad + 2(1-\varsigma_k)\big\|u_k - u^*\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| \\
&\quad + 2\big\|u^*\big\|\big\|q_k - u_{k+1}\big\| + 2\langle u^*, u^* - u_{k+1}\rangle\Big].
\end{aligned} \tag{3.26}$$

Combining equations (3.18) and (3.26), we obtain

$$\begin{aligned}
\big\|u_{k+1} - u^*\big\|^2 \leq (1-\varsigma_k)\big\|u_k - u^*\big\|^2 &+ \varsigma_k\Big[\theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| \\
&+ 2(1-\varsigma_k)\big\|u_k - u^*\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| \\
&+ 2\big\|u^*\big\|\big\|q_k - u_{k+1}\big\| + 2\langle u^*, u^* - u_{k+1}\rangle\Big].
\end{aligned} \tag{3.27}$$

Claim 4: The sequence $\big\|u_k - u^*\big\|^2$ converges to zero.

Let's define

$$c_k := \|u_k - u^*\|^2$$

and

$$\begin{aligned}
e_k := \theta_k\big\|u_k - u_{k-1}\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| + 2(1-\varsigma_k)\big\|u_k - u^*\big\|\frac{\theta_k}{\varsigma_k}\big\|u_k - u_{k-1}\big\| \\
+ 2\big\|u^*\big\|\big\|q_k - u_{k+1}\big\| + 2\langle u^*, u^* - u_{k+1}\rangle.
\end{aligned}$$

Then, Claim 4 can be restated as

$$c_{k+1} \leq (1-\varsigma_k)c_k + \varsigma_k e_k.$$

By Lemma 2.5, it's enough to prove that $\limsup_{j\to+\infty} e_{k_j} \leq 0$ for $\{c_{k_j}\}$ in $\{c_k\}$ such that

$$\liminf_{j\to+\infty}(c_{k_j+1} - c_{k_j}) \geq 0.$$

This is equivalent to stating

$$\limsup_{j\to+\infty}\langle u^*, u^* - u_{k_j+1}\rangle \leq 0$$

and

$$\limsup_{j\to+\infty}\big\|q_{k_j} - u_{k_j+1}\big\| \leq 0,$$

one from each subsequence $\{\|u_{k_j} - u^*\|\}$ of $\{\|u_k - u^*\|\}$ following

$$\liminf_{j\to+\infty} \left(\|u_{k_j+1} - u^*\| - \|u_{k_j} - u^*\|\right) \geq 0.$$

Suppose $\{\|u_{k_j} - u^*\|\}$ is such a subsequence of $\{\|u_k - u^*\|\}$ satisfying

$$\liminf_{j\to+\infty} \left(\|u_{k_j+1} - u^*\| - \|u_{k_j} - u^*\|\right) \geq 0.$$

Then, we have

$$\begin{aligned}
&\liminf_{j\to+\infty} \left(\|u_{k_j+1} - u^*\|^2 - \|u_{k_j} - u^*\|^2\right) \\
&= \liminf_{j\to+\infty} \left(\|u_{k_j+1} - u^*\| - \|u_{k_j} - u^*\|\right)\left(\|u_{k_j+1} - u^*\| + \|u_{k_j} - u^*\|\right) \\
&\geq 0.
\end{aligned} \tag{3.28}$$

As a result of Claim 2,

$$\begin{aligned}
&\limsup_{j\to+\infty} \left[\left(1 - \frac{\mu^2\eta_{k_j}^2}{\eta_{k_j+1}^2}\right)\|q_{k_j} - y_{k_j}\|^2 + \alpha_{k_j}(1-\rho-\alpha_{k_j})\left\|\mathcal{Q}(p_{k_j}) - p_{k_j}\right\|^2\right] \\
&\leq \limsup_{j\to+\infty} \left[\|u_{k_j} - u^*\|^2 - \|u_{k_j+1} - u^*\|^2\right] + \limsup_{j\to+\infty} \varsigma_{k_j} K_2 \\
&= -\liminf_{j\to+\infty} \left[\|u_{k_j+1} - u^*\|^2 - \|u_{k_j} - u^*\|^2\right] \\
&\leq 0.
\end{aligned} \tag{3.29}$$

