Nonlinear Functional Analysis and Applications
Vol. 18, No. 3 (2013), pp. 307-325

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AN ITERATIVE METHOD FOR MIXED EQUILIBRIUM PROBLEMS, FIXED POINT PROBLEMS OF STRICTLY PSEUDO-CONTRACTIVE MAPPINGS AND NONEXPANSIVE SEMI-GROUPS

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Abstract. The purpose of this paper is to introduce an iterative algorithm for finding a common element of the set of solutions of mixed equilibrium problems, the set of common fixed point for strictly pseudo-contractive mappings and the set of common fixed points for nonexpansive semi-groups in Hilbert space. Under suitable conditions, we prove some strong convergence theorems. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. We denote the sets of nonnegative integers and real numbers by N and R respectively.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$.

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[0]Received February 5, 2013. Revised April 28, 2013.
[0]2000 Mathematics Subject Classification: 47H10, 47H06, 47H17.
[0]Keywords: Mixed equilibrium problems, strictly pseudo-contractive mappings, nonexpansive semi-groups, fixed point.
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Let $\mathcal{S}$=$\{S(s) : 0 \leq s \leq \infty\}$ be a nonexpansive semigroup on C, if it satisfies the following conditions:

(i) $S(0)x = x$ for all $x \in C$;
(ii) $S(s+t) = S(s)S(t)$ for all $s, t \geq 0$;
(iii) $\|S(s)x - S(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C, s \mapsto S(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S} = \{S(s) : s \geq 0\}$, *i.e.*,

$$F(\mathcal{S}) = \bigcap_{s \geq 0} F(S(s)).$$

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. We consider the mixed equilibrium problem (MEP)which is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by Ω.

In particular, if $\varphi = 0$, this problem reduces to the equilibrium problem(EP), which is to find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

It is well-known that the MEP includes fixed point problem, variational inequality problems, Nash equilibrium problems and the equililbrium problems as special cases.

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x \in \overline{C}} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x) \tag{1.3}$$

where $\overline{C} = \bigcap_{n=1}^{\infty} C_n, C_1, C_2, \cdots$ are infinitely many closed subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset, u \in H, \mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e.,$h'(x) = \gamma f(x)$ for all $x \in H$).

In 2009, Wangkeeree [1] studied the strong convergence of the sequence $\{x_n\}$ defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds.$$

They prove that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{\beta_n\}$, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \;\; x \in F(\mathcal{S}).$$

Recently, Yao [2] introduced the following iterative scheme as below for finding a common of the solutions of mixed equilibrium problems and a set of fixed points of an infinite family of nonexpansive mappings in a Hilbert space.

Defined sequence $\{x_n\}$ by

$$\begin{cases} \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), x - y_n\rangle \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + (1 - \beta_n)x_n \\ \qquad\qquad +((1 - \beta_n)I - \alpha_n(I + \mu A))W_n y_n, \ \ \forall n \geq 1. \end{cases}$$

They prove that under certain appropriate conditions the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega \cap F(W_n)$, which is the solution of the following optimization problem:

$$\min_{x \in \Omega \cap F(W_n)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x).$$

Motivated and inspired by Wangkeeree [1] and Yao [2], the purpose of this paper is to introduce an iterative algorithm for finding a common element of the set of solutions for mixed equilibrium problems and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings and the set of common fixed points for nonexpansive semi-groups in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by " $\to$ " and " $\rightharpoonup$ " the strong convergence and weak convergence, respectively.

Let H be a real Hilbert space and let C be a closed convex subset of H, there exists a unique nearest point $u \in C$ such that

$$\|x - u\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping $P_C : x \to u$ is called the metric projection of H onto C. It is well known that P_C is nonexpansive, furthermore, for $x \in H$ and $u \in C$,

$$u = P_C(x) \Leftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall y \in C.$$

Banach space E is said to satisfy the Opial condition, if for any sequence $\{x_n\}$ in E with $x_n \rightharpoonup x \in E$, then for any $y \in E$ with $y \neq x$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Recall that a mapping $f : H \to H$ is said to be contractive, if there exists a constant $\xi \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \xi\|x - y\|, \quad \forall x, y \in H.$$

A mapping $V : C \to H$ is said to be k-strictly pseudo-contractive, if there exists a constant $k \in [0, 1)$ such that

$$\|V(x) - V(y)\|^2 \leq \|x - y\|^2 + k\|(I - V)x - (I - V)y\|^2, \quad \forall x, y \in C.$$

