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THE GLOBAL EXISTENCE AND BEHAVIOR OF RADIAL SOLUTIONS OF A NONLINEAR p-LAPLACIAN TYPE EQUATION WITH SINGULAR COEFFICIENTS

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Abstract. This paper is concerned with the radial solutions of a nonlinear elliptic equation

$$\Delta_p u + |x|^{l_1} |u|^{q_1-1} u + |x|^{l_2} |u|^{q_2-1} u = 0, \; x \in \mathbb{R}^N,$$

where $p > 2$, $N \geq 1$, $q_2 > q_1 \geq 1$, $-p < l_2 < l_1 \leq 0$ and $-N < l_2 < l_1 \leq 0$.
We prove the existence of global solutions, we give their classification and we present the explicit behavior of positive solutions near the origin and infinity.

1. Introduction

We are interested in the following elliptic equation involving the p-Laplace operator

$$\Delta_p u + |x|^{l_1} |u|^{q_1-1} u + |x|^{l_2} |u|^{q_2-1} u = 0, \;\; x \in \mathbb{R}^N, \tag{1.1}$$

where $p > 2$, $N \geq 1$, $q_2 > q_1 \geq 1$, $-p < l_2 < l_1 \leq 0$ and $-N < l_2 < l_1 \leq 0$.
As usual $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ is the p-Laplace operator and ∇u is the gradient of u. We restrict our considerations to solutions of the form $u = u(|x|)$

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for which the original equation (1.1) is reduced to the ordinary differential equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + r^{l_1}|u|^{q_1-1}u + r^{l_2}|u|^{q_2-1}u = 0,\ \ r > 0, \quad (1.2)$$

where $p > 2$, $N \geq 1$, $q_2 > q_1 \geq 1$, $-p < l_2 < l_1 \leq 0$ and $-N < l_2 < l_1 \leq 0$.

The study of elliptic equations having singular coefficients has been the subject of recent research by many authors. In the case $p = 2$, Lai et al. [13], studied the equation

$$u''(r) + \frac{N-1}{r}u'(r) + r^{l_1}u^{q_1} + r^{l_2}u^{q_2} = 0,\ \ r > 0, \quad (1.3)$$

where $N \geq 3$, $1 < q_1 < q_2$ and $-2 < l_2 < l_1 \leq 0$.
They gave the asymptotic behavior near the origin and infinity of positive solutions. More precisely, they showed that if $\dfrac{N+l_1}{N-2} < q_1 < q_2$, $q_1 \neq \dfrac{N+2+2l_1}{N-2}$ and $q_2 \neq \dfrac{N+2+2l_2}{N-2}$, then $\lim\limits_{r\to 0} r^{\frac{l_2+2}{q_2-1}}u(r)$ and $\lim\limits_{r\to+\infty} r^{\frac{l_1+2}{q_1-1}}u(r)$ always exist.

If $l_1 = l_2$ and $q_1 = q_2$, rescaling, equation (1.3) is reduced to

$$u''(r) + \frac{N-1}{r}u'(r) + r^{l_1}u^{q_1} = 0. \quad (1.4)$$

Equation (1.4) has been the subject of much literature. It appears in different physical models. It is involved in the analysis of Riemannian geometry, the standing wave of nonlinear Schrodinger equations and dynamics of globular cluster of stars [15]. If $l_1 = 0$, the equation (1.4) is recognized in astrophysics as the Emden-Fowler equation, as shown in [11]. In this context, the variable u denotes the density of an individual star. Moreover, in geometry, if $N \geq 3$ and $q_1 = \dfrac{N+2}{N-2}$, (1.4) is known as the conformal scalar curvature equation. The first existence results of equation (1.4) are due to Ni [17] in 1982, he proved that (1.4) has an infinite number of positive solutions, all of which are bounded by positive constants from below. Li and Ni [14] showed that, for positive solution u to (1.4), $\lim\limits_{r\to+\infty} u(r)$ still exists. The asymptotic behavior of positive solutions of (1.4) was described by Li [15]. Interesting results about this equation can be found in [2, 8, 9, 10, 12, 16, 18, 19].

We also notice that if $l_1 = l_2 = 0$, the equation (1.3) becomes

$$u''(r) + \frac{N-1}{r}u'(r) + u^{q_1}(r) + u^{q_2}(r) = 0,\ \ r > 0. \quad (1.5)$$

The equation (1.5) has been recently studied by Bamon et al. [3], when the powers q_1 and q_2 are respectively sub and supcritical, that is $1 < q_1 < \frac{N+2}{N-2} < q_2$. They proved that if q_2 is fixed and q_1 tends to $\frac{N+2}{N-2}$, then equation (1.5) exhibits a significant number of radial solutions. A similar outcome applies if we fix $q_1 > \frac{N}{N-2}$ and we tend q_2 to $\frac{N+2}{N-2}$. Moreover, they have shown that (1.5) has no solution if q_1 tends to $\frac{N}{N-2}$ and q_2 is fixed.

In the general case $p > 2$, Bouzelmate and Gmira [6, 7] proved that the equation (1.2) possesses an explicit positive solution when $l_1 = 0$ and $q_2 = 0$ or $l_2 = 0$ and $q_1 = 0$. Moreover, they gave some results concerning the existence, the nonexistence and the asymptotic behavior of singular solutions of equation (1.2) around the origin, that is the solutions that satisfy $\lim_{r\to 0} u(r) = +\infty$.

The purpose of this paper is to generalize the paper studied by Lai et al. [13] to the case $p > 2$. We study the existence and the asymptotic behavior near the origin and infinity and we give a classification of global solutions of equation (1.2) with positive initial data in 0. We show that for any solution u of equation (1.2) such that $u(0) = a > 0$, $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r)$ exists and is finite. Hence, we look for a solution u defined on $[0, +\infty[$ such that $u \in C^0([0, +\infty[) \cap C^1(]0, +\infty[)$, $|u'|^{p-2}u' \in C^1(]0, +\infty[)$ and satisfying

$$(P) \quad \begin{cases} \left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + r^{l_1}|u|^{q_1-1}u + r^{l_2}|u|^{q_2-1}u = 0, \quad r > 0, \\ u(0) = a, \ \lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = b, \end{cases}$$

where $p > 2$, $N \geq 1$, $q_2 > q_1 \geq 1$, $-p < l_2 < l_1 \leq 0$, $-N < l_2 < l_1 \leq 0$, $a > 0$ and $b \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we present the asymptotic behavior near the origin of solutions of problem (P). The local and global existence of solutions are proved in Section 3. Section 4 offers a classification of global solutions for problem (P), under certain conditions. There are two types of solutions: those that remain strictly positive and strictly decreasing, that is, $u(r) > 0$ and $u'(r) < 0$ for all $r > 0$, and those that undergo a sign change, meaning there exists an $r_0 > 0$ (the first zero of u) where u is strictly positive and decreasing on $(0, r_0)$, $u(r_0) = 0$, and $u'(r_0) < 0$. In Section 5, we elucidate the asymptotic behavior of positive solutions as they approach infinity. In particular, we prove that under some assumptions, $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r)$ always exists. Finally, in Section 6, we give the conclusions and we present the perspectives of this work.

2. Behavior near the origin

In the following section, we present the asymptotic behavior of solutions of (1.2) with positive initial data near the origin. More precisely, we give explicit equivalents of first and second derivatives of solutions near 0. The analysis is affected by the parameters N, p, q_2, l_2 and the initial data.

Proposition 2.1. *Consider a solution u to equation (1.2) with $u(0) = a > 0$. Then u exhibits strict monotonicity near the origin. Furthermore, if $N \geq p$ holds, then it can be deduced that u is strictly decreasing in the neighborhood of the origin.*

Proof. As $u(0) = a > 0$ and u is a continuous function then there exists a small $\eta > 0$ such that for $r \in]0, \eta[$, we have $u(r) > 0$. Suppose by contradiction that u oscillates on $]0, \eta[$ and let r_0 be the first zero of u' in $]0, \eta[$, so using the equation (1.2) we obtain

$$(|u'|^{p-2}u')'(r_0) = -r_0^{l_1}|u|^{q_1-1}u(r_0) - r_0^{l_2}|u|^{q_2-1}u(r_0) < 0, \tag{2.1}$$

which yields that $u'(r) \neq 0$ near 0.

