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A MODIFIED KRASNOSELSKII-TYPE SUBGRADIENT EXTRAGRADIENT ALGORITHM WITH INERTIAL EFFECTS FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEM

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Abstract. In this paper, we propose a new inertial subgradient extragradient algorithm with a new linesearch technique that combines the inertial subgradient extragradient algorithm and the KrasnoselskiiMann algorithm. Under some suitable conditions, we prove a weak convergence theorem of the proposed algorithm for finding a common element of the common solution set of a finitely many variational inequality problem and the fixed point set of a nonexpansive mapping in real Hilbert spaces. Moreover, using our main result, we derive some others involving systems of variational inequalities. Finally, we give some numerical examples to support our main result.

1. Introduction

Throughout this paper, let H be a real Hilbert space and C be a nonempty closed convex subset of H with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let

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$T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

In this paper, we consider the classical variational inequality problem, which consists in finding a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $VI(C, A)$. The problem (1.1) was introduced and studied by Stampacchia [28] in 1966. Additionally, various topics in economics, engineering mechanics, mathematical programming, transportation, and other fields can be solved using the problem (1.1). The solution of the problem (1.1) is well-known to be equivalent to the solution of the following fixed point equation for finding a point $x \in C$ such that

$$x = P_C(I - \lambda A)x, \tag{1.2}$$

where $\lambda > 0$ is an arbitrary constant and P_C is the metric projection from H onto C (see [18] for details). In 2012, Kangtunyakarn [14] modified the problem (1.1) and called it the combination of variational inequality problems, which consists in finding a point $x \in C$ such that

$$\langle y - z^*, (aA + (1 - a)B)z^*\rangle \geq 0, \quad \forall y \in C, \quad \forall a \in (0, 1), \tag{1.3}$$

where $A, B : C \to H$ are the mappings. The solution set of the problem (1.3) is denoted by $VI(C, aA + (1 - a)B)$. If $A \equiv B$, then the problem (1.3) reduces to the problem (1.1). Moreover, Kangtunyakarn [14] also introduced the mathematical tool related to the problem (1.3) and the problem (1.1) (See Lemma 2.11 in [14]).

The extragradient algorithm, first described by Korpelevich [20] in 1976, is one of the most widely used techniques for solving the problem (1.1). The extragradient algorithm, which consists of two steps, can be expressed as follows

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

where $\lambda \in (0, \frac{1}{L})$, $A : C \to \mathbb{R}^n$ is monotone and Lipschitz continuous with Lipschitz constant L. If a solution set of the problem (1.1) is nonempty, then the sequence $\{x_n\}$ generated by process (1.4) converges weakly to an element in a solution set of the problem (1.1). The extragradient algorithm has been considered by many authors in recent years (see [37, 39] and the references therein). However, the extragradient algorithm requires that two projections are calculated in each iteration. In order to further develop the extragradient algorithm for solving the problem (1.1) in Hilbert space, Censor et al. [7] modified the extragradient algorithm by substituting a projection onto a half-space for the second projection. This novel method is known as the subgradient extragradient algorithm, and it is described as follows

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where A is monotone, L-Lipschitz continuous, and $\lambda \in (0, 1/L)$. They also proved that the sequence $\{x_n\}$ generated by process (1.5) converges weakly to an element in a solution set of the problem (1.1).

The subgradient extragradient algorithm has been studied by numerous researchers in a number of different ways, for example [12, 17, 19, 22, 36] and the references therein. In 2021, Kheawborisut and Kangtunyakarn [17] defined the new half-space $T_n = \{x \in H : \langle x_n - \lambda \sum_{i=1}^N a_i A_i x_n - y_n, x - y_n \rangle \leq 0\}$ as a tool for proving the strong convergence theorem. They also introduced the modified subgradient extragradient algorithm as follows

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda \sum_{i=1}^N a_i A_i x_n), \\ T_n = \{x \in H : \langle x_n - \lambda \sum_{i=1}^N a_i A_i x_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = \alpha_n u + \beta_n P_{T_n}(x_n - \lambda \sum_{i=1}^N a_i A_i y_n) + \gamma_n G x_n, \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where for every $i = 1, 2, 3, ..., N$, A_i is α_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,3,...,N} \alpha_i$, G is a nonlinear operator, $\lambda \in (0, \eta)$ and $\sum_{i=1}^N a_i = 1$, $0 < a_i < 1$ for every $i = 1, 2, ..., N$. Furthermore, they demonstrated the strong convergence of the sequence generated from the suggested iterative methods for finding a common element of the set of common solutions of finitely many variational inequality problems and the set of solutions to nonlinear problems in Hilbert spaces.

There has recently been a rise in interest in the research of inertial type algorithms. The inertial extragradient algorithm [29], the inertial forward-backward splitting algorithm [8] and the inertial Douglas-Rachford splitting algorithm [9] are a few examples. In recent years there has been increasing interests in studying inertial subgradient extragradient algorithm, see [1, 2, 6, 27, 30, 32, 33, 34, 35].

Subsequently, the inertial subgradient extragradient algorithm was modified, as well as the stepsize, in order to avoid the Lipschitz constant, using techniques such as linesearch and self-adaptive techniques, see, for example, [5, 21, 23, 26, 31, 33]. For solving the variational inequality problem and fixed point problem of a quasi-nonexpansive mapping in real Hilbert spaces, Thong and Hieu [31] introduced the inertial subgradient extragradient algorithm with linesearch technique as follows: Given $\gamma > 0$, $l \in (0, 1), \mu \in (0, 1)$. Let $x_0, x_1 \in H$ be arbitrary and the sequence $\{x_n\}$ is generated by

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ T_n := \{z \in H : \langle w_n - \tau_n A w_n - y_n, y_n - z\rangle \geq 0\}, \\ z_n = P_{T_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \alpha_n w_n + \beta_n T z_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where τ_n is chosen to be largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\| \tag{1.8}$$

and $T : H \to H$ is a quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero, and $A : H \to H$ is monotone and Lipschitz continuous on H with the constant L. Under mild assumptions, the sequences generated by the proposed algorithm converge weakly an element of $F(T) \cap VI(C, A)$.

In 2017, Kanzow and Shehu [16] proposed the inexact KrasnoselskiiMann algorithm for finding a fixed point of a nonexpansive mapping T in a real Hilbert space as follows: For arbitrarily given $x_1 \in H$, let the sequences $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n, \qquad \forall n \geq 1, \tag{1.9}$$

where $T : H \to C$ is a nonexpansive mapping, r_n denotes the residual vector and $\{\alpha_n\}$, $\{\beta_n\}$ are two real number sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$. They proved that if $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, $\sum_{n=1}^{\infty} \|r_n\| < \infty$, and $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$, then the sequence $\{x_n\}$ generated by (1.9) converges weakly an element of $F(T)$. The Krasnoselskii-Mann algorithm has been the focus of increasing research in many different directions during the past few years, for example [10, 13, 25, 38] and the references therein.