A family of mappings $\{V_i : C \to C\}_{i=1}^{\infty}$ is callled a family of uniformly k-strictly pseudo-contractive, if there exists a constant $k \in [0, 1)$ such that

$$\|V_i x - V_i y\|^2 \leq \|x - y\|^2 + k\|(I - V_i)x - (I - V_i)y\|^2, \quad \forall x, y \in C, \quad \forall i \geq 1.$$

Lemma 2.1. ([3]) *Let $V : C \to H$ be a k-strictly pseudo-contractive. Then*

(1) *the fixed point set $F(V)$ of V is closed convex so that the projection $P_{F(V)}$ is well defined;*

(2) *define a mapping $T : C \to H$ by $Tx = \gamma x + (1 - \gamma)Vx, x \in C$. If $\gamma \in [k, 1)$, then T is a nonexpansive mapping such that $F(V) = F(T)$.*

Definition 2.1. Let $\{V_i : C \to C\}$ be a countable family of uniformly k-strictly pseudo-contractive. Let $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by

$$T_i x = \gamma x + (1 - \gamma)V_i x, \;\; \gamma \in [k, 1). \tag{2.1}$$

Let $\{\mu_i\}$ be a nonnegative real sequence with $0 \leq \mu_i < 1$, $\forall i \geq 1$. For any $n \geq 1$ define a mapping $W_n : C \to C$ as follows

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \mu_n T_n U_{n,n+1} + (1 - \mu_n)I, \\ U_{n,n-1} = \mu_{n-1} T_{n-1} U_{n,n} + (1 - \mu_{n-1})I, \\ \vdots \\ U_{n,k} = \mu_k T_k U_{n,k+1} + (1 - \mu_k)I, \\ U_{n,k-1} = \mu_{k-1} T_{k-1} U_{n,k} + (1 - \mu_{k-1})I, \\ \vdots \\ U_{n,2} = \mu_2 T_2 U_{n,3} + (1 - \mu_2)I, \\ W_n = U_{n,1} = \mu_1 T_1 U_{n,2} + (1 - \mu_1)I. \end{cases} \tag{2.2}$$

Such a mapping W_n is nonexpansive from C to C called the $W-$mapping generated by $T_n, T_{n-1}, \cdots, T_1$ and $\mu_n, \mu_{n-1}, \cdots, \mu_1$. We have the following Lemmas concerning W_n.

Lemma 2.2. ([4]) *Let C be a nonempty closed convex subset of a Hilbert space $H, \{T_i : C \to C\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings with*

$$\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset,$$

$\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$, $\forall i \geq 1$. Then

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{\infty} F(T_i)$ for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;*
(3) *the mapping $W : C \to C$ defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $x \in C$, is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.*

Lemma 2.3. ([5]) *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a countable family of nonexpansive mappings with*

$$\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset,$$

$\{\mu_i\}$ be a real sequence such that $0 < \mu_i \leq b < 1$, $\forall i \geq 1$. If D is any bounded subset of C, then

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Lemma 2.4. *Let H be a real Hilbert space. Then we have the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H.$$

Lemma 2.5. ([6]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$$

for all integer $n \geq 1$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6. ([7]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\Sigma_{n=1}^{\infty}|\delta_n| = \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

For nonexpansive semi-group $\mathcal{S}=\{S(s) : 0 \le s < \infty\}$, we need the following lemmas to prove our main result:

Lemma 2.7. ([8]) *Let* C *be a bounded closed convex subset of* H *and* $\mathcal{S}=\{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semi-group on* C. *Then for any* $h > 0$,

$$\lim_{s\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t S(s)xds - S(h)\left(\frac{1}{t}\int_0^t S(s)xds\right) \right\| = 0.$$

Lemma 2.8. ([9]) *Let* C *be a nonempty bounded closed convex subset of* H, $\{x_n\}$ *be a sequence in* C *and* $\mathcal{S}=\{S(s) : 0 \le s < \infty\}$ *be a nonexpansive semi-group on* C. *If the following conditions are satisfied:*

(i) $x_n \rightharpoonup z$;

(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|S(s)x_n - x_n\| = 0$.

Then $z \in F(\mathcal{S})$.

For solving the mixed equilibrium problem (1.1), let us assume that the function $\Theta : C \times C \to \mathbb{R}$ satisfied the following conditions:

(H1) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \le 0$ for all $x, y \in C$;

(H2) for each fixed $y \in C, x \mapsto \Theta(x,y)$ is concave and upper semicontinuous;

(H3) for each $x \in C, y \mapsto \Theta(x,y)$ is convex.