Suppose by contradiction that u is strictly increasing near 0. We note that the equation (1.2) can be expressed as follows

$$\left(r^{N-1}|u'|^{p-2}u'(r)\right)' = -r^{l_1+N-1}|u|^{q_1-1}u(r) - r^{l_2+N-1}|u|^{q_2-1}u(r). \tag{2.2}$$

Then $r^{N-1}|u'|^{p-2}u'(r)$ is strictly positive and $\left(r^{N-1}|u'|^{p-2}u'(r)\right)' < 0$ near the origin and so $\lim\limits_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \in]0, +\infty]$. This implies the existence of a small positive value denoted as η and a positive constant represented as C such that

$$u'(r) > C\, r^{\frac{1-N}{p-1}} \quad \text{for each } r \in]0, \eta[. \tag{2.3}$$

This is impossible because $u' \in L^1((0,\eta))$ and $r^{\frac{1-N}{p-1}} \notin L^1((0,\eta))$ for $N \geq p$. □

Proposition 2.2. *Consider a solution u to equation (1.2), with the initial condition $u(0) = a > 0$. Then $\lim\limits_{r\to 0} r^{\frac{N-1}{p-1}} u'(r)$ exists and is finite. In addition, if $N \geq p$, then $\lim\limits_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0$.*

Proof. By using (2.2), we get $\left(r^{N-1}|u'|^{p-2}u'\right)' < 0$ for small r. Then

$$\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \in]-\infty, +\infty].$$

So, we distinguish two cases.

Case 1. $N \geq p$. We suppose that $\lim\limits_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) \neq 0$, then there exist $\eta > 0$ sufficiently small and a constant $C > 0$ where

$$|u'(r)| > C\, r^{\frac{1-N}{p-1}} \quad \text{for any } r \in (0, \eta). \tag{2.4}$$

This yields a contradiction as in the proof of Proposition 2.1 because u' keeps a constant sign near 0, $u' \in L^1((0,\eta))$ and $r^{\frac{1-N}{p-1}} \notin L^1((0,\eta))$. Therefore,

$$\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0.$$

Case 2. $N < p$. Suppose that $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = +\infty$ and using again equation (2.2), we get

$$\lim_{r\to 0} r^{-l_2-N+1}\left(r^{N-1}|u'|^{p-2}u'(r)\right)' = -a^{q_2}, \tag{2.5}$$

then there are $\eta > 0$ and $C_1 > 0$ such that

$$\left(r^{N-1}|u'|^{p-2}u'(r)\right)' > -C_1(r^{l_2+N})' \tag{2.6}$$

for any $r \in (0,\eta)$. Integrating the last inequality on (r,ρ) for small $0 < \rho < \eta$, we obtain

$$\rho^{N-1}|u'(\rho)|^{p-2}u'(\rho) - r^{N-1}|u'(r)|^{p-2}u'(r) > -C_1(\rho^{l_2+N} - r^{l_2+N}). \tag{2.7}$$

Using the fact that $l_2 + N > 0$ and letting $r \to 0$, we get a contradiction. Hence $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r)$ is finite. □

Theorem 2.3. *Consider a solution u to equation* (1.2) *with the initial condition $u(0) = a > 0$. Additionally, suppose that $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = b \in \mathbb{R}$. Then*

(i) *for $b = 0$, we have*

$$\lim_{r\to 0} r^{-\frac{l_2+1}{p-1}} u'(r) = -\left(\frac{a^{q_2}}{l_2+N}\right)^{\frac{1}{p-1}} \tag{2.8}$$

and

$$\lim_{r\to 0} r^{\frac{(p-2-l_2)}{p-1}} u''(r) = \frac{-(l_2+1)}{p-1}\left(\frac{a^{q_2}}{l_2+N}\right)^{\frac{1}{p-1}}. \tag{2.9}$$

In particular, u is strictly decreasing near the origin.

(ii) *for $b \neq 0$, we have*

$$\lim_{r\to 0} r^{\frac{(N+p-2)}{p-1}} u''(r) = \frac{-(N-1)}{p-1} b. \tag{2.10}$$

Proof. The demonstration will proceed in two steps.

Step 1. Behavior of $u'(r)$. Since $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0$,

$$\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r)r^{N-1} = 0.$$

So using equation (2.2) and Hospital's rule, we get

$$\lim_{r\to 0} r^{-l_2-N}\left(r^{N-1}|u'|^{p-2}u'(r)\right) = \frac{-a^{q_2}}{l_2+N}. \tag{2.11}$$

That is,

$$\lim_{r\to 0} r^{-l_2-1}|u'|^{p-2}u'(r) = \frac{-a^{q_2}}{l_2+N}\,. \tag{2.12}$$

Step 2. Behavior of $u''(r)$. Note that as $u' \neq 0$ near 0, then u'' exists. Consequently, we proceed to differentiate between two distinct cases.

(i) When $b = 0$, multiply equation (1.2) by r^{-l_2}, then using (2.12), $l_1 > l_2$ and $u(0) > 0$, we obtain

$$\lim_{r\to 0} r^{-l_2}\left(|u'|^{p-2}u'\right)'(r) = \frac{-a^{q_2}(l_2+1)}{l_2+N}. \tag{2.13}$$

Using again the behavior of u' given by (2.12), we obtain

$$\lim_{r\to 0} r^{\frac{(p-2-l_2)}{p-1}} u''(r) = \frac{-(l_2+1)}{p-1}\left(\frac{a^{q_2}}{l_2+N}\right)^{\frac{1}{p-1}}. \tag{2.14}$$

(ii) When $b \neq 0$. In a manner analogous to (i), multiplying equation (1.2) by r^N and using the fact that $l_1 > l_2 > -N$, we get by letting $r \to 0$,

$$\lim_{r\to 0} r^{N}\left(|u'|^{p-2}u'\right)'(r) = -(N-1)|b|^{p-2}b. \tag{2.15}$$

Therefore

$$\lim_{r\to 0} r^{\frac{(p-2+N)}{p-1}} u''(r) = \frac{-(N-1)}{p-1}b. \tag{2.16}$$

□

Corollary 2.4. *Consider a solution u to equation* (1.2) *satisfies* $u(0) = a > 0$. *Then* $\lim_{r\to 0} ru'(r) = 0$.

Proof. If $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) = 0$, then using the behavior of u' given in Theorem 2.3 and taking account that $l_2 > -p$, we get $\lim_{r\to 0} ru'(r) = 0$. If $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r) \neq 0$, then by Proposition 2.2, we have certainly $N < p$ and we easily get the result. □

3. Global solutions

We are aware, based on Proposition 2.2, that if u is a solution to (1.2) with the condition $u(0) = a > 0$, then $\lim_{r\to 0} r^{\frac{N-1}{p-1}} u'(r)$ exists and is finite. Consequently, we aim to find a solution u defined on $[0, +\infty[$ such that $u \in C^0([0, +\infty[) \cap C^1(]0, +\infty[)$, $|u'|^{p-2}u' \in C^1(]0, +\infty[)$ and satisfying problem (P).

In order to establish local existence and uniqueness, we intend to transform problem (P) into a fixed-point problem involving a certain operator by building on the ideas introduced in [5] and [6].

Theorem 3.1. *Assume that $a > 0$ and $b \in \mathbb{R}$. Then (P) has a unique global solution u.*

Proof. Proofing will take place in two steps.
Step 1. Existence of a maximal solution.

By performing two integrations of (2.2) over the interval $[0, r]$, we can establish that problem (P) has become equivalent to the following integral equation:

$$u(r) = a - \int_0^r G(F(u)(s))\,\mathrm{d}s, \tag{3.1}$$

$$G(s) = |s|^{\frac{2-p}{p-1}} s, \;\; s \in \mathbb{R} \tag{3.2}$$

and the nonlinear function F is given by

$$F(u)(s) = -ds^{1-N} + s^{1-N}\int_0^s \Big(\sigma^{l_1+N-1}|u|^{q_1-1}(\sigma)u(\sigma) + \sigma^{l_2+N-1}|u|^{q_2-1}(\sigma)u(\sigma)\Big)\mathrm{d}\sigma, \tag{3.3}$$

where $d = |b|^{p-2}b = \lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r)$.

Let $R > 0$ and $0 < K < a$. We consider the following complete metric space

$$E = \{v \in C([0, R]) : \|v - a\|_0 \le K\}. \tag{3.4}$$

Here, $C([0, R])$ represents the Banach space comprising real continuous functions defined on the interval $[0, R]$, equipped with the uniform norm denoted as $\|\cdot\|_0$.

$$T(v)(r) = a - \int_0^r G(F(v)(s))\,\mathrm{d}s. \tag{3.5}$$

First, we prove that T maps E into itself for small R.

Let $v \in E$ and $r \in [0, R]$. Then $T(v)(r) \in C([0, R])$. So we obtain the following estimate

$$|T(v)(s) - a| \le \int_0^r |F(v)(s)|^{\frac{1}{p-1}}\, ds. \tag{3.6}$$

In order to establish our argument, it is adequate to bound the function $F(v)$ between two expressions that have the same sign for any $v \in E$. As $v(r) \in [a-K, a+K]$, it is easy to deduce that for any $s \in (0, R)$,

$$F(v)(s) \leq -ds^{1-N} + \frac{(K+a)^{q_1}}{l_1+N} s^{1+l_1} + \frac{(K+a)^{q_2}}{l_2+N} s^{1+l_2}. \tag{3.7}$$

Since $l_1 > l_2$, we can choose R small enough such that

$$F(v)(s) \leq \left(-d + \frac{2(K+a)^{q_2}}{l_2+N} s^{l_2+N}\right) s^{1-N}, \tag{3.8}$$

for any $s \in (0, R)$. Therefore

$$F(v)(s) \leq \begin{cases} \frac{2(K+a)^{q_2}}{l_2+N} s^{l_2+1} & \text{if } d = 0, \\ -\frac{d}{2} s^{1-N} & \text{if } d > 0, \\ -2ds^{1-N} & \text{if } d < 0, \end{cases} \tag{3.9}$$

for any $s \in (0, R)$.