Question. Can we design a linesearch technique for the algorithm (1.6) above?

In this paper, we give a positive answer to this question. Motivated and inspired by the works in literature, we introduce a modified inertial subgradient extragradient algorithm by using the concept of the solution sets of the modified variational inequality problem introduced by Kangtunyakarn [14] with the new linesearch technique, which combine the inertial subgradient extragradient algorithm [31] and the Krasnoselskii- Mann algorithm [16], for finding a common element of the common solution set of finitely many variational inequalities problems and the fixed point set of a nonexpansive mapping. In addition, our algorithm does not require the knowledge of the constant of inverse strongly monotone operators. Under some suitable conditions, we provide a weak convergence theorem of the proposed algorithm in real Hilbert spaces and, by use of Lemma 2.10 in [24], also get a strong convergence theorem. Furthermore, we obtain an additional result involving a system of variational inequalities by using our main result. Finally, we give some numerical examples to support our main result.

Following is an outline for this paper. Some lemmas that will be utilized as further proof are listed in Sect. 2. In Sect. 3, we proposed the new algorithms, then the weak convergence theorem is analyzed. In Sect. 4, we apply our main result to system of variational inequalities. Several numerical experiments are provided in Sect. 5.

2. Preliminaries

In this section, let C be a nonempty closed convex subset of a real Hilbert space H. We use the notations "$\rightharpoonup$" and "$\to$" to represent weak convergence and strong convergence, respectively. For every $x \in H$, there exists a unique nearest point $P_C x \in C$ such that

$$||x - P_C x|| \leq ||x - y||, \quad \forall y \in C$$

and P_C is called a metric projection of H onto C.

Next, we provide some useful lemmas that will be used to support our main result.

Lemma 2.1. ([4]) *Given $x \in H$ and $y \in C$. Then, $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2. *Let C be a closed convex subset in a real Hilbert space H, $x \in H$. Then*

(i) $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall y \in C,$

(ii) $||P_C x - y||^2 \leq ||x - y||^2 - ||x - P_C x||^2, \quad \forall y \in C.$

Definition 2.3. Let $A : C \to H$ and $T : C \to C$ be mappings. Then

(i) a mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C,$$

(ii) a mapping T is called *quasi-nonexpansive* if

$$\|Tx - y\| \leq \|x - y\|, \quad \text{for all } x, y \in C \text{ and } y \in F(T),$$

(iii) a mapping T is called *Lipschitz continuous* on C if there exists $L > 0$ such that
$$\|Tx - Ty\| \leq L\|x - y\|, \quad \text{for all } x, y \in C,$$
(iv) a mapping A is called *α-inverse strongly monotone* if there exists $\alpha > 0$ such that
$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C,$$
(v) a mapping A is called *firmly nonexpansive* if
$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \text{for all } x, y \in C.$$

It is also known that every α-inverse strongly monotone mapping is $\frac{1}{\alpha}$-Lipschitz continuous.

Lemma 2.4. ([11]) *Assume that $T : H \to H$ is a nonlinear operator with $F(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:*
$$x_n \rightharpoonup x \quad \textit{and} \quad \{(I - T)x_n\} \to 0 \quad \Rightarrow \quad x \in F(T).$$

Lemma 2.5. ([16]) *Let X be a real inner product space, the following results hold:*

(i) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$, *for all $x, y \in X$ and $\alpha \in [0, 1]$,*

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, *for all $x, y \in X$,*

(iii) $\|tx + ty\|^2 = t(t + s)\|x\|^2 + s(t + s)\|y\|^2 - st\|x - y\|^2$, *for all $x, y \in X$ and $s, t \in \mathbb{R}$.*

Lemma 2.6. ([3]) *Let $\{\sigma_n\}$ and $\{\gamma_n\}$ be nonnegative sequences satisfying $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\gamma_{n+1} \leq \gamma_n + \sigma_n$, $n = 1, 2, \ldots$. Then, $\{\gamma_n\}$ is a convergent sequence.*

Lemma 2.7. ([15]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $A, B : C \to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha, \beta > 0$ and*
$$VI(C, A) \cap VI(C, B) \neq 0.$$
Then
$$VI(C, aA + (1 - a)B) = VI(C, A) \cap VI(C, B), \ \ \forall a \in (0, 1).$$
Furthermore, if $0 < \gamma < \min\{2\alpha, 2\beta\}$, then we find that $I - \gamma(aA + (1 - a)B)$ is a nonexpansive mapping.

Remark 2.8. ([17]) *If for every $i = 1, 2, ..., N$ $A_i : C \to H$ are ξ_i-inverse strongly monotone mappings with $\eta = \min_{1,2,...,N}\{\xi_i\}$ and $\bigcap_{i=1}^{N} VI(C, A_i) \neq \emptyset$, then*

$$VI(C, \sum_{i=1}^{N} a_i A_i) = \bigcap_{i=1}^{N} VI(C, A_i), \tag{2.1}$$

where $\sum_{i=1}^{N} a_i = 1$ and $0 < a_i < 1$ for every $i = 1, 2, ..., N$. Moreover, we have $\sum_{i=1}^{N} a_i A_i$ is monotone and μ-Lipschitz continuous mapping.

The following lemma was provided by Kanyanee and Kangtunyakarn [24] to prove the strong convergence theorem.

Lemma 2.9. ([24]) *Let H be a real Hilbert space and let S be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in S$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\| + b_n$$

for every $n = 1, 2, ...$ and $\sum_{n=1}^{N} b_n < \infty$. Thus $\{P_S x_n\}$ converges strongly to some $z \in S$.

3. Main result

In this section, we propose the modified Krasnoselskii-type subgradient extragradient algorithm with inertial effects for finding a common element of the set of finite family variational inequalities problem and the fixed point set of a nonexpansive mapping. Under some suitable conditions, we provide a weak convergence theorem of the proposed algorithm in real Hilbert spaces.

Let H be a real Hilbert space. For $i = 1, 2, ..., N$, let $A_i : H \to H$ be ξ_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,...,N}\{\xi_i\}$ and let $T : H \to H$ be a nonexpansive mapping with

$$\Gamma = \bigcap_{i=1}^{N} VI(C, A_i) \cap F(T) \neq \emptyset.$$

Let r_n be the residual vector and let $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0, 1]$ such that $\alpha_n + \beta_n \leq 1$ satisfying the the following conditions:

(i) there are $c, d > 0$ with $0 < c \leq \beta_n \leq d < 1$ for all $n \geq 1$;
(ii) $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$;
(iii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$;
(iv) $\sum_{n=1}^{\infty} \|r_n\| < \infty$.