Let $F : C \to C$ and $\eta : C \times C \to C$ be two mappings. F is said to be:

(i) $\beta-$Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|F(x) - F(y)\| \le \beta\|x - y\|, \quad \forall x, y \in C;$$

(ii) $\eta-$monotone, if

$$\langle F(x) - F(y), \eta(x,y)\rangle \ge 0, \quad \forall x, y \in C;$$

(iii) $\eta-$strong monotone, if there exists a constant $\alpha > 0$, such that

$$\langle F(x) - F(y), \eta(x,y)\rangle \ge \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

A differentiable function $K : C \to \mathbb{R}$ is said to be:

(i) η convex, if

$$K(y) - K(x) \geq \langle K'(x), \eta(y,x)\rangle;$$

where $K'(x)$ is the Fréchet derivative of K at x.

(ii) $\eta-$strong convex, if there exists a constant $\mu > 0$, such that

$$K(y) - K(x) - \langle K'(x), \eta(y,x)\rangle \geq (\mu/2)\|x-y\|^2, \quad \forall x, y \in C.$$

Let C be a nonempty closed convex subset of H, $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. Let r be any given positive number. For a given point $x \in H$, consider the following auxiliary problem for MEP to find $y \in C$ such that

$$\Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \quad \forall z \in C,$$

where $\eta : H \times H \to H$ and $K'(x)$ is the Fréchet derivative of a functional $K : H \to \mathbb{R}$ at x. Let $S_r : H \to C$ be the mapping such that for each $x \in H, S_r(x)$ is the solution set of MEP, *i.e.*,

$$S_r(x) = \left\{ y \in C : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \forall z \in C \right\},$$

for all $x \in H$.

Lemma 2.9. ([10]) *Let C be a nonempty closed convex subset of H and $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying the conditions* (H1)-(H3). *Assume that*

(i) *$\eta : C \times C \to H$ is $\lambda-$ Lipschitz continuous such that*
   (a) *$\eta(x,y) + \eta(y,x) = 0, \quad \forall x, y \in H;$*
   (b) *$\eta(\cdot,\cdot)$ is affine in the first variable;*
   (c) *for each fixed $x \in H$, the mapping $y \mapsto \eta(x,y)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : H \to \mathbb{R}$ is η-strongly convex with constant $\mu > 0$, and its derivative K' is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each $x \in C$, there exists a bounded subset $D_x \subset C$ and a point $z_x \in C$ such that for any $y \in C \backslash D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Then the following hold:

(i) *S_r is single-valued;*

(ii) (a) *$\langle K'(x_1) - K'(x_2), \eta(u_1,u_2)\rangle \geq \langle K'(u_1) - K'(u_2), \eta(u_1,u_2)\rangle, \forall (x_1,x_2) \in H \times H$, where $u_i = S_r(x_i), i = 1,2;$*

(b) S_r *is nonexpansive if* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that* $\mu \geq \lambda\nu$;

(iii) $F(S_r) = \Omega$*;*

(iv) Ω *is closed and convex.*

Definition 2.2. Let $A : H \to H$ be a bounded self-adjoint linear operator, A is said to be a strongly positive operator if there exist a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

Lemma 2.10. ([11]) *Assume that* A *is a strongly positive bounded linear operator on* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$*. Then* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 2.11. ([12]) *Let* C *be an nonempty closed and convex subset of a real Hilbert space* H *and* $g : C \to \mathbb{R}\bigcup\{+\infty\}$ *be a proper lower semicontinuous differential convex function. If* x^* *is a solution to the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x)$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad x \in C.$$

In particular, if x^* *solves the optimization problem* (1.3)*, then*

$$\langle u + (\gamma f - (I + \mu A))x^*, x - x^*\rangle \leq 0, \quad x \in C,$$

where h *is a potential function for* γf.

3. Main Results

Theorem 3.1. *Let* C_1, C_2 *be two nonempty closed convex subsets of a real Hilbert space* H*,* $f : H \to H$ *be a contractive mapping with a contractive constant* $\xi \in (0,1)$*. Let* $\varphi : C_1 \to \mathbb{R}$ *be a lower semicontinous and convex function, and let* $\Theta : C_1 \times C_1 \to \mathbb{R}$ *be an equilibrium bifunction satisfying conditions* (H1)-(H3) *with* C *replaced by* C_1*. Let* $\{V_i : C_1 \to C_1\}_{i=1}^{\infty}$ *be a countable family of uniformly* $k-$*strict pseudo-contractions and* $\{T_i : C_1 \to C_1\}_{i=1}^{\infty}$ *be the countable family of nonexpansive mappings defined by* (2.1) *with* C *replaced by* C_1*. For each* $n \geq 1$*, let* $W_n : C_1 \to C_1$ *be the* $W-$*mapping define by* (2.2) *with* C *replaced by* C_1*. Let* $\mathcal{S}=\{S(s) : 0 \leq s < \infty\}$ *be a nonexpansive semi-group on* C_2*. Let* $\mu > 0$*,* $\gamma > 0$*,* $r > 0$ *be three constants and let* A *be a strongly positive bounded linear operator on* H *with coefficient* $\overline{\gamma} > 0$ *such that*