On the other hand, using again the fact that $l_1 > l_2$, we can choose R small enough such that

$$F(v)(s) \geq \left(-d + \frac{(a-K)^{q_2}}{l_2+N} s^{l_2+N}\right) s^{1-N} \tag{3.10}$$

for any $s \in (0, R)$. Therefore

$$F(v)(s) \geq \begin{cases} \frac{(a-K)^{q_2}}{2(l_2+N)} s^{l_2+1} & \text{if } d = 0, \\ -2ds^{1-N} & \text{if } d > 0, \\ -\dfrac{d}{2} s^{1-N} & \text{if } d < 0, \end{cases} \tag{3.11}$$

for any $s \in (0, R)$. Combining (3.9) and (3.11), we have

$$\begin{cases} \frac{(a-K)^{q_2}}{2(l_2+N)} s^{l_2+1} \leq F(v)(s) \leq \frac{2(K+a)^{q_2}}{l_2+N} s^{l_2+1} & \text{if } d = 0, \\ \frac{|d|}{2} s^{1-N} \leq |F(v)(s)| \leq 2|d| s^{1-N} & \text{if } d \neq 0, \end{cases} \tag{3.12}$$

for any $s \in (0, R)$. This yields that

$$|T(v)(r) - a| \leq \begin{cases} \frac{p-1}{p+l_2} \left(\frac{2(K+a)^{q_2}}{l_2+N}\right)^{\frac{1}{p-1}} r^{\frac{l_2+p}{p-1}} & \text{if } d = 0, \\ \frac{p-1}{p-N} (2|d|)^{\frac{1}{p-1}} r^{\frac{p-N}{p-1}} & \text{if } d \neq 0, \end{cases} \tag{3.13}$$

for each $r \in (0, R)$. As a consequence, for R sufficiently small, we have

$$|T(v)(r) - a| \leq K \quad \text{for } v \in E. \tag{3.14}$$

That is $T(v) \in E$.

Now, we claim that T is a contraction from E into itself for R sufficiently small. In fact for each $\varphi, \psi \in E$,

$$|T(\varphi)(r) - T(\psi)(r)| \leqslant \int_0^r |G(F(\varphi)(r)) - G(F(\psi)(r))| ds \tag{3.15}$$

such that $F(\varphi)$ is defined by (3.3). Next letting $\phi(s) = \min(|F(\varphi)(s)|, |F(\psi)(s)|)$. Then

$$|T(\varphi)(r) - T(\psi)(r)| \leqslant \int_0^r (\phi(s))^{\frac{2-p}{p-1}} |F(\varphi)(s) - F(\psi)(s)| ds. \tag{3.16}$$

Based on (3.3) and (3.4), we obtain

$$|F(\varphi)(s) - F(\psi)(s)| \leqslant \left(q_1 \frac{(a+K)^{q_1-1}}{l_1+N} s^{l_1+1} + q_2 \frac{(a+K)^{q_2-1}}{l_2+1} s^{l_2+1} \right) \|\varphi - \psi\|_0. \tag{3.17}$$

Since $l_1 > l_2$, we can choose R small enough such that

$$|F(\varphi)(s) - F(\psi)(s)| \leqslant \left(\frac{2q_2(a+K)^{q_2-1}}{l_2+N} \right) s^{l_2+1} \|\varphi - \psi\|_0 \tag{3.18}$$

for any $0 < s < r < R$. Therefore, using (3.16) and (3.18), we have that

- if $d = 0$, then

$$|T(\varphi)(r) - T(\psi)(r)| \leqslant \frac{2q_2(p-1)(a+K)^{q_2-1}}{(l_2+p)(l_2+N)} \left(\frac{(a-K)^{q_2}}{2(l_2+N)} \right)^{\frac{2-p}{p-1}} r^{\frac{l_2+p}{p-1}} \|\varphi - \psi\|_0 \tag{3.19}$$

for each $r \in (0, R)$.

- If $d \neq 0$, then $N < p$ and

$$|T(\varphi)(r) - T(\psi)(r)| \leqslant \frac{2q_2(p-1)(a+K)^{q_2-1}}{(l_2+N)\big((l_2+N)(p-1)+p-N\big)} \left(\frac{|d|}{2} \right)^{\frac{2-p}{p-1}} \times r^{\frac{(l_2+N)(p-1)+p-N}{p-1}} \|\varphi - \psi\|_0 \tag{3.20}$$

for any $r \in (0, R)$.

Consequently, in estimates (3.19) and (3.20), it is possible to choose R as being sufficiently small, which assures that T behaves as a contraction. Hence, by virtue of the Banach Fixed Point Theorem (as outlined in [1]), we establish the existence of a unique fixed point for the operator T, which serves as a solution to problem (P). Typically, this solution can be prolonged to a maximal interval $[0, R_{max}[$ with $0 < R_{max} \leq +\infty$.

Step 2. Existence of an entire solution.

Let u be a solution of problem (P) defined $[0, R_{max}[$. Assume that $R_{max} < +\infty$. Then

$$\lim_{r \to R_{max}} |u(r)| = \lim_{r \to R_{max}} |u'(r)| = +\infty.$$

For any $r \in (0, R_{max})$, we introduce the following function

$$Q(r) = \frac{p-1}{p}|u'|^p + \frac{r^{l_1}}{q_1+1}|u|^{q_1+1} + \frac{r^{l_2}}{q_2+1}|u|^{q_2+1}. \tag{3.21}$$

According to equation (1.2), we get

$$Q'(r) = -\frac{N-1}{r}|u'|^p + \frac{l_1}{q_1+1}r^{l_1-1}|u|^{q_1+1} + \frac{l_2}{q_2+1}r^{l_2-1}|u|^{q_2+1}. \tag{3.22}$$

Since $N \geq 1$ and $l_2 < l_1 \leq 0$, then $\lim_{r\to R_{max}} Q'(r) = -\infty$ and $\lim_{r\to R_{max}} Q(r) = +\infty$. Which is a contradiction. Therefore, $R_{max} = +\infty$ and u is global. □

4. Classification of solutions

In the sequel, we offer a classification of solutions to problem (P). Before presenting the theorems, we need some basic results.

Lemma 4.1. *Consider a solution u to problem (P). If $r_0 > 0$ represents the first zero of u, it follows that $u'(r_0) < 0$.*

Proof. Let $r_0 > 0$ denote the first zero of u. Given that $u(0) > 0$, it is necessary that $u'(r_0) \leqslant 0$. Suppose for contradiction, that $u'(r_0) = 0$. Using relation (2.2) and the observation that u is strictly positive in $(0, r_0)$, we can deduce that $\left(r^{N-1}|u'|^{p-2}u'\right)'(r) < 0$ on $(0, r_0)$. Consequently

$$0 = r_0^{N-1}|u'|^{p-2}u'(r_0) < r^{N-1}|u'|^{p-2}u'(r).$$

So u is strictly increasing on $(0, r_0)$. But this is in contradiction with the fact that $u(r_0) = 0$. Consequently $u'(r_0) < 0$. □

Proposition 4.2. *Assume that $N \geq p$. Consider a solution u to problem (P). Then u is strictly decreasing on $(0, +\infty)$.*

Proof. Since $N \geq p$, then $u'(r) < 0$ for small r. Suppose by contradiction that there exists an $r_0 > 0$ representing the first zero of u'. This would imply that $\left(|u'|^{p-2}u'\right)'(r_0) \geq 0$. But this contradicts the fact that $\left(|u'|^{p-2}u'\right)'(r_0) < 0$ by equation (1.2). Hence u is strictly decreasing for all $r > 0$. □

Proposition 4.3. *Assuming that $N \geq p$ and $q_1 > p-1$, and considering a strictly positive solution u to problem (P), it follows that for any $r > 0$, we have*

$$u(r) \leqslant C(N, p, q_1, l_1)\, r^{\frac{-(l_1+p)}{q_1+1-p}}, \tag{4.1}$$

where

$$C(N,p,q_1,l_1)=\left[(q_1+1-p)\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}\left(\frac{1-2^{-\frac{l_1+p}{p-1}}}{l_1+p}\right)\right]^{\frac{-(p-1)}{q_1+1-p}}. \tag{4.2}$$

Proof. By using (2.2), we have for each $r>0$,

$$\left(r^{N-1}|u'|^{p-2}u'\right)'<-r^{l_1+N-1}u^{q_1}(r). \tag{4.3}$$

Integrating (4.3) in $\left(\frac{r}{2},r\right)$ for any $r>0$, we obtain

$$r^{N-1}|u'|^{p-2}u'(r)-\left(\frac{r}{2}\right)^{N-1}|u'|^{p-2}u'\left(\frac{r}{2}\right)<-\int_{\frac{r}{2}}^{r}s^{l_1+N-1}u^{q_1}(s)ds. \tag{4.4}$$