The above expressions imply the following mathematical relationships

$$\lim_{j\to+\infty} \|q_{k_j} - y_{k_j}\| = 0, \quad \lim_{j\to+\infty} \left\|\mathcal{Q}(p_{k_j}) - p_{k_j}\right\| = 0. \tag{3.30}$$

Consequently, we have

$$\|p_{k_j} - y_{k_j}\| = \|y_{k_j} + \eta_{k_j}[\mathcal{N}(q_{k_j}) - \mathcal{N}(y_{k_j})] - y_{k_j}\| \leq \eta_{k_j} L \|q_{k_j} - y_{k_j}\|. \tag{3.31}$$

Thus, it follows that

$$\lim_{j\to+\infty} \|p_{k_j} - y_{k_j}\| = 0. \tag{3.32}$$

Hence, we can deduce

$$\lim_{j\to+\infty} \|p_{k_j} - q_{k_j}\| = 0. \tag{3.33}$$

Moreover, it's straightforward to derive

$$\lim_{j\to+\infty} \|q_{k_j} - u_{k_j}\| = 0. \tag{3.34}$$

Combining equations (3.33) and (3.34) leads to

$$\lim_{j\to+\infty} \|p_{k_j} - u_{k_j}\| = 0. \tag{3.35}$$

Given the equation $u_{k_j+1} = (1-\alpha_{k_j})p_{k_j} + \alpha_{k_j}\mathcal{Q}(p_{k_j})$, we observe:

$$\lim_{j\to+\infty} \|u_{k_j+1} - p_{k_j}\| = \alpha_{k_j}\|\mathcal{Q}(p_{k_j}) - p_{k_j}\| \le (1-\rho)\|\mathcal{Q}(p_{k_j}) - p_{k_j}\|. \tag{3.36}$$

Thus, it follows that

$$\lim_{j\to+\infty} \|u_{k_j+1} - p_{k_j}\| = 0. \tag{3.37}$$

The expressions above imply that

$$\lim_{j\to+\infty} \|u_{k_j} - u_{k_j+1}\| \le \lim_{j\to+\infty} \|u_{k_j} - p_{k_j}\| + \lim_{j\to+\infty} \|p_{k_j} - u_{k_j+1}\| = 0 \tag{3.38}$$

and

$$\lim_{j\to+\infty} \|q_{k_j} - u_{k_j+1}\| \le \lim_{j\to+\infty} \|q_{k_j} - p_{k_j}\| + \lim_{j\to+\infty} \|p_{k_j} - u_{k_j+1}\| = 0. \tag{3.39}$$

Given that the sequence $\{u_{k_j}\}$ is bounded, without loss of generality, we assume that $\{u_{k_j}\}$ converges weakly to some $\hat{u} \in \mathcal{X}$.

Next, we need to prove that $\hat{u} \in VI(\mathcal{K},\mathcal{N})$. Since $\{q_{k_j}\}$ converges weakly to $\hat{u}$ and because $\lim_{j\to+\infty} \|q_{k_j} - y_{k_j}\| = 0$, the sequence $\{y_{k_j}\}$ also converges weakly to $\hat{u}$. We then need to prove that $\hat{u} \in VI(\mathcal{K},\mathcal{N})$. This implies

$$y_{k_j} = P_{\mathcal{X}}[q_{k_j} - \eta_{k_j}\mathcal{N}(q_{k_j})],$$

which is equivalent to

$$\langle q_{k_j} - \eta_{k_j}\mathcal{N}(q_{k_j}) - y_{k_j}, y - y_{k_j}\rangle \le 0,\ \forall\, y \in \mathcal{X}. \tag{3.40}$$

From the inequality above, we have

$$\langle q_{k_j} - y_{k_j}, y - y_{k_j}\rangle \le \eta_{k_j}\langle \mathcal{N}(q_{k_j}), y - y_{k_j}\rangle,\ \forall\, y \in \mathcal{X}. \tag{3.41}$$

Consequently, we obtain

$$\frac{1}{\eta_{k_j}}\langle q_{k_j} - y_{k_j}, y - y_{k_j}\rangle + \langle \mathcal{N}(q_{k_j}), y_{k_j} - q_{k_j}\rangle \le \langle \mathcal{N}(q_{k_j}), y - q_{k_j}\rangle,\ \forall\, y \in \mathcal{X}. \tag{3.42}$$