$0 < \gamma < (1+\mu\overline{\gamma})/\xi$. *For given* $x_0 \in H$ *arbitrarily and fixed* $u \in H$, $\{x_n\}, \{y_n\}$ *are the sequences generated by*

$$\begin{cases} y_n \in C_1 : \Theta(y_n, x) + \varphi(x) - \varphi(y_n) \\ \qquad\qquad +\frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0, \quad \forall x \in C_1, \\ x_{n+1} = \alpha_n(u + \gamma f(y_n)) + \beta_n x_n \\ \qquad +((1-\beta_n)I - \alpha_n(I+\mu A))\frac{1}{t_n}\int_0^{t_n} S(s)P_{C_2}W_n y_n ds, \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ *are two sequences in* $(0,1)$ *and* $\{t_n\}$ *is a sequence in* $(0,\infty)$. *If the following conditions are satisfied:*

(i) $\eta : C_1 \times C_1 \to H$ *is* $\lambda-$*Lipschitz continuous such that*
  (a) $\eta(x, y) + \eta(y, x) = 0, \quad \forall x, y \in C_1$;
  (b) $\eta(\cdot,\cdot)$ *is affine in the first variable;*
  (c) *for each fixed* $x \in C_1$, *the mapping* $y \mapsto \eta(x, y)$ *is sequentially continuous from the weak topology to the weak topology;*

(ii) $K : H \to R$ *is* η*-strongly convex with constant* $\mu > 0$, *and its derivative* K' *is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant* $\nu > 0$ *and* $\mu \geq \lambda\nu$;

(iii) *for each* $x \in C_1$, *there exists a bounded subset* $D_x \subset C_1$ *and a point* $z_x \in C_1$ *such that for any* $y \in C_1 \setminus D_x$,

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0;$$

(iv) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(v) $0 < \liminf\limits_{n\to\infty} \beta \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(vi) $\lim\limits_{n\to\infty} t_n = \infty$ *with bounded* $\sup_{n\geq 1} |t_n - t_{n+1}|$;

(vii) $\Gamma := F(W)\bigcap \Omega \bigcap F(\mathcal{S}) \neq \emptyset$, $F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *defined by* (3.1) *converge strongly to* $x^* \in \Gamma$ *which solves the following optimization problem* (OP).

$$\min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)$$

provided S_r *is firmly nonexpansive, i.e.,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle, \quad \forall x, y \in H.$$

Proof. We split the proof into five steps.

Step1. We show that $\{x_n\}$ is bounded.

By the conditions (iv) and (v), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)(1+\mu\|A\|)^{-1}$ for all $n \geq 1$. Since A is a strongly positive bounded self-adjoint linear operator on H and Lemma 2.10, we have

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.$$

Obeserve that

$$\begin{aligned}\langle((1-\beta_n)I-\alpha_n(1+\mu A))u,u\rangle &= 1-\beta_n-\alpha_n-\alpha_n\mu\langle Au,u\rangle\\ &\geq 1-\beta_n-\alpha_n-\alpha_n\mu\|A\|\\ &\geq 0.\end{aligned}$$

This shows that $(1-\beta_n)I-\alpha_n(1+\mu A)$ is positive. It follows that

$$\begin{aligned}&\|(1-\beta_n)I-\alpha_n(1+\mu A)\|\\ =\ &\sup\{\langle((1-\beta_n)I-\alpha_n(1+\mu A))u,u\rangle| : u\in H, \|u\|=1\}\\ =\ &\sup\{1-\beta_n-\alpha_n-\alpha_n\mu\langle Au,u\rangle : u\in H,\|u\|=1\}\\ \leq\ &1-\beta_n-\alpha_n-\alpha_n\mu\overline{\gamma}.\end{aligned} \tag{3.2}$$