Now, we propose the modified Krasnoselskii-type subgradient extragradient algorithm with inertial effects. The purposed algorithm is written as follows:

Algorithm 1:

Initialization: Let $x_0, x_1 \in H$ be arbitrary. Given $\gamma > 0$, $l \in (0,1), \mu \in (0,1)$, $\sum_{i=1}^{N} a_i = 1$, $0 < a_i < 1$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1: Set $w_n = x_n + \theta_n(x_n - x_{n-1})$ and compute

$$y_n = P_C\Big(w_n - \tau_n \sum_{i=1}^{N} a_i A_i w_n\Big),$$

where τ_n is chosen to be largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau \sum_{i=1}^{N} a_i \|A_i w_n - A_i y_n\| \leq \mu \|w_n - y_n\|. \tag{3.1}$$

Step 2: Compute

$$z_n = P_{Q_n}\Big(w_n - \tau_n \sum_{i=1}^{N} a_i A_i y_n\Big),$$

where $Q_n := \{z \in H : \langle w_n - \tau_n \sum_{i=1}^{N} a_i A_i w_n - y_n, y_n - z\rangle \geq 0\}$.

Step 3: Compute

$$x_{n+1} = \alpha_n w_n + \beta_n T z_n + r_n. \tag{3.2}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.1. *The Armijo-like search rule* (3.1) *is well defined and*

$$\min\{\gamma, \mu l \eta\} < \tau_n \leq \gamma.$$

Proof. Since A_i is ξ_i-inverse strongly monotone with $\eta = \min_{i=1,2,...,N}\{\xi_i\}$, for all $i = 1, 2, ..., N$ on H, we have

$$\begin{aligned}
&\sum_{i=1}^{N} a_i \|A_i w_n - A_i(P_C(w_n - \gamma l^m \sum_{i=1}^{N} a_i A_i w_n))\| \\
&\leq \sum_{i=1}^{N} a_i \left(\frac{1}{\xi_i}\|w_n - (P_C(w_n - \gamma l^m \sum_{i=1}^{N} a_i A_i w_n))\|\right) \\
&\leq \frac{1}{\eta}\|w_n - P_C(w_n - \gamma l^m \sum_{i=1}^{N} a_i A_i w_n)\|,
\end{aligned}$$

this is equivalent to

$$\begin{aligned}
&\eta\mu\sum_{i=1}^{N}a_i\|A_iw_n - A_i(P_C(w_n-\gamma l^m\sum_{i=1}^{N}a_iA_iw_n))\| \\
&\leq \mu\|w_n - P_C(w_n-\gamma l^m\sum_{i=1}^{N}a_iA_iw_n)\|.
\end{aligned}$$

This implies that (3.1) holds for all $\gamma l^m \leq \eta\mu$, so τ_n is well defined.

Obviously, $\tau_n \leq \gamma$. If $\tau_n = \gamma$, then this lemma is proved; otherwise, if $\tau_n < \gamma$. In this case, we find that $\frac{\tau_n}{l}$ dose not satisfy the search rule (3.1), that is,

$$\begin{aligned}
&\sum_{i=1}^{N}a_i\frac{\tau_n}{l}\|A_iw_n - A_i(P_C(w_n-\frac{\tau_n}{l}\sum_{i=1}^{N}a_iA_iw_n))\| \\
&> \mu\|w_n - P_C(w_n-\frac{\tau_n}{l}\sum_{i=1}^{N}a_iA_iw_n)\|;
\end{aligned}$$

combining this with A_i is ξ_i-inverse strongly monotone with $\eta = \min\{\xi_i\}$, for all $i = 1, 2, ..., N$, we obtain

$$\tau_n > \mu l\eta.$$

This completes the proof. □

Lemma 3.2. *Let H be a real Hilbert space, for every $i = 1, 2, ..., N$, let $A_i : H \to H$ be ξ_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,...,N}\{\xi_i\}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by Algorithm 1. Then*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - (1-\mu)\|y_n - w_n\|^2 - (1-\mu)\|z_n - y_n\|^2$$

for all $p \in \bigcap_{i=1}^{N} VI(C, A_i)$, where $\sum_{i=1}^{N} a_i = 1$ and $0 < a_i < 1$.

Proof. Since $p \in \bigcap_{i=1}^{N} VI(C, A_i) \subset C \subset Q_n$, we have $p \in VI(C, A_i)$ for every $i = 1, 2, ..., N$ and Lemma 2.2 (i), we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_{Q_n}(w_n - \tau_n\sum_{i=1}^{N}a_iA_iy_n) - P_{Q_n}p\|^2 \\
&\leq \langle P_{Q_n}(w_n - \tau_n\sum_{i=1}^{N}a_iA_iy_n) - P_{Q_n}p, w_n - \tau_n\sum_{i=1}^{N}a_iA_iy_n - p\rangle \\
&= \langle z_n - p, w_n - \tau_n\sum_{i=1}^{N}a_iA_iy_n - p\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - \tau_n \sum_{i=1}^{N} a_i A_i y_n - p\|^2 - \frac{1}{2}\|z_n - w_n + \tau_n \sum_{i=1}^{N} a_i A_i y_n\|^2 \\
&= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}[\|w_n - p\|^2 + \tau_n^2 \|\sum_{i=1}^{N} a_i A_i y_n\|^2 - 2\langle w_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle] \\
&\quad - \frac{1}{2}[\|z_n - w_n\|^2 + \tau_n^2 \|\sum_{i=1}^{N} a_i A_i y_n\|^2 - 2\langle z_n - w_n, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle] \\
&= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle.
\end{aligned}
$$

It implies that

$$
\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle. \tag{3.3}
$$

From monotonicity of $\sum_{i=1}^{N} a_i A_i$, we have

$$
2\tau_n \langle \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i p, y_n - p\rangle \geq 0. \tag{3.4}
$$

From (3.3) and (3.4), we get

$$
\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle \\
&\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - y_n + y_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle \\
&\quad + 2\tau_n \langle \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i p, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - y_n, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle \\
&\quad - 2\langle y_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle + 2\tau_n \langle \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i p, y_n - p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\langle y_n - z_n, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle \\
&\quad - 2\langle y_n - p, \tau_n \sum_{i=1}^{N} a_i A_i y_n\rangle + 2\tau_n\langle \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i p, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|z_n - w_n\|^2 \\
&\quad + 2\langle y_n - z_n, \tau_n \sum_{i=1}^{N} a_i A_i y_n - \tau_n \sum_{i=1}^{N} a_i A_i w_n + \tau_n \sum_{i=1}^{N} a_i A_i w_n\rangle \\
&\quad - 2\langle y_n - p, \tau_n \sum_{i=1}^{N} a_i A_i p\rangle \\
&= \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i w_n\rangle \\
&\quad + 2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_i A_i w_n\rangle - 2\tau_n\langle y_n - p, \sum_{i=1}^{N} a_i A_i p\rangle. \qquad (3.5)
\end{aligned}
$$