Given that $\min\left\{\frac{\mu}{L}, \eta_1\right\} \le \eta \le \eta_1$ and $\{q_{k_j}\}$ forms a bounded sequence, utilizing the limits $\lim_{j\to+\infty} \|q_{k_j} - y_{k_j}\| = 0$ and $j \to +\infty$ in (3.42), we deduce

$$\liminf_{j\to+\infty} \langle \mathcal{N}(q_{k_j}), y - q_{k_j}\rangle \ge 0,\ \forall\, y \in \mathcal{X}. \tag{3.43}$$

Moreover, it implies

$$\begin{aligned}\langle \mathcal{N}(y_{k_j}), y - y_{k_j}\rangle &= \langle \mathcal{N}(y_{k_j}) - \mathcal{N}(q_{k_j}), y - q_{k_j}\rangle \\ &\quad + \langle \mathcal{N}(q_{k_j}), y - q_{k_j}\rangle + \langle \mathcal{N}(y_{k_j}), q_{k_j} - y_{k_j}\rangle.\end{aligned} \tag{3.44}$$

As $\lim_{j\to+\infty} \|q_{k_j} - y_{k_j}\| = 0$ and under the Lipschitz condition on the mapping $\mathcal{N}$, we obtain

$$\lim_{j\to+\infty} \|\mathcal{N}(q_{k_j}) - \mathcal{N}(y_{k_j})\| = 0, \tag{3.45}$$

which, coupled with (3.44) and (3.45), yields

$$\liminf_{j\to+\infty}\langle \mathcal{N}(y_{k_j}), y - y_{k_j}\rangle \geq 0, \ \forall\, y \in \mathcal{X}. \tag{3.46}$$

For further proof, let's consider a positive sequence $\{\epsilon_j\}$ that converges to zero and decreases. For each $\{\epsilon_j\}$, there exists a least positive integer denoted by m_j such that

$$\langle \mathcal{N}(q_{k_i}), y - q_{k_i}\rangle + \epsilon_j > 0, \ \forall\, i \geq m_j, \tag{3.47}$$

where the existence of m_j follows from expression (3.46). As $\{\epsilon_j\}$ is decreasing, the sequence m_j is evidently increasing. If there exists a natural number $N_0 \in \mathbb{N}$ such that $\mathcal{N}((u_{n_{m_k}}) \neq 0$ for all $n_{m_k} \geq N_0$, then we consider

$$\hbar_{k_{m_n}} = \frac{\mathcal{N}(q_{k_{m_n}})}{\|\mathcal{N}(q_{k_{m_n}})\|^2}, \ \forall\, k_{m_n} \geq N_0. \tag{3.48}$$

Using the provided value of $\hbar_{k_{m_n}}$, we derive the following equation

$$\langle \mathcal{N}(q_{k_{m_n}}), \hbar_{k_{m_n}}\rangle = 1, \quad \forall\, k_{m_n} \geq N_0. \tag{3.49}$$

Combining equations (3.47) and (3.49), we obtain

$$\langle \mathcal{N}(q_{k_{m_n}}), y + \epsilon_k \hbar_{k_{m_n}} - q_{k_{m_n}}\rangle > 0. \tag{3.50}$$

Utilizing the definition of the pseudomonotone mapping $\mathcal{N}$, we express

$$\langle \mathcal{N}(y + \epsilon_k \hbar_{k_{m_n}}), y + \epsilon_k \hbar_{k_{m_n}} - q_{k_{m_n}}\rangle > 0. \tag{3.51}$$

For all $k_{m_n} \geq N_0$, we have

$$\begin{aligned}\langle \mathcal{N}(y), y - q_{k_{m_n}}\rangle \geq{}& \langle \mathcal{N}(y) - \mathcal{N}(y + \epsilon_k \hbar_{k_{m_n}}), y + \epsilon_k \hbar_{k_{m_n}} - q_{k_{m_n}}\rangle \\ &- \epsilon_k \langle \mathcal{N}(y), \hbar_{k_{m_n}}\rangle.\end{aligned} \tag{3.52}$$