Since u is strictly decreasing by Proposition 4.2, then

$$r^{N-1}|u'|^{p-2}u'(r)<-\frac{(1-2^{-l_1-N})}{l_1+N}r^{l_1+N}u^{q_1}(r). \tag{4.5}$$

It follows that for any $r>0$,

$$u'(r)<-\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}r^{\frac{l_1+1}{p-1}}u^{\frac{q_1}{p-1}}(r). \tag{4.6}$$

Since u is strictly positive and $q_1>p-1$, then

$$\frac{p-1}{p-q_1-1}\left(u^{\frac{p-q_1-1}{p-1}}\right)'<-\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}r^{\frac{l_1+1}{p-1}}. \tag{4.7}$$

Therefore

$$\left(u^{\frac{p-q_1-1}{p-1}}\right)'>\frac{q_1+1-p}{p-1}\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}r^{\frac{l_1+1}{p-1}}. \tag{4.8}$$

We integrate inequality (4.8) on $\left(\frac{r}{2},r\right)$ for $r>0$, we get

$$u^{\frac{p-q_1-1}{p-1}}(r)>(q_1+1-p)\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}\left(\frac{1-2^{-\frac{l_1+p}{p-1}}}{l_1+p}\right)r^{\frac{l_1+p}{p-1}}. \tag{4.9}$$

Consequently we obtain the inequality (4.1). □

Before introducing other results, let us define for all real $c\neq 0$ the following function

$$E_c(r)=cu(r)+ru'(r),\ \ r>0. \tag{4.10}$$

It is evident that for each $r>0$,

$$(r^c u(r))'=r^{c-1}E_c(r). \tag{4.11}$$

Therefore, the analysis of the variation of $r^c u(r)$ hinges on the sign of $E_c(r)$. Using the equation (1.2). We obtain, for all $r > 0$ verifies $u'(r) \neq 0$, that

$$(p-1)|u'|^{p-2}E_c'(r) = (p - N + c(p-1))\,|u'|^{p-2}u'(r) - r^{l_1+1}|u|^{q_1-1}u \\ - r^{l_2+1}|u|^{q_2-1}u. \tag{4.12}$$

If we have $E_c(r_0) = 0$ for a certain $r_0 > 0$, then we can deduce that

$$(p-1)|u'|^{p-2}E_c'(r_0) = (N - p - c(p-1))\,|c|^{p-2}c\, r_0^{1-p}|u|^{p-2}u(r_0) \\ - r_0^{l_1+1}|u|^{q_1-1}u(r_0) - r_0^{l_2+1}|u|^{q_2-1}u(r_0). \tag{4.13}$$

Proposition 4.4. *Assuming that $N > p$ and $c \geq \dfrac{N-p}{p-1}$. Consider a strictly positive solution u to problem (P). We can conclude that $E_c'(r) < 0$ and $E_c(r) > 0$ for all $r > 0$.*

Proof. By exploiting the fact that u is strictly decreasing for any $r > 0$ (by Proposition 4.2), we have according to relation (4.12),

$$(p-1)\frac{E_c'(r)}{u'(r)} = (p - N + c(p-1)) + \frac{r^{l_1+1}u^{q_1}}{|u'|^{p-1}} + \frac{r^{l_2+1}u^{q_2}}{|u'|^{p-1}}. \tag{4.14}$$

This yields that for $c \geq \dfrac{N-p}{p-1}$, we have $E_c'(r) < 0$ for any $r > 0$.

Using Corollary 2.4, we have $E_c(0) = c\,u(0) > 0$. Suppose by contradiction that there is $r_0 > 0$ the first zero of E_c. Then $E_c(r) < E_c(r_0) = 0$ for $r > r_0$. Therefore, $\lim\limits_{r\to+\infty} r^c u(r)$ exists and finite (by (4.11)) and $\lim\limits_{r\to+\infty} E_c(r) \in [-\infty, 0[$. Hence, $\lim\limits_{r\to+\infty} u(r) = 0$ and by expression (4.10), $\lim\limits_{r\to+\infty} ru'(r) \in [-\infty, 0[$. Which is impossible. □

Theorem 4.5. *Assuming that $N > p$. Consider a solution u to problem (P), then u changes sign in the following cases:*

(i) $q_1 \leq p - 1$.

(ii) $p - 1 < q_1 < \dfrac{(l_1 + N)(p-1)}{N-p}$.

Proof. Assume by contradiction that u is strictly positive, then by Proposition 4.2, $\lim\limits_{r\to+\infty} u(r)$ exists and is finite. We distinguish the following cases.

(i) **Case 1**. Let $q_1 < p-1$. Using relation (4.6) and the fact that $q_1 < p-1$, we obtain also (4.7). Integrating this inequality from 0 to r with $l_1 + p > 0$, we get

$$\frac{p-1}{p-q_1-1}u^{\frac{p-q_1-1}{p-1}}(r)-\frac{p-1}{p-q_1-1}a^{\frac{p-q_1-1}{p-1}}<-\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}\frac{p-1}{l_1+p}r^{\frac{l_1+p}{p-1}}.$$

By tending r to $+\infty$ in the last inequality, we obtain a contradiction because the right-hand side tends to $-\infty$, but the left converges.

Case 2 Let $q_1=p-1$.

By equation (4.6) we obtain

$$\frac{u'}{u}\leq-\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}r^{\frac{l_1+1}{p-1}}.$$

We integrate this previous inequality on $(0,r)$ for all $r>0$, we obtain

$$u(r)<u(0)\,exp\left(-\frac{p-1}{l_1+p}\left(\frac{1-2^{-l_1-N}}{l_1+N}\right)^{\frac{1}{p-1}}r^{\frac{l_1+p}{p-1}}\right).$$

Thus, $\lim_{r\to+\infty}r^{\frac{N-p}{p-1}}u(r)=0$. But using relation (4.11) and Proposition 4.4, we have $r^{\frac{N-p}{p-1}}u(r)$ is strictly increasing. This is a contradiction

(ii) If $p-1<q_1<\dfrac{(l_1+N)(p-1)}{N-p}$, then $\dfrac{N-p}{p-1}<\dfrac{l_1+p}{q_1+1-p}$. Using Proposition 4.3, we obtain $\lim_{r\to+\infty}r^{\frac{N-p}{p-1}}u(r)=0$. Which is impossible like the second case of (i).

We deduce that u is not strictly positive in previous cases, that is there exists $r_0>0$ the first zero of u, then by Lemma 4.1, $u'(r_0)<0$ and so u changes sign. The proof is complete. □

Theorem 4.6. *Assuming that $N\geq p$ and $l_2>-\dfrac{p(N-1)}{p-1}$. Let u be a solution of (P). Then there is $a_0>0$ such that for each $a\in(0,a_0)$ and for all $r\in(0,+\infty)$, $u(r)>0$.*

Proof. Assume for contradiction that u is not strictly positive and let r_0 denote the first zero of u. According to Lemma 4.1, we have $u'(r_0)<0$. Let us define the function H on $(0,r_0)$ by

$$H(r)=\frac{p-1}{p}|u'(r)|^p+\frac{1}{q_1+1}r^{l_1}u^{q_1+1}(r)-\frac{1}{q_2+1}r^{l_2}u^{q_2+1}(r).\qquad(4.15)$$

Hence, based on (1.2), we get for each $r \in (0, r_0)$,

$$H'(r) = -\frac{N-1}{r}|u'|^p + \frac{l_1}{q_1+1} r^{l_1-1} u^{q_1+1}(r) - \frac{l_2}{q_2+1} r^{l_2-1} u^{q_2+1}(r) - 2r^{l_2} u^{q_2} u'(r). \tag{4.16}$$

Using Proposition 2.3, we have $\lim_{r\to 0} r^{-l_2}|u'(r)|^p = 0$. Therefore, since $l_2 < l_1$, we obtain

$$\lim_{r\to 0} r^{-l_2} H(r) = -\frac{a^{q_2+1}}{q_2+1}.$$

This gives that H is strictly negative for small r.