We estimate $2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i w_n\rangle$. It follows that

$$
\begin{aligned}
&2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i w_n\rangle \\
&\le 2\tau_n \|\sum_{i=1}^{N} a_i A_i y_n - \sum_{i=1}^{N} a_i A_i w_n\| \|y_n - z_n\| \\
&\le 2\tau_n \sum_{i=1}^{N} a_i \|A_i y_n - A_i w_n\| \|y_n - z_n\| \\
&\le 2\mu \|y_n - w_n\| \|y_n - z_n\| \\
&\le \mu \|y_n - w_n\|^2 + \mu \|y_n - z_n\|^2. \qquad (3.6)
\end{aligned}
$$

As $y_n = P_C(w_n - \tau_n \sum_{i=1}^{N} a_i A_i w_n)$ and $z_n \in Q_n$, we have

$$
\begin{aligned}
0 &\ge \langle w_n - \tau_n \sum_{i=1}^{N} a_i A_i w_n - y_n, z_n - y_n\rangle \\
&= \langle w_n - y_n, z_n - y_n\rangle - \tau_n\langle \sum_{i=1}^{N} a_i A_i w_n, z_n - y_n\rangle.
\end{aligned}
$$

This implies that

$$2\tau_n\langle\sum_{i=1}^{N} a_iA_iw_n, y_n - z_n\rangle \le 2\langle w_n - y_n, y_n - z_n\rangle$$
$$= \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2. \quad (3.7)$$

Using (3.6) and (3.7), we deduce in (3.5) that

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_iA_iy_n - \sum_{i=1}^{N} a_iA_iw_n\rangle \\ &\quad + 2\tau_n\langle y_n - z_n, \sum_{i=1}^{N} a_iA_iw_n\rangle - 2\tau_n\langle y_n - p, \sum_{i=1}^{N} a_iA_ip\rangle \\ &\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + \mu\|y_n - w_n\|^2 + \mu\|y_n - z_n\|^2 \\ &\quad + \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2 - 2\tau_n\langle y_n - p, \sum_{i=1}^{N} a_iA_ip\rangle \\ &= \|w_n - p\|^2 - (1-\mu)\|y_n - w_n\|^2 - (1-\mu)\|z_n - y_n\|^2 \\ &\quad - 2\tau_n\langle y_n - p, \sum_{i=1}^{N} a_iA_ip\rangle. \end{aligned} \quad (3.8)$$

Sine $p \in \bigcap_{i=1}^{N} VI(C, A_i)$ and Remark 2.8, we obtain $\langle y_n - p, \sum_{i=1}^{N} a_iA_ip\rangle \ge 0$. From (3.8) and $\langle y_n - p, \sum_{i=1}^{N} a_iA_ip\rangle \ge 0$, we have

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - (1-\mu)\|y_n - w_n\|^2 - (1-\mu)\|z_n - y_n\|^2.$$

□

Theorem 3.3. *Let H be a real Hilbert space. For $i = 1, 2, ..., N$, let $A_i : H \to H$ be ξ_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,\dots,N}\{\xi_i\}$ and let $T : H \to H$ be a nonexpansive mapping with*

$$\Gamma = \bigcap_{i=1}^{N} VI(C, A_i) \cap F(T) \ne \emptyset.$$

Let $\{x_n\}$ be a sequence generated by Algorithm 1, where r_n is the residual vector, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ such that $\alpha_n + \beta_n \le 1$, $\sum_{i=1}^{N} a_i = 1, 0 < a_i < 1$. Assume that conditions $(i) - (iv)$ hold. Then $\{x_n\}$ converges weakly to $z \in \Gamma$ and furthermore,

$$z = \lim_{n\to\infty} P_\Gamma x_n.$$

Proof. First, we must show that $\{x_n\}$ is bounded.

Let $p \in \bigcap_{i=1}^N VI(C, A_i) \cap F(T)$. From the definition of w_n, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.9}$$

From Lemma 3.2, we have

$$\|z_n - p\| \le \|w_n - p\|. \tag{3.10}$$

From the definition of x_n, Lemma 2.5 (iii), (3.9) and (3.10), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n w_n + \beta_n T z_n + r_n - p\| \\ &= \|\alpha_n(w_n - p) + \beta_n(Tz_n - p) + r_n - (1 - \alpha_n - \beta_n)p\| \\ &\le \alpha_n\|w_n - p\| + \beta_n\|Tz_n - p\| + \|r_n - (1 - \alpha_n - \beta_n)p\| \\ &= \alpha_n\|w_n - p\| + \beta_n\|Tz_n - p\| \\ &\quad + \|(1 - \alpha_n - \beta_n)r_n - (1 - \alpha_n - \beta_n)p + (\alpha_n + \beta_n)r_n\| \\ &\le \alpha_n\|w_n - p\| + \beta_n\|z_n - p\| + (1 - \alpha_n - \beta_n)\|r_n - p\| \\ &\quad + (\alpha_n + \beta_n)\|r_n\| \\ &\le \alpha_n\|w_n - p\| + \beta_n\|w_n - p\| + (1 - \alpha_n - \beta_n)\|r_n - p\| \\ &\quad + (\alpha_n + \beta_n)\|r_n\| \\ &= (\alpha_n + \beta_n)\|w_n - p\| + (1 - \alpha_n - \beta_n)\|r_n - p\| + (\alpha_n + \beta_n)\|r_n\| \\ &\le \|w_n - p\| + (1 - \alpha_n - \beta_n)\|r_n - p\| + \|r_n\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| + (1 - \alpha_n - \beta_n)\|r_n - p\| + \|r_n\| \\ &\le \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| + (1 - \alpha_n - \beta_n)M + \|r_n\|, \end{aligned} \tag{3.11}$$

for some $M > 0$. Applying Lemma 2.6 and using conditions (i), (ii) and (iii), we have $\lim_{n\to\infty} \|x_n - p\|$ exists. In particular, this implies that $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

From the definition of w_n and Lemma 2.5 (ii), we obtain

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, w_n - p\rangle \\ &\le \|x_n - p\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|w_n - p\|. \end{aligned} \tag{3.12}$$