Taking $p\in\Gamma$, we have

$$\|y_n-p\|=\|S_rx_n-S_rp\|\leq\|x_n-p\|. \tag{3.3}$$

Set $S_nx:=\frac{1}{t_n}\int_0^{t_n}S(s)xds$ and $\overline{A}:=(I+\mu A)$. Since

$$\begin{aligned}\|S_nP_{C_2}x-S_nP_{C_2}y\| &= \left\|\frac{1}{t_n}\int_0^{t_n}S(s)P_{C_2}xds-\frac{1}{t_n}\int_0^{t_n}S(s)P_{C_2}yds\right\|\\ &\leq\|x-y\|,\end{aligned} \tag{3.4}$$

so $S_nP_{C_2}$ is nonexpansive. Put $M_1=\max\{\|x_1-p\|,\frac{\|\gamma f(p)-\overline{A}p\|+\|u\|}{1+\mu\overline{\gamma}-\gamma\xi}\}$. Clearly, $\|x_1-p\|\leq M_1$. Suppose that $\|x_n-p\|\leq M_1$, it follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}&\|x_{n+1}-p\|\\ =\ &\|\alpha_n(u+\gamma f(y_n))+\beta_nx_n\\ &+((1-\beta_n)I-\alpha_n\overline{A}))S_nP_{C_2}W_ny_n-p\|\\ =\ &\|\alpha_nu+\alpha_n(\gamma f(y_n)-\overline{A}p)+\beta_n(x_n-p)\\ &+((1-\beta_n)I-\alpha_n\overline{A}))(S_nP_{C_2}W_ny_n-p)\|\\ \leq\ &\alpha_n\|u\|+\alpha_n\|\gamma f(y_n)-\overline{A}p\|+\beta_n\|x_n-p\|\\ &+(1-\beta_n-\alpha_n-\alpha_n\mu\overline{\gamma}))\|x_n-p\|\|\\ \leq\ &\alpha_n\|u\|+\alpha_n\|\gamma f(y_n)-\overline{A}p\|+(1-\alpha_n-\alpha_n\mu\overline{\gamma}))\|x_n-p\|\|\\ \leq\ &\alpha_n\|u\|+\alpha_n\|\gamma f(y_n)-\gamma f(p)\|+\alpha_n\|\gamma f(p)-\overline{A}p\|\\ &+(1-\alpha_n-\alpha_n\mu\overline{\gamma}))\|x_n-p\|\|\\ \leq\ &\alpha_n\|u\|+\alpha_n\gamma\xi\|x_n-p\|+\alpha_n\|\gamma f(p)-\overline{A}p\|\\ &+(1-\alpha_n-\alpha_n\mu\overline{\gamma}))\|x_n-p\|\\ =\ &[1-(1+\mu\overline{\gamma}-\gamma\xi)\alpha_n]\|x_n-p\|+\alpha_n(\|u\|+\|\gamma f(p)-\overline{A}p\|)\\ =\ &[1-(1+\mu\overline{\gamma}-\gamma\xi)\alpha_n]\|x_n-p\|\\ &+(1+\mu\overline{\gamma}-\gamma\xi)\alpha_n\frac{(\|u\|+\|\gamma f(p)-\overline{A}p\|)}{1+\mu\overline{\gamma}-\gamma\xi}\\ =\ &[1-(1+\mu\overline{\gamma}-\gamma\xi)\alpha_n]M_1+(1+\mu\overline{\gamma}-\gamma\xi)\alpha_nM_1\\ \leq\ &M_1.\end{aligned} \tag{3.5}$$

It follows from (3.5) by induction that $\|x_n - p\| \le M_1$, for all $n \ge 1$ and hence $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{W_n x_n\}$, $\{S_n P_{C_2} W_n x_n\}$, $\{W_n y_n\}$, $\{S_n P_{C_2} W_n y_n\}$ and $\{f(y_n)\}$ are bounded. Denote by

$$\begin{aligned} M_2 = \sup_{n\ge 1}\{&\|x_n\|, \|y_n\|, \|f(y_n)\|, \|W_n x_n\|, \|W_n y_n\|,\\ &\|S(s)P_{C_2}W_n x_n\|, \|S(s)P_{C_2}W_n y_n\|\}. \end{aligned} \tag{3.6}$$

Step 2. We prove that $\|x_{n+1} - x_n\| \to 0$ and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$. Define $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ for all $n \ge 1$. We observe that

$$\begin{aligned} z_{n+1} - z_n &= \tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\ &= \tfrac{\alpha_{n+1}(u+\gamma f(y_{n+1}))+[(1-\beta_{n+1})I-\alpha_{n+1}\overline{A}]S_{n+1}P_{C_2}W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\ &\quad -\tfrac{\alpha_n(u+\gamma f(y_n))+[(1-\beta_n)I-\alpha_n\overline{A}]S_nP_{C_2}W_n y_n}{1-\beta_n}\\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[u+\gamma f(y_{n+1}) - \overline{A}S_{n+1}P_{C_2}W_{n+1}y_{n+1}]\\ &\quad +\tfrac{\alpha_n}{1-\beta_n}[\overline{A}S_nP_{C_2}W_n y_n - u - \gamma f(y_n)]\\ &\quad +S_{n+1}P_{C_2}W_{n+1}y_{n+1} - S_nP_{C_2}W_n y_n. \end{aligned}$$