Meanwhile, considering the fact that $H(r_0) > 0$, then there is $r_1 \in (0, r_0)$ such that $H(r_1) = 0$ and $H'(r_1) \geqslant 0$. Then, at $r = r_1$, we have

$$\begin{aligned} H'(r_1) = r_1^{l_2-1} u^{q_2+1}(r_1)\Big[&-\frac{p(N-1)}{(p-1)(q_2+1)} - \frac{l_2}{q_2+1} \\ &+ \left(\frac{p(N-1)}{(p-1)(q_1+1)} + \frac{l_1}{q_1+1}\right) r_1^{l_1-l_2} u^{q_1-q_2}(r_1) - 2\frac{r_1 u'(r_1)}{u(r_1)}\Big]. \end{aligned} \tag{4.17}$$

Using again the fact that $H(r_1) = 0$, we obtain

$$r_1 = r_1(a) < \left(\frac{q_1+1}{q_2+1}\right)^{\frac{1}{l_1-l_2}} u^{\frac{q_2-q_1}{l_1-l_2}}(r_1). \tag{4.18}$$

Since $l_2 < l_1,\ q_1 < q_2$ and $0 < u(r_1) < u(0) = a$, then

$$r_1(a) < \left(\frac{q_1+1}{q_2+1}\right)^{\frac{1}{l_1-l_2}} a^{\frac{q_2-q_1}{l_1-l_2}}. \tag{4.19}$$

Hence $\lim_{a\to 0} r_1(a) = 0$. Then, combining this inequality with $l_2 > -\frac{p(N-1)}{p-1}$, $\lim_{r\to 0} r^{l_1-l_2} u^{q_1-q_2}(r) = 0$ and $\lim_{r\to 0} r u'(r) = 0$ (by Corollary 2.4), we get

$$\lim_{a\to 0} r_1^{l_1-l_2} u^{q_1-q_2}(r_1) = 0$$

and

$$\lim_{a\to 0} \frac{r_1 u'(r_1)}{u(r_1)} = 0.$$

So $H'(r_1) < 0$ for values of a that are sufficiently small. This yields a contradiction with $H'(r_1) \geqslant 0$. Hence, $u = u(,a)$ is strictly positive for sufficiently small a. □

Theorem 4.7. *Assuming that* $N > p$ *and* $q_1 \geq \frac{N(p-1)+p(l_1+1)}{N-p}$. *Consider* u *as a solution to problem* (P). *Then* $u(r) > 0$ *for each* $r > 0$.

Proof. Suppose by contradiction that there exists a positive value r_0 which represents the first zero of u. Then u is strictly positive and strictly decreasing on $(0, r_0)$ and $u'(r_0) < 0$ by Lemma 4.1.

Using Pohozaev identity, we define for $r \in (0, r_0)$ the following function

$$\begin{aligned} I(r) =& \frac{p-1}{p} r^N |u'|^p + \frac{1}{q_1+1} r^{l_1+N} u^{q_1+1}(r) + \frac{1}{q_2+1} r^{l_2+N} u^{q_2+1}(r) \\ &+ \frac{N-p}{p} r^{N-1} |u'|^{p-2} u' u(r). \end{aligned} \tag{4.20}$$

Then by equation (1.2),

$$\begin{aligned} I'(r) =& \left(\frac{l_1+N}{q_1+1} - \frac{N-p}{p} \right) r^{l_1+N-1} u^{q_1+1}(r) \\ &+ \left(\frac{l_2+N}{q_2+1} - \frac{N-p}{p} \right) r^{l_2+N-1} u^{q_2+1}(r). \end{aligned} \tag{4.21}$$

Integrating (4.21) on $(0, r)$ for $0 < r < r_0$ and since $\lim_{r \to 0} r u'(r) = 0$ (by Corollary 2.4), we get

$$\begin{aligned} I(r) =& \left(\frac{l_1+N}{q_1+1} - \frac{N-p}{p} \right) \int_0^r s^{l_1+N-1} u^{q_1+1}(s) ds \\ &+ \left(\frac{l_2+N}{q_2+1} - \frac{N-p}{p} \right) \int_0^r s^{l_2+N-1} u^{q_2+1}(s) ds. \end{aligned} \tag{4.22}$$

Since $q_1 \geq \frac{N(p-1)+p}{N-p} + \frac{pl_1}{N-p}$, then $\frac{l_2+N}{q_2+1} < \frac{l_1+N}{q_1+1} \leq \frac{N-p}{p}$. Letting $r \to r_0$ in (4.22), we obtain $I(r_0) \leq 0$. On the other hand, since $u'(r_0) < 0$, then in (4.20), we have $I(r_0) = \frac{p-1}{p} r_0^p |u'(r_0)|^p > 0$. This is a contradiction. Consequently, u is strictly positive. □

5. Behavior near infinity

Within this section, we delve into the asymptotic behavior near the infinity of positive solutions pertaining to problem (P). To undertake this study, we draw upon concepts and insights presented in papers [4, 7].

Theorem 5.1. *Suppose that $N > p$ and $q_1 > \frac{(l_1+N)(p-1)}{N-p}$. Consider u as a strictly positive solution of (P). If $q_1 \neq \frac{N(p-1)+p(l_1+1)}{N-p}$, then u has one of the following behaviors near infinity:*

(i)

$$\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = 0. \tag{5.1}$$

(ii)

$$\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = \left(\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1) \right) \left(\frac{l_1+p}{q_1+1-p} \right)^{p-1} \right)^{\frac{1}{q_1+1-p}}. \tag{5.2}$$

To prove Theorem 5.1 we need some interesting results.

Proposition 5.2. *Assuming that $N > p$ and $q_1 > p-1$. consider a strictly positive solution u to problem (P). In such a case, it follows that the function $r^{\frac{l_1+p}{q_1+1-p}+1} u'(r)$ remains bounded as r approaches infinity.*

Proof. We know by Proposition 4.4 that $E_{\frac{N-p}{p-1}}(r) > 0$ for large r, since u is decreasing, we have for large r,

$$r|u'(r)| < \frac{N-p}{p-1} u(r).$$

Since $r^{\frac{l_1+p}{q_1+1-p}} u(r)$ is bounded by Proposition 4.3, $r^{\frac{l_1+p}{q_1+1-p}+1} u'(r)$ is bounded for large r. □

Now we use the logarithmic change introduced in [4, 7] which plays an important role in the proofs of the next theorems. We set

$$v_c(t) = r^c u(r), \text{ where } t = \ln(r). \tag{5.3}$$

Then v_c satisfies

$$w_c'(t) + Aw_c(t) + e^{-B_c t} v_c^{q_1}(t) + e^{-D_c t} v_c^{q_2}(t) = 0, \tag{5.4}$$

where

$$h_c(t) = v_c'(t) - cv(t), \tag{5.5}$$

$$w_c(t) = |h_c|^{p-2} h_c(t), \tag{5.6}$$

$$A_c = N - p - c(p-1), \tag{5.7}$$

$$B_c = c(q_1 + 1 - p) - (l_1 + p) \tag{5.8}$$

and

$$D_c = c(q_2 + 1 - p) - (l_2 + p). \tag{5.9}$$

We note that

$$h_c(t) = r^{c+1} u'(r). \tag{5.10}$$

Proposition 5.3. *Assuming that $N > p$ and $q_1 > \dfrac{(l_1+N)(p-1)}{N-p}$. Consider a strictly positive solution u to problem (P). If $r^{\frac{l_1+p}{q_1+1-p}}u(r)$ converges when $r \to +\infty$, then the function $r^{\frac{l_1+p}{q_1+1-p}+1}u'(r)$ converges also when r goes to $+\infty$.*

Proof. Using the logarithmic change (5.3) with $c = \dfrac{l_1+p}{q_1+1-p}$, we have $v(t) = r^{\frac{l_1+p}{q_1+1-p}}u(r)$ is bounded by Proposition 4.3. Then by (5.10), we have $h(t) = r^{\frac{l_1+p}{q_1+1-p}+1}u'(r)$ is bounded for large t by Proposition 5.2. Therefore, $w(t) = |h|^{p-2}(t)h(t)$ is bounded for large t. Assume by contradiction that there are two sequences k_i and s_i that both tend to $+\infty$ as i approaches infinity. Such that k_i represents a local minimum, s_i represents a local maximum of the function $w(t)$, and for all i, we have $k_i < s_i < k_{i+1}$, and additionally, they satisfy

$$-\infty < \liminf_{t\to+\infty} w(t) = \lim_{i\to+\infty} w(k_i) < \lim_{i\to+\infty} w(s_i) = \limsup_{t\to+\infty} w(t) \leq 0 \tag{5.11}$$

and

$$w'(k_i) = w'(s_i) = 0. \tag{5.12}$$

According to equation (5.4), v satisfies the following equation

$$w'(t) + Aw(t) + v^{q_1}(t) + e^{-Dt}v^{q_2}(t) = 0, \tag{5.13}$$

where

$$A = N - p - \frac{l_1+p}{q_1+1-p}(p-1) \tag{5.14}$$

and

$$D = \frac{l_1+p}{q_1+1-p}(q_2+1-p) - (l_2+p). \tag{5.15}$$

Therefore

$$Aw(k_i) + v^{q_1}(k_i) + e^{-Dk_i}v^{q_2}(k_i) = Aw(s_i) + v^{q_1}(s_i) + e^{-Ds_i}v^{q_2}(s_i).$$

Since v converges, $A > 0$ and $D > 0$ (because $q_2 > q_1 > \frac{(l_1+N)(p-1)}{N-p}$ and $l_2 < l_1$), then

$$\lim_{i\to+\infty} w(k_i) = \lim_{i\to+\infty} w(s_i). \tag{5.16}$$

Which contradicts (5.11). Therefore, w converges when t tends to $+\infty$ and consequently $r^{\frac{l_1+p}{q_1+1-p}}u'(r)$ converges when r tends to $+\infty$. □

We now present the following lemma, which is a classical result by Gidas and Spruck [12].