From the definition of x_n, Lemma 2.5 (ii), Lemma 3.2 and (3.12), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n w_n + \beta_n T z_n + r_n - p\|^2 \\
&= \|\alpha_n(w_n-p) + \beta_n(Tz_n-p) + r_n - (1-\alpha_n-\beta_n)p\|^2 \\
&\le \|\alpha_n(w_n-p) + \beta_n(Tz_n-p)\|^2 \\
&\quad + 2\langle r_n - (1-\alpha_n-\beta_n)p, x_{n+1}-p\rangle \\
&= \alpha_n(\alpha_n+\beta_n)\|w_n-p\|^2 + \beta_n(\alpha_n+\beta_n)\|Tz_n-p\|^2 \\
&\quad - \alpha_n\beta_n\|w_n-p-(Tz_n-p)\|^2 \\
&\quad + 2\langle r_n - (1-\alpha_n-\beta_n)p, x_{n+1}-p\rangle \\
&\le \alpha_n(\alpha_n+\beta_n)\|w_n-p\|^2 + \beta_n(\alpha_n+\beta_n)\|z_n-p\|^2 \\
&\quad - \alpha_n\beta_n\|w_n-Tz_n\|^2 + 2\langle r_n - (1-\alpha_n-\beta_n)p, x_{n+1}-p\rangle \\
&\le \alpha_n(\alpha_n+\beta_n)\|w_n-p\|^2 + \beta_n(\alpha_n+\beta_n) \\
&\quad \times [\|w_n-p\|^2 - (1-\mu)\|y_n-w_n\|^2 - (1-\mu)\|z_n-y_n\|^2] \\
&\quad - \alpha_n\beta_n\|w_n-Tz_n\|^2 + 2\langle r_n - (1-\alpha_n-\beta_n)p, x_{n+1}-p\rangle \\
&= (\alpha_n+\beta_n)^2\|w_n-p\|^2 - \beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2 \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2 - \alpha_n\beta_n\|w_n-Tz_n\|^2 \\
&\quad + 2\langle r_n - (1-\alpha_n-\beta_n)p, x_{n+1}-p\rangle \\
&\le \|w_n-p\|^2 - \beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2 \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2 - \alpha_n\beta_n\|w_n-Tz_n\|^2 \\
&\quad + 2\langle (1-\alpha_n-\beta_n)r_n - (1-\alpha_n-\beta_n)p + (\alpha_n+\beta_n)r_n, x_{n+1}-p\rangle \\
&= \|w_n-p\|^2 - \beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2 \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2 - \alpha_n\beta_n\|w_n-Tz_n\|^2 \\
&\quad + 2(1-\alpha_n-\beta_n)\langle r_n-p, x_{n+1}-p\rangle + 2(\alpha_n+\beta_n)\langle r_n, x_{n+1}-p\rangle \\
&\le \|w_n-p\|^2 - \beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2 \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2 - \alpha_n\beta_n\|w_n-Tz_n\|^2 \\
&\quad + 2(1-\alpha_n-\beta_n)\|r_n-p\|\|x_{n+1}-p\| \\
&\quad + 2(\alpha_n+\beta_n)\|r_n\|\|x_{n+1}-p\| \\
&\le \|x_n-p\|^2 + 2\theta_n\|x_n-x_{n-1}\|\|w_n-p\| \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2 \\
&\quad - \beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2 - \alpha_n\beta_n\|w_n-Tz_n\|^2 \\
&\quad + [2(1-\alpha_n-\beta_n)\|r_n-p\| + 2(\alpha_n+\beta_n)\|r_n\|]\|x_{n+1}-p\|.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\beta_n(\alpha_n+\beta_n)(1-\mu)\|y_n-w_n\|^2\\
&\quad+\beta_n(\alpha_n+\beta_n)(1-\mu)\|z_n-y_n\|^2+\alpha_n\beta_n\|w_n-Tz_n\|^2\\
&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\theta_n\|x_n-x_{n-1}\|\|w_n-p\|\\
&\quad+[2(1-\alpha_n-\beta_n)\|r_n-p\|+2(\alpha_n+\beta_n)\|r_n\|]\|x_{n+1}-p\|. \qquad (3.13)
\end{aligned}$$

From (3.13), $\lim_{n\to\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)=0$, condition (ii), (iii) and (iv), we have

$$\lim_{n\to\infty}\|y_n-w_n\|^2=\lim_{n\to\infty}\|z_n-y_n\|^2=\lim_{n\to\infty}\|w_n-Tz_n\|^2=0. \qquad (3.14)$$

Since

$$\begin{aligned}
\|x_n-y_n\|&\le\|x_n-w_n\|+\|w_n-y_n\|\\
&=\|x_n-(x_n+\theta_n(x_n-x_{n-1}))\|+\|w_n-y_n\|\\
&=\theta_n\|x_n-x_{n-1}\|+\|w_n-y_n\|,
\end{aligned}$$

condition (ii) and (3.14), we obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \qquad (3.15)$$

Since

$$\|x_n-w_n\| \quad\le\quad \|x_n-y_n\|+\|y_n-w_n\|,$$

(3.14) and (3.15), we have

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \qquad (3.16)$$

Since

$$\|x_n-z_n\| \quad\le\quad \|x_n-y_n\|+\|y_n-z_n\|,$$

(3.14) and (3.15), we get

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \qquad (3.17)$$

From (3.14), (3.17) and

$$\begin{aligned}
\|Tz_n-z_n\|&\le\|Tz_n-w_n\|+\|w_n-z_n\|\\
&\le\|Tz_n-w_n\|+\|w_n-y_n\|+\|y_n-z_n\|,
\end{aligned}$$

we have

$$\lim_{n\to\infty}\|Tz_n-z_n\|=0. \qquad (3.18)$$

Now, we have to show that $\{x_n\}$ converge weakly to $z\in\Gamma$ and $z=\lim_{n\to\infty}P_\Gamma x_n$, that is, it has a least one weak accumulation point. If ϵ is a weak limit point of some subsequence $\{x_{n_j}\}$ of $\{x_n\}$, then $x_{n_j}\rightharpoonup\epsilon$ as $j\to\infty$.

From (3.17) and $x_{n_j} \rightharpoonup \epsilon$ as $j \to \infty$, we have $z_{n_j} \rightharpoonup \epsilon$ as $j \to \infty$. From (3.17) and Lemma 2.4, we obtain

$$\epsilon \in F(T). \tag{3.19}$$

From (3.16) and $x_{n_j} \rightharpoonup \epsilon$ as $j \to \infty$, we have $w_{n_j} \rightharpoonup \epsilon$ as $j \to \infty$.

On the other hand, since $\|x_n - y_n\| \to 0$ as $n \to \infty$ implies that $y_{n_j} \rightharpoonup p$ and since $y_{n_j} \in C$, we have $\epsilon \in C$. For all $x \in C$ and using the property of the projection P_C, we have ($\sum_{i=1}^N a_i A_i$ is monotone by Remark 2.5 in [17])

$$\begin{aligned}
0 &\leq \langle y_{n_j} - x_{n_j} + \tau_{n_j} \sum_{i=1}^N a_i A_i x_{n_j}, x - y_{n_j} \rangle \\
&= \langle y_{n_j} - x_{n_j}, x - y_{n_j} \rangle + \tau_{n_j} \langle \sum_{i=1}^N a_i A_i x_{n_j}, x_{n_j} - y_{n_j} \rangle + \tau_{n_j} \langle \sum_{i=1}^N a_i A_i x_{n_j}, x - x_{n_j} \rangle \\
&\leq \langle y_{n_j} - x_{n_j}, x - y_{n_j} \rangle + \tau_{n_j} \langle \sum_{i=1}^N a_i A_i x_{n_j}, x_{n_j} - y_{n_j} \rangle + \tau_{n_j} \langle \sum_{i=1}^N a_i A_i x_{n_j}, x - x_{n_j} \rangle
\end{aligned}$$

Let $j \to \infty$, using (3.15) and the facts $\tau_n > \min\{\gamma, \mu l \eta\}$ for all $n \in \mathbb{N}$, we get

$$\langle \sum_{i=1}^N a_i A_i \epsilon, x - \epsilon \rangle \geq 0, \quad \forall x \in C.$$

This implies that $\epsilon \in VI(C, \sum_{i=1}^N a_i A_i)$. Applying Remark 2.8, we have

$$\epsilon \in \bigcap_{i=1}^N VI(C, A_i). \tag{3.20}$$

From (3.19) and (3.20), we have $\epsilon \in \bigcap_{i=1}^N VI(C, A_i) \cap F(T) = \Gamma$.