That is

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\|u\| + \|\gamma f(y_{n+1}\|)\\ &\quad +\|\overline{A}S_{n+1}P_{C_2}W_{n+1}y_{n+1}\|]\\ &\quad +\tfrac{\alpha_n}{1-\beta_n}[\|\overline{A}S_nP_{C_2}W_n y_n\| + \|u\| + \|\gamma f(y_n)\|]\\ &\quad +\|S_{n+1}P_{C_2}W_{n+1}y_{n+1} - S_nP_{C_2}W_n y_n\|. \end{aligned} \tag{3.7}$$

On the other hand, it follows from (2.2) that

$$\begin{aligned} &\|W_{n+1}y_n - W_n y_n\|\\ =\ &\|\mu_1 T_1 U_{n+1,2}x_n + (1-\mu_1)x_n - \mu_1 T_1 U_{n,2}x_n + (1-\mu_1)x_n\|\\ \le\ &\mu_1\|U_{n+1,2}x_n - U_{n,2}x_n\|\\ \le\ &\mu_1\mu_2\|U_{n+1,3}x_n - U_{n,3}x_n\|\\ &\vdots\\ \le\ &\mu_1\mu_2\cdots\mu_n\|U_{n+1,n+1}x_n - x_n\|\\ =\ &\mu_1\mu_2\cdots\mu_n\mu_{n+1}\|T_{n+1}x_n - x_n\|\\ \le\ &b^{n+1}M_3, \end{aligned} \tag{3.8}$$

where $M_3 = \sup_{n\geq 1} \|T_{n+1}x_n - x_n\|$ is some positive constant. Notice (3.6) and (3.8), we obtain

$$\begin{aligned}
& \|S_{n+1}P_{C_2}W_{n+1}y_{n+1} - S_nP_{C_2}W_ny_n\| \\
= & \|\tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)P_{C_2}W_{n+1}y_{n+1}ds - \tfrac{1}{t_n}\int_0^{t_n} S(s)P_{C_2}W_ny_nds\| \\
= & \|\tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)P_{C_2}W_{n+1}y_{n+1}ds + \tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)P_{C_2}W_ny_nds \\
& -\tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} S(s)P_{C_2}W_ny_nds - \tfrac{1}{t_n}\int_0^{t_n} S(s)P_{C_2}W_ny_nds\| \\
\leq & \tfrac{1}{t_{n+1}}\int_0^{t_{n+1}} \|S(s)P_{C_2}W_{n+1}y_{n+1} - S(s)P_{C_2}W_ny_n\|ds \\
& +\|\tfrac{1}{t_{n+1}}(\int_0^{t_n} S(s)P_{C_2}W_ny_nds + \int_{t_n}^{t_{n+1}} S(s)P_{C_2}W_ny_nds) \\
& -\tfrac{1}{t_n}\int_0^{t_n} S(s)P_{C_2}W_ny_nds\| \\
\leq & \|W_{n+1}y_{n+1} - W_ny_n\| + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2 \\
\leq & \|W_{n+1}y_{n+1} - W_{n+1}y_n\| + \|W_{n+1}y_n - W_ny_n\| + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2 \\
\leq & \|y_{n+1} - y_n\| + \|W_{n+1}y_n - W_ny_n\| + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2 \\
\leq & \|x_{n+1} - x_n\| + b^{n+1}M_3 + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2.
\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7), we get

$$\begin{aligned}
& \|z_{n+1} - z_n\| \\
\leq & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\|u\| + \|\gamma f(y_{n+1}\|) + \|\overline{A}S_{n+1}P_{C_2}W_{n+1}y_{n+1}\|] \\
& +\tfrac{\alpha_n}{1-\beta_n}[\|\overline{A}S_nP_{C_2}W_ny_n\| + \|u\| + \|\gamma f(y_n)\|] \\
& +\|x_{n+1} - x_n\| + b^{n+1}M_3 + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2.
\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}
& \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
\leq & \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\|u\| + \|\gamma f(y_{n+1}\|) + \|\overline{A}S_{n+1}P_{C_2}W_{n+1}y_{n+1}\|] \\
& +\tfrac{\alpha_n}{1-\beta_n}[\|\overline{A}S_nP_{C_2}W_ny_n\| + \|u\| + \|\gamma f(y_n)\|] \\
& +b^{n+1}M_3 + 2\tfrac{|t_{n+1}-t_n|}{t_{n+1}}M_2,
\end{aligned} \tag{3.11}$$

which implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) = 0.$$

Hence by Lemma 2.5, we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

By the definition of $\{z_n\}$, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|z_n - x_n\| = 0. \tag{3.12}$$