Lemma 5.4. ([12]) *Consider a positive differential function g such that*

(1) $\int_{t_0}^{+\infty} g(t)dt < +\infty$ *for large* t_0.

(ii) $g'(t)$ *is bounded for large* t.

Then $\lim_{t\to+\infty} g(t) = 0$.

Now we are able to prove Theorem 5.1.

Proof. Recalling the logarithmic change (5.3), we have $v(t) = r^{\frac{l_1+p}{q_1+1-p}}u(r)$ and defining the following energy function relied with (5.13)

$$\begin{aligned}E(t) =&\frac{p-1}{p}|h(t)|^p + Aw(t)v(t) + \frac{q_1}{q_1+1}\left(A - \frac{l_1+p}{q_1+1-p}\right)A^{\frac{1}{q_1}}|w(t)|^{\frac{q_1+1}{q_1}}\\ &+\frac{v^{q_1+1}}{q_1+1},\end{aligned} \tag{5.17}$$

where A is given by (5.14).

As $v(t)$, $h(t)$ and $w(t)$ are bounded for large t, then $E(t)$ is bounded for large t. In addition, we have

$$\begin{aligned}E'(t) =& \left(\frac{l_1+p}{q_1+1-p} - A\right)Y(t) - e^{-Dt}v^{q_2}(t)v'(t)\\ &-\left(\frac{l_1+p}{q_1+1-p} - A\right)e^{-Dt}v^{q_2}(t)\left(A^{\frac{1}{q_1}}|w|^{\frac{1}{q_1}} - v(t)\right),\end{aligned} \tag{5.18}$$

where D is given by (5.15) and

$$Y(t) = \left(v(t) - A^{\frac{1}{q_1}}|w(t)|^{\frac{1}{q_1}}\right)\left(v^{q_1} - A|w(t)|\right). \tag{5.19}$$

We shall split the remainder of the proof into three distinct steps.

Step 1. $E(t)$ is convergent as $t \to +\infty$.

Integrating (5.18) in (T, t) for T large enough, we get

$$\begin{aligned}E(t) =& E(T) + \left(\frac{l_1+p}{q_1+1-p} - A\right)S(t) - \int_T^t e^{-Ds}v^{q_2}(s)v'(s)ds\\ &-\left(\frac{l_1+p}{q_1+1-p} - A\right)\int_T^t e^{-Ds}v^{q_2}\left(A^{\frac{1}{q_1}}|w(s)|^{\frac{1}{q_1}} - v(s)\right)ds,\end{aligned} \tag{5.20}$$

where

$$S(t) = \int_T^t Y(s)ds. \tag{5.21}$$

As the function $s \to s^{q_1}$ is increasing, then $Y(t)$ is positive. Thus S is positive and increasing. We prove that $S(t)$ is bounded for large t. Since $q_1 \neq \dfrac{N(p-1)+p(l_1+1)}{N-p}$, then $A - \dfrac{l_1+p}{q_1+1-p} \neq 0$ and so

$$\begin{aligned} S(t) =& \frac{1}{\frac{l_1+p}{q_1+1-p} - A} \left(E(t) - E(T)\right) \\ &+ \frac{1}{\frac{l_1+p}{q_1+1-p} - A} \left(\frac{e^{-Dt}}{q_2+1} v^{q_2+1} - \frac{e^{-DT}}{q_2+1} v^{q_2+1} + \frac{D}{q_2+1} \int_T^t e^{-Ds} v^{q_2+1}(s) ds \right) \\ &+ \int_T^t e^{-Ds} v^{q_2} \left(A^{\frac{1}{q_1}} |w(s)|^{\frac{1}{q_1}} - v(s) \right) ds. \end{aligned} \tag{5.22}$$

Recall that $v(t)$, $w(t)$ and $E(t)$ are bounded for large t and $D > 0$, then $S(t)$ is bounded for large t. As follows $S(t)$ converges when $t \to +\infty$. Therefore, $E(t)$ converges when t tends to $+\infty$.

Step 2. $\lim\limits_{r \to +\infty} w'(t) = 0$.

Note that for any $1 < \delta \leqslant 2$, there exists $C_\delta > 0$ such that

$$\left(|a_1|^{\delta-2} a_1 - |a_2|^{\delta-2} a_2 \right) (a_1 - a_2) \geqslant C_\delta (a_1 - a_2)^2 \left(|a_1| + |a_2| \right)^{\delta-2} \tag{5.23}$$

for any $a_1, a_2 \in \mathbb{R}$ such that $|a_1| + |a_2| > 0$. In particular for $\delta = \dfrac{q_1+1}{q_1}$, there exists $C_1 > 0$, such that

$$\begin{aligned} \left(v(t) - A^{\frac{1}{q_1}} |w(s)|^{\frac{1}{q_1}} \right) \left(v^{q_1}(t) - A|w(t)| \right) \geqslant& C_1 \left(v^{q_1}(t) - A|w(t)| \right)^2 \\ & \times \left(v^{q_1}(t) + A|w(t)| \right)^{\frac{1}{q_1}-1}. \end{aligned} \tag{5.24}$$

Since $w(t)$ is strictly negative for large t, by (5.13) we get

$$\left(v^{q_1} + A|w(t)| \right)^{1-\frac{1}{q_1}} Y(t) \geqslant C_1 \left(w'(t) + e^{-Dt} v^{q_2}(t) \right)^2 .$$

Using again the fact that $v(t)$ and $w(t)$ are bounded for large t and $q_1 > 1$, then there is a constant $C_2 > 0$ such that for large t

$$\left(w'(t) + e^{-Dt} v^{q_2}(t) \right)^2 \leqslant C_2 Y(t).$$

Which yields that

$$\int_T^t \left(w'(s) + e^{-Ds} v^{q_2}(s) \right)^2 ds \leqslant C_2 S(t).$$

Consequently, we get

$$\int_T^t w'^2(s) ds \leqslant C_2 S(t) - 2 \int_T^t e^{-Ds} w'(s) v^{q_2}(s) ds.$$

Since $S(t)$, $v(t)$, $w'(t)$ are bounded for large t and $D > 0$, then $\int_T^t w'^2(s)ds$ is also bounded. Moreover, since $\int_T^t w'^2(s)ds$ is increasing, then

$$\int_T^{+\infty} w'^2(t)dt < +\infty.$$

On the other hand, by deriving equation (5.13) we get

$$w''(t) + Aw'(t) + q_1 v^{q_1-1}v'(t) - De^{-Dt}v^{q_2}(t) + q_2 e^{-Dt}v^{q_2-1}(t)v'(t) = 0. \quad (5.25)$$

Since $w'(t)$, $v'(t)$ and $v(t)$ are bounded for large t, then $w''(t)$ is bounded for large t. Thus, by Lemma 5.4, we have $\lim_{t\to+\infty} w'(t) = 0$.

Step 3. $v(t)$ converges when $t \to +\infty$.

Since v is bounded, $\lim_{t\to+\infty} w'(t) = 0$ and $D > 0$ and then by (5.13), we have

$$\lim_{t\to+\infty} (Aw(t) + v^{q_1}(t)) = 0. \quad (5.26)$$

Assume now by absurd that $v(t)$ oscillates for large t, then there are two sequences $\{\eta_i\}$ and $\{\zeta_i\}$ that go to $+\infty$ as $i \to +\infty$ such that $\{\eta_i\}$ and $\{\zeta_i\}$ are respectively local minimum and local maximum of v, verifying $\eta_i < \zeta_i < \eta_{i+1}$ and the relation

$$\begin{aligned} 0 \leqslant \lim_{i\to+\infty} v(\eta_i) &= \liminf_{t\to+\infty} v(t) = \alpha \\ < \lim_{i\to+\infty} v(\zeta_i) &= \limsup_{t\to+\infty} v(t) = \beta < +\infty. \end{aligned} \quad (5.27)$$

Since $v'(\eta_i) = v'(\zeta_i) = 0$, we have

$$h(\eta_i) = -\frac{l_1 + p}{q_1 + 1 - p} v(\eta_i) \quad \text{and} \quad h(\zeta_i) = -\frac{l_1 + p}{q_1 + 1 - p} v(\zeta_i).$$

It follows that

$$w(\eta_i) = -\left(\frac{l_1 + p}{q_1 + 1 - p}\right)^{p-1} v^{p-1}(\eta_i) \quad \text{and} \quad w(\zeta_i) = -\left(\frac{l_1 + p}{q_1 + 1 - p}\right)^{p-1} v^{p-1}(\zeta_i).$$

Combining with relation (5.26) and letting $i \to +\infty$, we get

$$\alpha^{p-1}\left(\alpha^{q_1+1-p} - \left(N - p - \frac{l_1 + p}{q_1 + 1 - p}(p-1)\right)\left(\frac{l_1 + p}{q_1 + 1 - p}\right)^{p-1}\right) = 0$$

and

$$\beta^{p-1}\left(\beta^{q_1+1-p} - \left(N - p - \frac{l_1 + p}{q_1 + 1 - p}(p-1)\right)\left(\frac{l_1 + p}{q_1 + 1 - p}\right)^{p-1}\right) = 0.$$

As $\alpha < \beta$, then $\alpha = 0$ and

$$\beta = \left(\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1)\right)\right)^{\frac{1}{q_1+1-p}}.$$

Which gives by (5.17) that $\lim\limits_{i\to+\infty} E(\eta_i) = 0$ and

$$\lim_{i\to+\infty} E(\zeta_i) = \frac{-(l_1+p)}{p(q_1+1)}\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\beta^p < 0.$$

This cannot take place because $E(t)$ converges when $t \to +\infty$. Consequently $v(t)$ converges when $t \to +\infty$. Now using Proposition 5.3, we have $h(t)$ converges and by (5.5), $v'(t)$ converges necessarily to 0 as $t \to +\infty$.