In order to show that the entire sequence $\{x_n\}$ weakly converges to ϵ, assume $x_{n_j} \rightharpoonup \epsilon'$ as $j \to \infty$, with $\epsilon' \neq \epsilon$ and $\epsilon' \in \bigcap_{i=1}^N VI(C, A_i) \cap F(T)$.

By Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \epsilon\| &= \liminf_{j\to\infty} \|x_{n_j} - \epsilon\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \epsilon'\| \\
&= \lim_{n\to\infty} \|x_n - \epsilon'\| \\
&= \liminf_{j\to\infty} \|x_{n_j} - \epsilon'\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - \epsilon\| \\
&= \lim_{n\to\infty} \|x_n - \epsilon\|.
\end{aligned}$$

This is a contradiction, thus $\epsilon = \epsilon'$. This implies that the sequence $\{x_n\}$ converge weakly to the same point $\epsilon \in \Gamma$.

Finally, if we take

$$u_n = P_\Gamma x_n, \tag{3.21}$$

then by (3.11) and Lemma 2.9, we have $\{P_\Gamma x_n\}$ converges strongly to some $z \in \Gamma$. From (3.21) and Lemma 2.1, we get

$$\langle x_n - u_n, u_n - \epsilon\rangle \geq 0, \quad \forall \epsilon \in \Gamma.$$

Take $n \to \infty$, we have

$$\langle \epsilon - z, z - \epsilon\rangle \geq 0,$$

and hence $z = \epsilon$. Therefore $\{P_\Gamma x_n\}$ converges strongly to $\epsilon \in \Gamma$, this completes the proof. □

Remark 3.4. It is worth mentioning that there are some advantages of our main result as follows:

(1) Theorem 3.3 is more convenient than the results of Kheawborisut and Kangtunyakarn [17] in practice. In fact, we do not require to know the constant of inverse strongly monotone operators which is not easy in computation.

(2) The traditional "Two Cases Method" which is widely used in various studies in order to guarantee convergence analysis (See, for example, [17, 22]), is not used in proof of Theorem 3.3.

Taking $A \equiv A_i$ for all $i = 1, 2, 3, ..., N$ and $r_n = 0$ in Theorem 3.3, then we have the following corollary:

Corollary 3.5. *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone mapping and let $T : H \to H$ be a nonexpansive mapping with*

$$\bar{\Gamma} = VI(C, A) \cap F(T) \neq \emptyset.$$

Let $\{w_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ be sequences generated by $x_0, x_1 \in H$ and

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ Q_n := \{z \in H : \langle w_n - \tau_n A w_n - y_n, y_n - z\rangle \geq 0\}, \\ z_n = P_{Q_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \alpha_n w_n + \beta_n T z_n, \ \ \forall n \geq 1, \end{cases} \tag{3.22}$$

where τ_n is chosen to be largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|,$$

and $\{\alpha_n\},\ \{\beta_n\} \subset [0,\ 1]$ *with* $\alpha_n + \beta_n \leq 1$, $\gamma > 0$, $l \in (0,1)$ *and* $\mu \in (0,1)$. *Suppose the following conditions hold:*

(i) *there are* $c, d > 0$ *with* $0 < c \leq \beta_n \leq d < 1$ *for all* $n \geq 1$*;*

(ii) $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$*;*

(iii) $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$.

Then $\{x_n\}$ *converges weakly to* $z \in \bar{\Gamma}$ *and furthermore,*

$$z = \lim_{n\to\infty} P_{\bar{\Gamma}}(x_n).$$

Remark 3.6. We remark that Corollary 3.5 is modified from Algorithm 1 in [31] in the following aspects:

(1) From a monotone and Lipschitz continuous operator to a inverse strongly monotone operator.
(2) From a quasi-nonexpansive mapping to a nonexpansive mapping.

4. Application

In this section, we introduce a weak convergence theorem for finding a common element of the set of a finite family of variational inequalities problems and the solution of the modification of system of variational inequality problems using our main result Theorem 3.3.

In 2013, Kangtunyakarn [15] introduced a modification of system of variational inequalities as follows: finding $(x^*, z^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda_1 D_1)(ax^* + (1-a)z^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \lambda_2 D_2)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{4.1}$$

where $D_1, D_2 : C \to H$ be two mappings, for every $\lambda_1, \lambda_2 \geq 0$ and $a \in [0,1]$.

Lemma 4.1. ([15]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $D_1, D_2 : C \to H$ *be mappings. Then, for every* $\lambda_1, \lambda_2 > 0$ *and* $b \in [0,1]$*, the following statements are equivalent:*

(1) $(x^*, z^*) \in C \times C$ *is a solution of problem* (4.1)*,*
(2) x^* *is a fixed point of mapping* $\widehat{G} : C \to C$*, i.e.,* $x^* \in F(\widehat{G})$*, defined by*

$$\widehat{G}(x) = P_C(I - \lambda_1 D_1)(bx + (1-b)P_C(I - \lambda_2 D_2)x), \tag{4.2}$$

where $z^* = P_C(I - \lambda_2 D_2)x^*$. *Moreover* $\widehat{G}$ *is nonexpansive mapping.*

Algorithm 2:

Initialization: Let $x_0, x_1 \in H$ be arbitrary. Given $\gamma > 0$, $l \in (0,1), \mu \in (0,1)$, $\sum_{i=1}^N a_i = 1$, $0 < a_i < 1$, $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1: Set $w_n = x_n + \theta_n(x_n - x_{n-1})$ and compute

$$y_n = P_C(w_n - \tau_n \sum_{i=1}^N a_i A_i w_n),$$

where τ_n is chosen to be largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau \sum_{i=1}^N a_i \|A_i w_n - A_i y_n\| \le \mu \|w_n - y_n\|.$$

Step 2: Compute

$$z_n = P_{Q_n}(w_n - \tau_n \sum_{i=1}^N a_i A_i y_n),$$

where $Q_n := \{z \in H : \langle w_n - \tau_n \sum_{i=1}^N a_i A_i w_n - y_n, y_n - z\rangle \ge 0\}$.