Notice

$$\|y_{n+1} - y_n\| = \|S_rx_{n+1} - S_rx_n\| \leq \|x_{n+1} - x_n\|. \tag{3.13}$$

This implies that

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Step 3. We show

$$\lim_{n\to\infty} \|x_n - S(s)P_{C_2}x_n\| = 0, \quad \lim_{n\to\infty} \|y_n - S(s)P_{C_2}y_n\| = 0, \ \ \forall s \geq 0.$$

Note that

$$x_{n+1} = \alpha_n(u + \gamma f(y_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n\overline{A})S_nP_{C_2}W_ny_n,$$

we obtain that

$$\begin{aligned}\|x_n - S_nP_{C_2}W_ny_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_nP_{C_2}W_ny_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|u + \gamma f(y_n) - \overline{A}S_nP_{C_2}W_ny_n\| \\ &\quad +\beta_n\|x_n - S_nP_{C_2}W_ny_n\|,\end{aligned}$$

that is,

$$\|x_n - S_nP_{C_2}W_ny_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u + \gamma f(y_n) - \overline{A}S_nP_{C_2}W_ny_n\|.$$

By the assumpation (iv) and (3.12), we get

$$\lim_{n\to\infty} \|x_n - S_nP_{C_2}W_ny_n\| = 0. \tag{3.14}$$

Since $\{P_{C_2}W_ny_n\}$ is a bounded sequence in C_2, from Lemma 2.7, we can obtain

$$\lim_{n\to\infty} \|S_nP_{C_2}W_ny_n - S(h)S_nP_{C_2}W_ny_n\| = 0, \ \ \forall h \geq 0. \tag{3.15}$$

So for each $s \geq 0$ we have

$$\begin{aligned} & \|x_n - S(s)P_{C_2}x_n\| \\ \leq\ & \|x_n - S_nP_{C_2}W_ny_n\| + \|S_nP_{C_2}W_ny_n - S(s)S_nP_{C_2}W_ny_n\| \\ & +\|S(s)S_nP_{C_2}W_ny_n - S(s)P_{C_2}x_n\| \\ \leq\ & \|x_n - S_nP_{C_2}W_ny_n\| + \|S_nP_{C_2}W_ny_n - S(s)S_nP_{C_2}W_ny_n\| \\ & +\|S_nP_{C_2}W_ny_n - P_{C_2}x_n\| \\ =\ & \|x_n - S_nP_{C_2}W_ny_n\| + \|S_nP_{C_2}W_ny_n - S(s)S_nP_{C_2}W_ny_n\| \\ & +\|P_{C_2}S_nP_{C_2}W_ny_n - P_{C_2}x_n\| \\ \leq\ & 2\|x_n - S_nP_{C_2}W_ny_n\| + \|S_nP_{C_2}W_ny_n - S(s)S_nP_{C_2}W_ny_n\|. \end{aligned}$$

Hence from (3.14) and (3.15), we have

$$\lim_{n\to\infty} \|x_n - S(s)P_{C_2}x_n\| = 0. \tag{3.16}$$

For given $p \in \Gamma$, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|S_rx_n - p\|^2 \\ &\leq \langle S_rx_n - S_rp, x_n - p\rangle \\ &\leq \langle y_n - p, x_n - p\rangle \\ &= \tfrac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2),\end{aligned}$$