Given that the sequence $\{q_{k_n}\}$ weakly converges to $\hat{u} \in \mathcal{X}$, it follows that $\{\mathcal{N}(q_{k_n})\}$ weakly converges to $\mathcal{N}(\hat{u})$. Suppose $\mathcal{N}(\hat{u}) \neq 0$, implying

$$\|\mathcal{N}(\hat{u})\| \leq \liminf_{n\to+\infty} \|\mathcal{N}(q_{k_n})\|. \tag{3.53}$$

Since $\{q_{k_{m_n}}\} \subset \{q_{k_n}\}$ and $\lim_{k\to+\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{n\to+\infty} \|\epsilon_k \hbar_{k_{m_n}}\| = \lim_{n\to+\infty} \frac{\epsilon_k}{\|\mathcal{N}(q_{k_{m_n}})\|} \leq \frac{0}{\|\mathcal{N}(\hat{u})\|} = 0. \tag{3.54}$$

By letting $n \to +\infty$ in equation (3.52), we obtain

$$\langle \mathcal{N}(y), y - \hat{u}\rangle \geq 0, \quad \forall\, y \in \mathcal{X}. \tag{3.55}$$

Let $u \in \mathcal{X}$ be an arbitrary element and $0 < \psi \leq 1$. Consider the expression

$$\hat{u}_\psi = \psi u + (1-\psi)\hat{u}.$$

Then, $\hat{u}_\psi \in \mathcal{X}$. From Equation (3.55), we have

$$\psi\, \langle \mathcal{N}(\hat{u}_\psi), u - \hat{u}\rangle \geq 0.$$

Thus,

$$\langle \mathcal{N}(\hat{u}_\psi), u - \hat{u} \rangle \geq 0.$$

As $\psi \to 0$, $\hat{u}_\psi \to \hat{u}$ along a line segment. By the continuity of the operator, $\mathcal{N}(\hat{u}_\psi)$ converges to $\mathcal{N}(\hat{u})$ as $\psi \to 0$. Hence, we have:

$$\langle \mathcal{N}(\hat{u}), u - \hat{u} \rangle \geq 0.$$

Therefore, $\hat{u}$ is a solution of problem (VIP). Given $u^* = P_{VI(\mathcal{K},\mathcal{N}) \cap Fix(\mathcal{Q})}(0)$, we have:

$$\langle 0 - u^*, y - u^* \rangle \leq 0, \quad \forall\, y \in VI(\mathcal{K}, \mathcal{N}) \cap Fix(\mathcal{Q}).$$

From Equation (3.34), it follows that $\{q_{k_j}\}$ converges weakly to $\hat{u} \in \mathcal{X}$. Similarly, from Equation (3.35), $\{p_{k_j}\}$ converges weakly to $\hat{u} \in \mathcal{X}$. By the demiclosedness of $(I - \mathcal{Q})$, we conclude $\hat{u} \in Fix(\mathcal{Q})$. Therefore, $\hat{u} \in VI(\mathcal{K}, \mathcal{N}) \cap Fix(\mathcal{Q})$, leading to:

$$\lim_{j \to +\infty} \left\langle u^*, u^* - u_{k_j} \right\rangle = \langle u^*, u^* - \hat{u} \rangle \leq 0.$$

Furthermore, since $\lim_{j \to +\infty} \left\| u_{k_j+1} - u_{k_j} \right\| = 0$, we have

$$\limsup_{j \to +\infty} \left\langle u^*, u^* - u_{k_j+1} \right\rangle \leq \limsup_{j \to +\infty} \left\langle u^*, u^* - u_{k_j} \right\rangle + \limsup_{j \to +\infty} \left\langle u^*, u_{k_j} - u_{k_j+1} \right\rangle \leq 0.$$

Combining Claim 3 and Lemma 2.5, we observe that $u_k \to u^*$ as $k \to +\infty$. This completes the proof. □

4. Numerical illustrations

This section investigates the algorithmic implications of the provided methodologies, as well as explores how variations in control parameters affect the numerical efficacy of the suggested algorithms.