Let $\lim\limits_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = d$. Then $\lim\limits_{t\to+\infty} h(t) = \dfrac{-(l_1+p)}{q_1+1-p}\, d$ and by relation (5.6), we have $\lim\limits_{t\to+\infty} w(t) = -\left(\dfrac{l_1+p}{q_1+1-p}\right)^{p-1} d^{p-1}$. Since $v(t)$ and $w(t)$ converge, then by equation (5.13) we see that $w'(t)$ converges necessarily to 0. By letting $t \to +\infty$ in the same equation, we get

$$d^{p-1}\left(d^{q_1+1-p} - \left(N-p-\frac{l_1+p}{q_1+1-p}(p-1)\right)\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\right) = 0.$$

Therefore, $d = 0$ or

$$d = \left(\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1)\right)\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\right)^{\frac{1}{q_1+1-p}}.$$

The proof has been completed. □

Based on the previous theorem, we can derive the following result.

Proposition 5.5. *Assuming that* $N > p$ *and* $q_1 > \dfrac{(l_1+N)(p-1)}{N-p}$. *Let* u *be a strictly positive solution of problem* (P). *If* $q_1 \neq \dfrac{N(p-1)+p(l_1+1)}{N-p}$, *then* u' *exhibits one of the following behaviors near infinity:*

(i)

$$\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}+1} u'(r) = 0. \tag{5.28}$$

(ii)

$$\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}+1}u'(r) = \frac{-(l_1+p)}{q_1+1-p}\left(\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1)\right)\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\right)^{\frac{1}{q_1+1-p}}. \tag{5.29}$$

Proof. Since $v(t) = r^{\frac{l_1+p}{q_1+1-p}}u(r)$ converges when t tends to $+\infty$, then by Proposition 5.3, $h(t) = r^{\frac{l_1+p}{q_1+1-p}+1}u'(r)$ converges also when t tends to $+\infty$, this implies that $\lim_{t\to+\infty} v'(t) = 0$. Therefore

$$\lim_{t\to+\infty} h(t) = 0$$

or

$$\lim_{t\to+\infty} h(t) = \frac{-(l_1+p)}{q_1+1-p}\left(\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1)\right)\left(\frac{l_1+p}{q_1+1-p}\right)^{p-1}\right)^{\frac{1}{q_1+1-p}}.$$

This completes the proof. □

Theorem 5.6. *Suppose that $N > p$ and $q_1 = \frac{N(p-1)+p(l_1+1)}{N-p}$. Let u be a strictly positive solution of (P). In that case, the function u conforms to one of the following cases:*

(i) $r^{\frac{l_1+p}{q_1+1-p}}u(r)$ *converges when r tends to $+\infty$.*

(ii) $r^{\frac{l_1+p}{q_1+1-p}}u(r)$ *oscillates near $+\infty$ and satisfies*

$$\begin{aligned} 0 \le \alpha = \liminf_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}}u(r) &< \left(\frac{N-p}{p}\right)^{\frac{p}{q_1+1-p}} \\ < \beta = \limsup_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}}u(r) &\le \left(\left(\frac{q_1+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{q_1+1-p}}. \end{aligned} \tag{5.30}$$

Moreover, if $\alpha = 0$, then $\beta = \left(\left(\frac{q_1+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{q_1+1-p}}$.

Proof. As v remains bounded, there are two potential cases to consider, either $v(t)$ converges when t tends to $+\infty$ and then Theorem 5.1 holds, either $v(t)$ oscillates. Therefore, there exists two sequences $\{\eta_i\}$ and $\{\zeta_i\}$ satisfying $\eta_i < \zeta_i < \eta_{i+1}$ and

$$\alpha = \lim_{i\to+\infty} v(\eta_i) = \liminf_{t\to+\infty} v(t) < \lim_{i\to+\infty} v(\zeta_i) = \limsup_{t\to+\infty} v(t) = \beta. \tag{5.31}$$

Therefore, by equation (5.13) and the fact that $A = \dfrac{l_1 + p}{q_1 + 1 - p} = \dfrac{N-p}{p}$, we have

$$0 \leqslant w'(\eta_i) = -\frac{N-p}{p} w(\eta_i) - v^{q_1}(\eta_i) - e^{-D\eta_i} v^{q_2}(\eta_i) \tag{5.32}$$

and

$$0 \geqslant w'(\zeta_i) = -\frac{N-p}{p} w(\zeta_i) - v^{q_1}(\zeta_i) - e^{-D\zeta_i} v^{q_2}(\zeta_i). \tag{5.33}$$

Letting $i \to +\infty$ in (5.32) and (5.33), we obtain

$$\alpha \leq \left(\frac{N-p}{p}\right)^{\frac{p}{q_1+1-p}} \leq \beta. \tag{5.34}$$

On the other hand, the relation (5.20) gives

$$E(t) = E(T) - \frac{e^{-Dt}}{q_2+1} v^{q_2+1} + \frac{e^{-DT}}{q_2+1} v^{q_2+1} - \frac{D}{q_2+1} \int_T^t e^{-Ds} v^{q_2+1}(s) ds. \tag{5.35}$$

Since v is bounded and $D > 0$ we obtain that $E(t)$ converges when $t \to +\infty$. Hence, by relation (5.17),

$$\lim_{t \to +\infty} E(t) = \psi(\alpha) = \psi(\beta), \tag{5.36}$$

where

$$\psi(s) = \frac{s^{q_1+1}}{q_1+1} - \frac{1}{p} \left(\frac{N-p}{p}\right)^p s^p, \quad s \geq 0. \tag{5.37}$$

A straightforward study of the function ψ reveals that

$$\psi(0) = \psi\left(\left(\left(\frac{q_1+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{q_1+1-p}}\right) = 0,$$

$$\psi'(0) = \psi'\left(\left(\frac{N-p}{p}\right)^{\frac{p}{q_1+1-p}}\right) = 0,$$

$$\psi'(s) < 0 \text{ for } 0 < s < \left(\frac{N-p}{p}\right)^{\frac{p}{q_1+1-p}} \text{ and } \psi'(s) > 0 \text{ for } s > \left(\frac{N-p}{p}\right)^{\frac{p}{q_1+1-p}},$$

$$\psi(s) < 0 \text{ for } \ 0 < s < \left(\left(\frac{q_1+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{q_1+1-p}},$$

$$\psi(s) > 0 \text{ for } \ s > \left(\left(\frac{q_1+1}{p}\right)\left(\frac{N-p}{p}\right)^p\right)^{\frac{1}{q_1+1-p}}.$$

Since $\alpha < \beta$, then the study of the function ψ with the two estimates (5.34) and (5.36) give easily the estimate (5.30) and ensure that if $\alpha = 0$, then

$$\beta = \left(\left(\frac{q_1 + 1}{p} \right) \left(\frac{N-p}{p} \right)^p \right)^{\frac{1}{q_1+1-p}}.$$

□

Theorem 5.7. *Assuming that $N > p$ and $q_1 > \dfrac{(l_1+N)(p-1)}{N-p}$. Let u be a strictly positive solution of* (1.2)*. If $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = 0$, then there is a constant $L > 0$ such that*

$$\lim_{r\to+\infty} r^{\frac{N-p}{p-1}} u(r) = L \tag{5.38}$$

and

$$\lim_{r\to+\infty} r^{\frac{N-1}{p-1}} u'(r) = \frac{-(N-p)}{p-1} L. \tag{5.39}$$

Before proving this Theorem, we need the following result.

Proposition 5.8. *Assume that $N > p$, $0 < c < \dfrac{N-p}{p-1}$ and $q_1 > \frac{(l_1+N)(p-1)}{N-p}$. Let u be a strictly positive solution of problem (P) such that*

$$\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = 0.$$

Then $\lim_{r\to+\infty} r^c u(r) = 0$.

Proof. The proof will be completed in three sequential steps.

Step 1. $E_{\frac{l_1+p}{q_1+1-p}}(r) < 0$ for large r.