Step 3: Compute

$$x_{n+1} = \alpha_n w_n + \beta_n \widehat{G} z_n + r_n. \tag{4.3}$$

Set $n := n+1$ and go to **Step 1**.

Theorem 4.2. *Let H be a real Hilbert space. For $i = 1, 2, ..., N$, let $A_i : H \to H$ be ξ_i-inverse strongly monotone mappings with $\eta = \min_{i=1,2,...,N}\{\xi_i\}$ and let $D_1, D_2 : C \to H$ be d_1, d_2-inverse strongly monotone mappings, respectively. Define the mapping $\widehat{G} : H \to H$ by (4.2). Assume that*

$$\Omega = \bigcap_{i=1}^N VI(C, A_i) \cap F(\widehat{G}) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by Algorithm 2, where r_n is the residual vector, $\sum_{i=1}^N a_i = 1, 0 < a_i < 1, \{\alpha_n\}$, $\{\beta_n\} \subset [0,\ 1]$ such that $\alpha_n + \beta_n \le 1$, $\lambda_1 \in (0, 2d_1)$, $\lambda_2 \in (0, 2d_2)$ and $b \in [0,1]$. Suppose the following conditions hold:

(i) *there are $c, d > 0$ with $0 < c \le \beta_n \le d < 1$ for all $n \ge 1$;*
(ii) $\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0$;
(iii) $\sum_{n=1}^\infty (1 - \alpha_n - \beta_n) < \infty$;
(iv) $\sum_{n=1}^\infty \|r_n\| < \infty$.

Then $\{x_n\}$ *converges weakly to* $z \in \Omega$ *and furthermore,*

$$z = \lim_{n\to\infty} P_\Omega x_n.$$

Proof. Putting $T \equiv \widehat{G}$ in Theorem 3.3, we obtain the desired conclusion. $\square$

5. Example and numerical results

The following examples are provided in this section to illustrate our main theorem.

Example 5.1. Let $H = \mathcal{R}^2$ be the two dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2$ and a usual norm $|| \cdot || : \mathcal{R}^2 \times \mathcal{R}^2 \to \mathcal{R}$ give by $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2}$ for all $\mathbf{x} = (x_1, x_2) \in \mathcal{R}^2$ and $\mathbf{y} = (y_1, y_2) \in \mathcal{R}^2$.

Let

$$C_1 = \{(x_1, x_2) \in H | -2x_1 + x_2 \leq 1\}$$

and

$$C_2 = \{(x_1, x_2) \in H | 4x_1 - 2x_2 \leq 3\}.$$

Define the mapping $A_1 : C_1 \to \mathcal{R}^2$ by $A_1(x_1, x_2) = (\frac{3x_1}{2}, \frac{3x_2}{2})$. Define the mapping $A_2 : C_2 \to \mathcal{R}^2$ by $A_2(x_1, x_2) = (2x_1, 2x_2)$. Let the mapping $T : \mathcal{R}^2 \to \mathcal{R}^2$ defined by $T(x_1, x_2) = (\frac{x_1}{2}, \frac{x_2}{2})$. Let $C = C_1 \cap C_2$. Also, it is well known that

$$P_C(x_1, x_2) = \begin{cases} (-1999x_1 + 1000x_2 + 750, 4000x_1 - 1999x_2 - 1500), \\ \qquad \text{if } -40x_1 + 20x_2 < -15; \\ (x_1, x_2), \\ \qquad \text{if } -15 \leq -40x_1 + 20x_2 \leq 5; \\ (-1999x_1 + 1000x_2 - 250, 4000x_1 - 1999x_2 - 500), \\ \qquad \text{if } -40x_1 + 20x_2 > 5. \end{cases}$$

Let $\mathbf{x}^0 = (x_1^0, x_2^0), \mathbf{x}^1 = (x_1^1, x_2^1) \in \mathcal{R}^2$, and $\{\mathbf{x}^n\}$ be a sequence generated by Algorithm 1. Let $\{\theta_n\} = \frac{1}{n^3}$, $\{\alpha_n\} = \frac{1}{n+1}$, $\{\beta_n\} = 1 - \frac{2}{n+1} \subset [0,\ 1]$ and $a = 0.5 \in (0,1)$. Show that $\{\mathbf{x}^n\}$ converge strongly to $\mathbf{0} = (0,0)$.

In fact, since A_1 and A_2 are $\frac{2}{3}$ and $\frac{1}{2}$-inverse strongly monotone mappings, respectively, then $\eta = \frac{1}{2}$. From definition of T, we have T is nonexpansive mapping. It is easy to see that the sequences $\{\theta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ satisfy all conditions in Theorem 3.3 and

$$(0,0) \in VI(C, A_1) \cap VI(C, A_2) \cap F(T).$$

From Theorem 3.3, we can conclude that the sequence $\{\mathbf{x}^n\}$ converge strongly to $(0,0)$. The numerical and graphical results of Example 5.1 are shown in Table 1 and Figure 1.

n	$\mathbf{x}_1^n$	$\mathbf{x}_2^n$
0	5.0000	4.0000
1	4.0000	3.0000
2	1.9062	1.8568
3	0.8894	1.0890
$\vdots$	$\vdots$	$\vdots$
14	0.0129	0.0177

TABLE 1. Computational results of $\mathbf{x}^n = (x_1^n, x_2^n)$ for Example 5.1 with $\mathbf{x}^0 = (5,4)$, $\mathbf{x}^1 = (4,3)$ and $n = 14$.

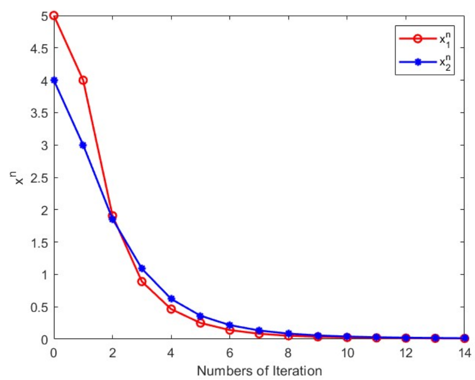

FIGURE 1. The convergence behaviour of $\mathbf{x}^n = (x_1^n, x_2^n)$ for Example 5.1 with $\mathbf{x}^0 = (5,4)$, $\mathbf{x}^1 = (4,3)$ and $n = 14$.

Next, we consider the problem in the infinite-dimensional Hilbert space (L_2).