Example 4.1. Consider a mapping $\mathcal{N} : \mathbb{R}^m \to \mathbb{R}^m$ defined as

$$\mathcal{N}(u) = Mu,$$

where $q = 0$. Here, $M = NN^T + B + D$, with $N = \text{rand}(m)$ and $K = \text{rand}(m)$ chosen randomly, and $B = 0.5K - 0.5K^T$, $D = \text{diag}(\text{rand}(m, 1))$. The feasible set $\mathcal{K}$ is defined as

$$\mathcal{K} = \{u \in \mathbb{R}^m : -10 \leq u_i \leq 10\}.$$

It is evident that the mapping $\mathcal{N}$ is monotone and Lipschitz continuous with constant $L = \|M\|$. Additionally, a function $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ is defined as

$$\mathcal{Q}(u) = \frac{1}{2} u.$$

The initial points for the tests are $u_0 = u_1 = (2, 2, \cdots, 2)$. Different dimensions of the Hilbert space are considered to study its behavior in higher dimensions. The stopping condition for these experiments is $D_k = \|q_k - y_k\| \leq 10^{-10}$.

Figures 1 to 6 and Tables 1 to 2 depict empirical observations for Example 2. The following control criteria are enforced:

(1) Algorithm 1 (**Algo1**):

$$\begin{aligned}&\eta_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64,\\&\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)};\end{aligned}$$

(2) Algorithm 2 (**Algo2**):

$$\begin{aligned}&\eta_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64,\\&\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)};\end{aligned}$$

(3) Algorithm 1 in [32] (**Algo3.1**):

$$\begin{aligned}&\tau_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64,\\&\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)},\\&f(u) = \frac{u}{3};\end{aligned}$$

(4) Algorithm 2 in [32] (**Algo3.2**):

$$\begin{aligned}&\tau_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64,\\&\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{k}{(3k+2)},\\&f(u) = \frac{u}{3}.\end{aligned}$$

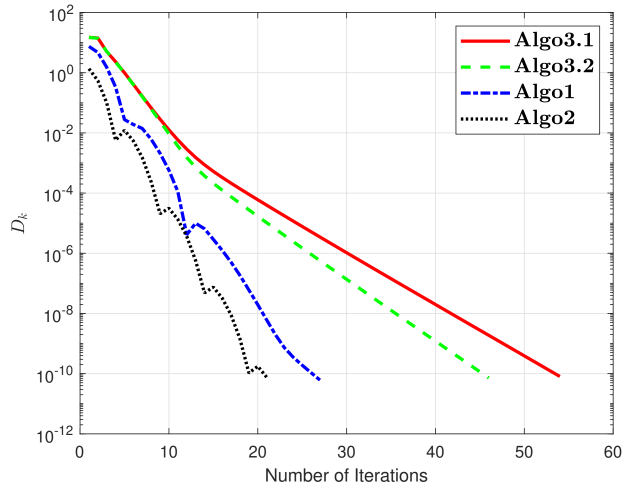

FIGURE 1. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 5$.

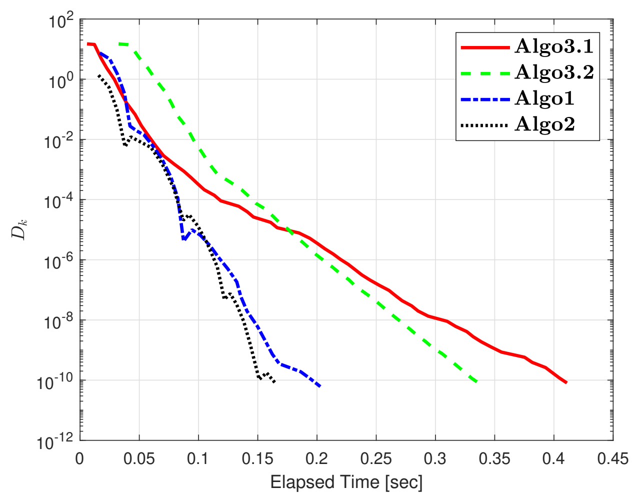

FIGURE 2. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 5$.

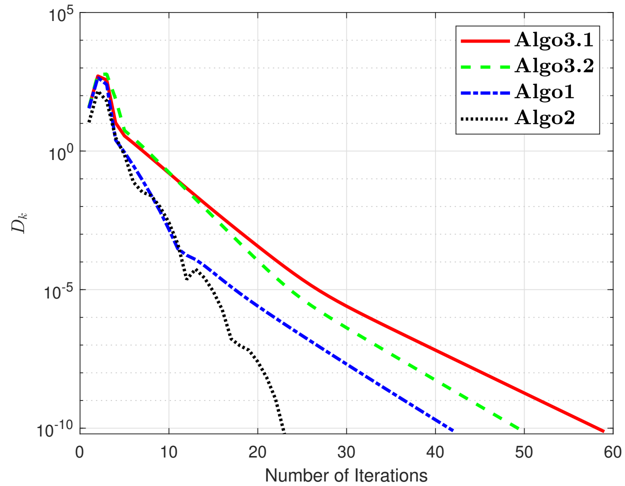

FIGURE 3. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 10$.