At first we prove that $E_{\frac{l_1+p}{q_1+1-p}}(r) \neq 0$ for large r. Suppose by contradiction that there is a large r such that $E_{\frac{l_1+p}{q_1+1-p}}(r) = 0$. Choosing $c = \dfrac{l_1+p}{q_1+1-p}$ in (4.13), we get

$$\begin{aligned}(p-1)|u'|^{p-2} E'_{\frac{l_1+p}{q_1+1-p}}(r) &= r^{1-p} u^{p-1}(r) \\ &\times \Bigg[\left(N-p-\frac{l_1+p}{q_1+1-p}(p-1) \right) \left(\frac{l_1+p}{q_1+1-p} \right)^{p-1} \\ &- r^{l_1+p} u^{q_1+1-p} - r^{l_2+p} u^{q_2+1-p} \Bigg].\end{aligned} \tag{5.40}$$

Since $q_1 > \dfrac{(l_1+N)(p-1)}{N-p}$, $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}}u(r)=0$ and $\lim_{r\to+\infty} r^{\frac{l_2+p}{q_2+1-p}}u(r)=0$ (because $\dfrac{l_2+p}{q_2+1-p}<\dfrac{l_1+p}{q_1+1-p}$), then $E'_{\frac{l_1+p}{q_1+1-p}}(r)>0$ for large r. Therefore, $E_{\frac{l_1+p}{q_1+1-p}}(r)\neq 0$ for large r. Consequently by relation (4.11) and the fact that $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}}u(r)=0$, we have necessarily $E_{\frac{l_1+p}{q_1+1-p}}(r)<0$ for large r.

Step 2. $E'_c(r)>0$ for large r.

Now using relation (4.12) and the fact that $u'(r)<0$ on $(0,+\infty)$ by Proposition 4.2, we obtain

$$(p-1)\frac{E'_c(r)}{u'(r)}=(p-N+c(p-1))+\frac{r^{l_1+1}u^{q_1}}{|u'|^{p-1}}+\frac{r^{l_2+1}u^{q_2}}{|u'|^{p-1}}. \tag{5.41}$$

Since $E_{\frac{l_1+p}{q_1+1-p}}(r)<0$ by the first step and $E_{\frac{N-p}{p-1}}(r)>0$ by Proposition 4.4, then

$$\left(\frac{p-1}{N-p}\right)^{p-1} r^{l_1+p}u^{q_1+1-p}<\frac{r^{l_1+1}u^{q_1}}{|u'|^{p-1}}<\left(\frac{q_1+1-p}{l_1+p}\right)^{p-1} r^{l_1+p}u^{q_1+1-p}. \tag{5.42}$$

This implies that $\lim_{r\to+\infty}\dfrac{r^{l_1+1}u^{q_1}}{|u'|^{p-1}}=0$ and so $\lim_{r\to+\infty}\dfrac{r^{l_2+1}u^{q_2}}{|u'|^{p-1}}=0$, because $l_2<l_1$, $q_2>q_1$ and $\lim_{r\to+\infty}u(r)=0$. Consequently, since $c<\dfrac{N-p}{p-1}$ and $u'<0$ on $(0,+\infty)$, we obtain in (5.41), that $E'_c(r)>0$ for large r.

Step 3. $\lim_{r\to+\infty} r^c u(r)=0$.

- If $0<c\le \dfrac{l_1+p}{q_1+1-p}$, the result is obvious since $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}}u(r)=0$.
- If $\dfrac{l_1+p}{q_1+1-p}<c<\dfrac{N-p}{p-1}$.

  Since $u'<0$ on $(0,+\infty)$, then

$$\left(r^{c+1}u'(r)\right)'=r^cE'_c(r). \tag{5.43}$$

As $E'_c(r)>0$ for large r then the function $r^{c+1}u'$ is increasing for r large enough and so it converges when r tends to $+\infty$ (because $u'<0$). Using the fact that $\lim_{r\to+\infty}u(r)=0$, then by the Hospital's rule we have also $\lim_{r\to+\infty}r^cu(r)$ exists and is finite. Suppose by absurd that $\lim_{r\to+\infty}r^cu(r)=d_1>0$. Using the logarithmic change (5.3) and our hypotheses, we have $\lim_{t\to+\infty}v_c(t)=d_1>0$ and $h_c(t)$ converges when $t\to+\infty$, therefore necessarily by (5.5), $\lim_{t\to+\infty}v'_c(t)=0$.

Therefore, $\lim_{t\to+\infty} h_c(t) = -cd_1 < 0$ and so by relation (5.6), equation (5.4) and the fact that $D_c > B_c > 0$, we get $\lim_{t\to+\infty} w_c'(t) = Ac^{p-1}d_1^{p-1} > 0$. However, this is contrary to the fact that w_c converges when $t \to +\infty$. Therefore, $\lim_{r\to+\infty} r^c u(r) = 0$. □

We now proceed to establish the proof of Theorem 5.7.

Proof. Using (2.2) and Proposition 4.2, we have the function $r^{N-1}|u'|^{p-2}u'$ is decreasing and negative. Therefore $\lim_{r\to+\infty} r^{N-1}|u'|^{p-2}u'(r) \in [-\infty, 0[$. Assume by contradiction that $\lim_{r\to+\infty} r^{N-1}|u'|^{p-2}u'(r) = -\infty$.

Define the following function

$$\varphi(r) = r^{N-1}|u'|^{p-1}. \tag{5.44}$$

Then $\lim_{r\to+\infty} \varphi(r) = +\infty$. Let $0 < m < \dfrac{q_1(N-p)}{p-1} - l_1 - N$ (this is possible because $q_1 > \dfrac{(l_1+N)(p-1)}{N-p}$). We show that $\lim_{r\to+\infty} r^{m+1}\varphi'(r) = 0$.
Using again relation (2.2) we have

$$r^{m+1}\varphi'(r) = r^{l_1+N+m}u^{q_1} + r^{l_2+N+m}u^{q_2}. \tag{5.45}$$

Since $0 < \dfrac{l_1+N+m}{q_1} < \dfrac{N-p}{p-1}$, $q_2 > q_1$ and $l_2 < l_1$, we have by Proposition 5.8

$$\lim_{r\to+\infty} r^{l_1+N+m}u^{q_1} = \lim_{r\to+\infty} r^{l_2+N+m}u^{q_2} = 0.$$

Hence we have by (5.45), $\lim_{r\to+\infty} r^{m+1}\varphi'(r) = 0$. Therefore, since φ' is strictly positive, there is a constant $C_0 > 0$ such that for large r,

$$\varphi'(r) < C_0 r^{-m-1}. \tag{5.46}$$

Integrating inequality (5.46) on (R, r) for large R and as $m > 0$, we get

$$\varphi(r) - \varphi(R) < -\frac{C_0}{m} r^{-m} + \frac{C_0}{m} R^{-m}.$$

By tending $r \to +\infty$, we obtain a contradiction with the fact that

$$\lim_{r\to+\infty} \varphi(r) = +\infty.$$

Hence $\lim_{r\to+\infty} r^{N-1}|u'|^{p-2}u'(r) \in]-\infty, 0[$, which implies that

$$\lim_{r\to+\infty} r^{\frac{N-1}{p-1}} u'(r) \in]-\infty, 0[\,.$$

Therefore, thanks to the Hospital's rule (because $\lim_{r\to+\infty} u(r) = 0$ and $N > p$), we have $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}} u(r)$ is finite and strictly positive. Let $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}} u(r) = L > 0$. Then $\lim_{r\to+\infty} r^{\frac{N-1}{p-1}} u'(r) = \dfrac{-(N-p)}{p-1} L < 0$. □

6. Conclusions

The purpose of this paper lies in the fact of studying a nonlinear elliptic equation involving two singular coefficients that are comparable near the origin and infinity. We have shown existence results of positive solutions and we have given their behavior near infinity. During the study, critical values appear, more exactly we have shown that in the case where $\dfrac{(l_1+N)(p-1)}{N-p} < q_1 \neq \dfrac{N(p-1)+p(l_1+1)}{N-p}$, the positive solution u has one of the following behaviors:

(i) $\lim_{r\to+\infty} r^{\frac{l_1+p}{q_1+1-p}} u(r) = \left(\left(N-p-\dfrac{l_1+p}{q_1+1-p}(p-1)\right)\left(\dfrac{l_1+p}{q_1+1-p}\right)^{p-1}\right)^{\frac{1}{q_1+1-p}}$,

(ii) $\lim_{r\to+\infty} r^{\frac{N-p}{p-1}} u(r) = L > 0$.

In the case where $q_1 = \dfrac{N(p-1)+p(l_1+1)}{N-p}$, we have shown that the function $r^{\frac{l_1+p}{q_1+1-p}} u(r)$, either converges at infinity to the explicit limit mentioned above, or oscillates with details on its $\liminf$ and $\limsup$ near infinity.

In conclusion, the choice between the possibilities of the behavior near the infinity of positive solutions mentioned above remains an open question and will be the subject of a future paper.

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