Example 5.2. Let $H = L_2([-1,1])$ with product $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the associated norm given as $||f|| := \sqrt{\int_{-1}^{1} f(t)g(t)dt}$ for all $f, g \in L_2([-1,1])$. Take $C = \{x \in H : ||x|| \leq 2\}$. Define the mapping $A_1 : L_2([-1,1]) \to L_2([-1,1])$ by $A_1(h(t)) = h(t) - 2t$ for all $t \in [-1,1]$. Define the mapping $A_2 : L_2([-1,1]) \to L_2([-1,1])$ by $A_2(h(t)) = \frac{3}{2}h(t) - 3t$ for all $t \in [-1,1]$. Let the mapping $T : L_2([-1,1]) \to L_2([-1,1])$ defined by $T(h(t)) = \frac{h(t)}{16} - \frac{15t}{8}$ for all $t \in [-1,1]$. Also, it is well known that

$$P_C(f(t)) = \begin{cases} f(t), & \text{if} \quad ||f(t)|| \leq 2; \\ \frac{2f(t)}{||f(t)||}, & \text{if} \quad ||f(t)|| > 2. \end{cases}$$

Let $x_0, x_1 \in L_2$ and $\{x_n\}$ be a sequence generated by Algorithm 1. Let $\{\theta_n\} = \frac{1}{n^3}$, $\{\alpha_n\} = \frac{1}{n+1}$, $\{\beta_n\} = 1 - \frac{2}{n+1} \subset [0,\ 1]$ and $a = 0.5 \in (0,1)$. Show that $\{x_n\}$ and $\{y_n\}$ converge strongly to $2t$ for all $t \in [-1,1]$.

Since A_1 and A_2 are $\frac{2}{3}$ and $\frac{1}{2}$-inverse strongly monotone mappings, respectively, then $\eta = \frac{1}{2}$. From definition of T, T is nonexpansive mapping. It is easy to see that the sequences $\{\theta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ satisfy all conditions in Theorem 3.3 and $2t \in VI(C, A_1) \cap VI(C, A_2) \cap F(T)$. From Theorem 3.3, we can conclude that the sequence $\{x_n\}$ converge strongly to $2t$. The numerical and graphical results of Example 5.1 are shown in Table 2 and Figure 2.

n	$E(x_n)$
1	8.0000
2	2.7406
3	0.5499
4	0.1110
$\vdots$	$\vdots$
14	0.0094

TABLE 2. Computational results of $E(x_n)$ for Example 5.2 with $x_0 = -3t$, $x_1 = 5t$ and $n = 14$.

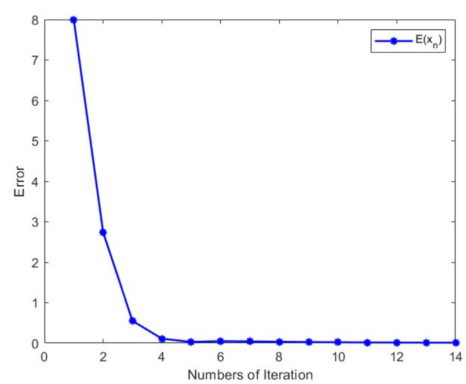

FIGURE 2. The convergence behaviour of $E(x_n)$ for Example 5.2 with $x_0 = -3t$, $x_1 = 5t$ and $n = 14$.

We next give a comparison between Algorithm (3.22) in Corollary 3.5 and Algorithm 1 in [31].

Example 5.3. Let $H = \mathcal{R}$. Take $C = [-1, 1]$. Define the mapping $A_1 : C_1 \to \mathcal{R}$ by $A_1(x) = \frac{3x}{2}$. Let the mapping $T : \mathcal{R} \to \mathcal{R}$ defined by $T(x) = \frac{x}{2}$. It is easy to see that the solution $0 \in VI(C, A_1) \cap F(T)$ and moreover, A_1 is $\frac{2}{3}$-inverse strongly monotone and T is nonexpansive mapping. The parameters are chosen as follows:
Algorithm (3.22): $\theta_n = \frac{1}{n^3}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = 1 - \frac{2}{n+1}$, $\tau_n = \frac{2}{3}$, $r_n = (\frac{1}{3})^n$.
Algorithm 1 in [31]: $\alpha_n = \frac{1}{n^3}$, $\beta_n = 1 - \frac{2}{n+1}$, $\tau_n = \frac{2}{3}$.

We now make comparisons of two algorithms with the starting point $x_0 = 2$ and $x_1 = 1$. From Table 3 and Figure 3, it is observed that Algorithm (3.22) is greatly better than Algorithm 1 in [31].

n	x_n : Algorithm (3.22)	x_n : Algorithm 1 in [31]
0	2.0000	2.0000
1	1.0000	1.0000
2	0.4028	0.2917
3	0.1322	0.1327
4	0.0379	0.0781
$\vdots$	$\vdots$	$\vdots$
12	0.0000	0.0117
13	0.0000	0.0099

TABLE 3. Detailed analysis of computational methods (3.22) and Algorithm 1 in [31] for Example 5.3 with $x_0 = 2$, $x_0 = 1$ and $n = 13$.

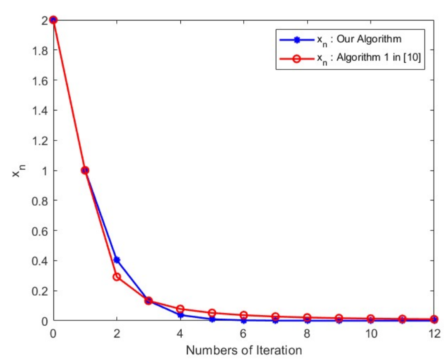

FIGURE 3. Comparison between algorithms 3.22 and Algorithm 1 in [31] for Example 5.3 with $x_0 = 2$, $x_1 = 1$ and $n = 13$.

Remark 5.4. From our numerical experiments in Example 5.1, 5.2 and 5.3, we make the following observations.

(1) Table 1 and Figure 1 show that $\{\mathbf{x}^n\}$ converge strongly to $(0,0)$, where $(0,0) \in VI(C,A_1) \cap VI(C,A_2) \cap F(T)$. The convergence of $\{\mathbf{x}^n\}$ of Example 5.1 can be guaranteed by Theorem 3.3.

(2) Table 2 and Figure 2 show that $\{x_n\}$ converge strongly to $2t$. The convergence of $\{x_n\}$ of Example 5.2 can be guaranteed by Theorem 3.3.

(3) From Table 3 and Figure 3, we see that Algorithm (3.22) is greatly better than Algorithm 1 in [31], in terms of the number of iterations.

6. Conclusion

We apply inertia technique and the Krasnoselskii-Mann algorithm to the subgradient extragradient algorithm for solving a common element of the set of a finite family of variational inequalities problems and the fixed point set of a nonexpansive mapping in real Hilbert spaces. the weak convergence of the algorithm has been proved. Furthermore, the constant of inverse strongly monotone operators are not required as an input parameter in our algorithms. Moreover, using our main result, we obtain the additional results involving system of variational inequalities. Some computational results have been reported to demonstrate the algorithm's efficacy.

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