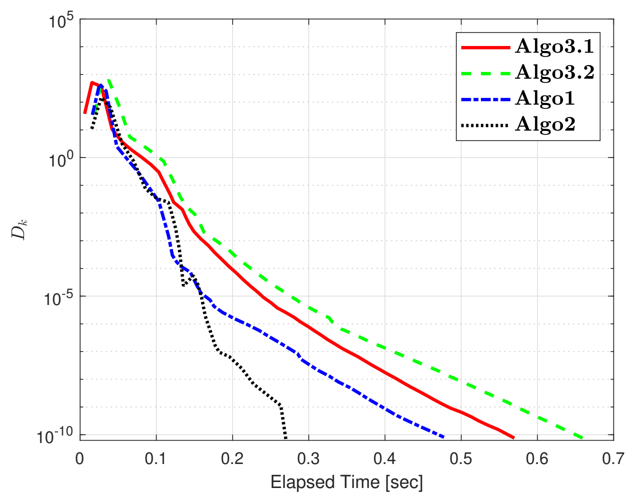

FIGURE 4. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 10$.

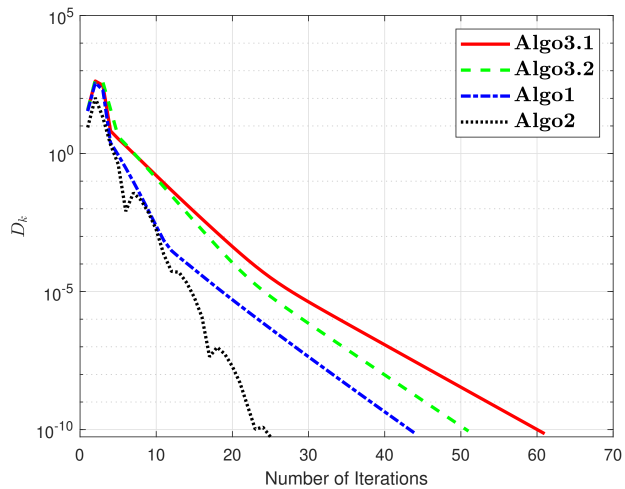

FIGURE 5. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 20$.

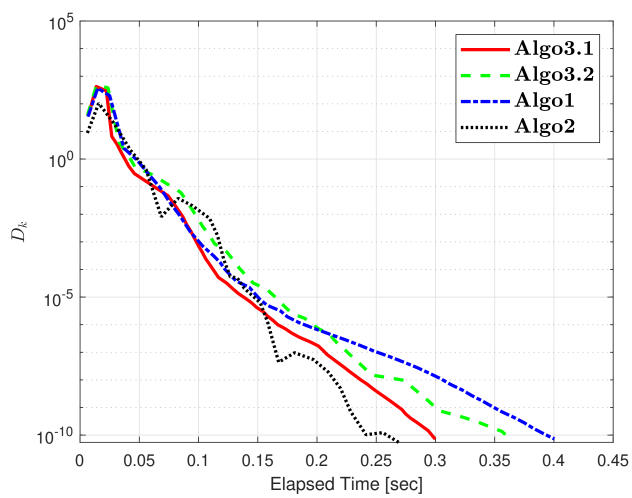

FIGURE 6. Numerical comparison of Algorithm 1 and Algorithm 2 with Algorithm 1 in [32], Algorithm 2 in [32] when $m = 20$.

Table 1. Numerical values for Figures 1–6.

The number of iterations				
m	**Algo3.1**	**Algo3.2**	**Algo1**	**Algo2**
5	54	46	27	21
10	59	49	42	23
20	61	51	44	25
50	67	56	45	34
100	73	61	48	36

Table 2. Numerical values for Figures 1–6.

Time required to complete				
m	**Algo3.1**	**Algo3.2**	**Algo1**	**Algo2**
5	0.41096080	0.33812320	0.20280990	0.16518980
10	0.64628270	0.57689310	0.49204840	0.29402830
20	0.48693460	0.36085080	0.30036720	0.26893690
50	0.57369690	0.40373730	0.37593220	0.29847481
100	0.67855063	0.53929280	0.41928400	0.30193762

Example 4.2. Let $\mathcal{K}$ denote the feasible set defined as follows:

$$\mathcal{K} := \{u \in L^2([0,1]) : \|u\|_{L^2} \leq 1\}.$$

We define an operator $\mathcal{N} : \mathcal{K} \to \mathcal{X}$ by

$$\mathcal{N}(u)(t) = \max\big\{u(t), 0\big\} = \frac{u(t) + |u(t)|}{2}.$$

Here, $\mathcal{X} = L^2([0,1])$ represents a real Hilbert space with its inner product and norm given by

$$\langle u, y \rangle = \int_0^1 u(t)y(t)dt, \quad \forall\, u, y \in \mathcal{X}$$

and

$$\|u\|_{L^2} = \sqrt{\int_0^1 |u(t)|^2 dt}.$$

It can be readily verified that $\mathcal{N}$ is monotone and 1-Lipschitz continuous. The projection onto $\mathcal{K}$ is straightforward

$$P_{\mathcal{K}}(u) = \begin{cases} \frac{u}{\|u\|_{L^2}} & \text{if} \quad \|u\|_{L^2} > 1, \\ u & \text{if} \quad \|u\|_{L^2} \leq 1. \end{cases}$$

A mapping $\mathcal{Q} : L^2([0,1]) \to L^2([0,1])$ takes the form

$$\mathcal{Q}(u)(t) = \int_0^1 tu(s)ds, \quad t \in [0,1].$$

A simple analysis shows that $\mathcal{Q}$ is 0-demicontractive, with the solution being $u^*(t) = 0$. These trials commence with a halting requirement $D_k = \|q_k - y_k\|_{L^2} \leq 10^{-6}$. The tables 3 and 4 present numerical results pertaining to Example 4.2. The following conditions serve as control criteria:

(1) Algorithm 1 (**Algo1**):

$$\begin{aligned} &\eta_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64, \\ &\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)}; \end{aligned}$$

(2) Algorithm 2 (**Algo2**):

$$\begin{aligned} &\eta_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64, \\ &\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)}; \end{aligned}$$

(3) Algorithm 1 in [32] (**Algo3.1**):

$$\begin{aligned} &\tau_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64, \\ &\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{2k}{(3k+2)}, \\ &f(u) = \frac{u}{3}; \end{aligned}$$

(4) Algorithm 2 in [32] (**Algo3.2**):

$$\begin{aligned} &\tau_1 = 0.43, \quad \theta = 0.56, \quad \mu = 0.64, \\ &\chi_k = \frac{10}{(1+k)^2}, \quad \varsigma_k = \frac{1}{(3k+5)}, \quad \alpha_k = \frac{k}{(3k+2)}, \\ &f(u) = \frac{u}{3}. \end{aligned}$$

Table 3. Numerical values for Example 4.2.

Total number of iterations				
$u_0 = u_1$	**Algo3.1**	**Algo3.2**	**Algo1**	**Algo2**
t^3	76	65	45	32
$t\cos(t)$	87	68	49	33
$t\exp(t)$	92	74	57	41
$t2^t$	98	79	61	43

Table 4. Numerical values for Example 4.2.

Required CPU time				
$u_0 = u_1$	**Algo3.1**	**Algo3.2**	**Algo1**	**Algo2**
t^3	1.1745382	1.1275749	0.5867949	0.3525294
$t\cos(t)$	1.1976944	1.4869759	0.8563924	0.6273644
$t\exp(t)$	2.0575325	1.5486922	0.9949494	0.7264347
$t2^t$	2.0025344	1.69576969	1.2207376	1.03811294

5. Conclusion

The paper introduces two explicit extragradient-like algorithms designed to address an equilibrium problem within a real Hilbert space, featuring a pseudomonotone and Lipschitz-type bifunction, constrained by fixed points. A novel stepsize rule is proposed, independent of Lipschitz-type constants. Various experiments are conducted to demonstrate the numerical performance of the proposed algorithms and compare them with established approaches in the literature.

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