and hence

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2. \tag{3.17}$$

From Lemma 2.9 and (3.2), we have

$$\begin{aligned}
& \|x_{n+1} - p\|^2 \\
= \ & \|\alpha_n(u + \gamma f(y_n) - \overline{A}p) + \beta_n(x_n - S_nP_{C_2}W_ny_n) \\
& +(I - \alpha_n\overline{A})(S_nP_{C_2}W_ny_n - p)\|^2 \\
\le \ & \|\beta_n(x_n - S_nP_{C_2}W_ny_n) + (I - \alpha_n\overline{A})(S_nP_{C_2}W_ny_n - p)\|^2 \\
& +2\alpha_n\langle u + \gamma f(y_n) - \overline{A}p, x_{n+1} - p\rangle \\
\le \ & \|\beta_n(x_n - S_nP_{C_2}W_ny_n) + (1 - \alpha_n - \alpha_n\mu\overline{\gamma})(y_n - p)\|^2 \\
& +2\alpha_n\langle u + \gamma f(y_n) - \overline{A}p, x_{n+1} - p\rangle \\
\le \ & \beta_n^2\|(x_n - S_nP_{C_2}W_ny_n\|^2 + (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|y_n - p\|^2 \\
& +2\alpha_n\|u + \gamma f(y_n) - \overline{A}p\|\|x_{n+1} - p\| \\
& +2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|x_n - S_nP_{C_2}W_ny_n\|\|y_n - p\| \\
\le \ & \beta_n^2\|(x_n - S_nP_{C_2}W_ny_n\|^2 + (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2[\|x_n - p\|^2 - \|x_n - y_n\|^2] \\
& +2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|x_n - S_nP_{C_2}W_ny_n\|\|y_n - p\| \\
& +2\alpha_n\|u + \gamma f(y_n) - \overline{A}p\|\|x_{n+1} - p\| \\
\le \ & \|x_n - p\|^2 + \alpha_n^2(1 + \mu\overline{\gamma})^2\|x_n - p\|^2 - (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|x_n - y_n\|^2 \\
& +2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|x_n - S_nP_{C_2}W_ny_n\|\|y_n - p\| \\
& +2\alpha_n\|u + \gamma f(y_n) - \overline{A}p\|\|x_{n+1} - p\| \\
& +\beta_n^2\|x_n - S_nP_{C_2}W_ny_n\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
& (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|x_n - y_n\|^2 \\
\le \ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2(1 + \mu\overline{\gamma})^2\|x_n - p\|^2 \\
& +2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|x_n - S_nP_{C_2}W_ny_n\|\|y_n - p\| \\
& +2\alpha_n\|u + \gamma f(y_n) - \overline{A}p\|\|x_{n+1} - p\| + \beta_n^2\|x_n - S_nW_ny_n\|^2 \\
\le \ & [\|x_n - p\| + \|x_{n+1} - p\|][x_{n+1} - x_n] + \alpha_n^2(1 + \mu\overline{\gamma})^2\|x_n - p\|^2 \\
& +2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|x_n - S_nP_{C_2}W_ny_n\|\|y_n - p\| \\
& +2\alpha_n\|u + \gamma f(y_n) - \overline{A}p\|\|x_{n+1} - p\| + \beta_n^2\|x_n - S_nP_{C_2}W_ny_n\|^2.
\end{aligned}$$

From (3.12) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.18}$$

Since

$$\|y_n - S_nP_{C_2}W_ny_n\| \le \|y_n - x_n\| + \|x_n - S_nP_{C_2}W_ny_n\|,$$

from (3.14) and (3.18), we have

$$\lim_{n\to\infty} \|y_n - S_nP_{C_2}W_ny_n\| = 0. \tag{3.19}$$

Since

$$\begin{aligned}\|y_n - S(s)P_{C_2}y_n\| &\leq \|y_n - x_n\| + \|x_n - S(s)P_{C_2}x_n\| \\ &\quad + \|S(s)P_{C_2}x_n - S(s)P_{C_2}y_n\| \\ &\leq 2\|y_n - x_n\| + \|x_n - S(s)P_{C_2}x_n\|,\end{aligned}$$

from (3.16) and (3.18), we have

$$\lim_{n\to\infty} \|y_n - S(s)P_{C_2}y_n\| = 0. \tag{3.20}$$

Step 4. Now we show that $\limsup_{n\to\infty}\langle u + [\gamma f - (I + \mu A)]x^*, x_n - x^*\rangle \leq 0$, where x^* is a solution of (OP).

To show this, we can choose a subsequence $\{y_{n_i}\} \subset \{y_n\}$ such that

$$\begin{aligned}&\lim_{i\to\infty}\langle u + [\gamma f - (I + \mu A)]x^*, y_{n_i} - x^*\rangle \\ &= \limsup_{n\to\infty}\langle u + [\gamma f - (I + \mu A)]x^*, y_n - x^*\rangle.\end{aligned} \tag{3.21}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to p. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup p$. From (3.19), we have $S_{n_i}P_{C_2}W_{n_i}y_{n_i} \rightharpoonup p$. Since $\{y_n\} \subset C_1$ and $\{S_nP_{C_2}W_ny_n\} \subset C_2$ and C_1, C_2 are two closed convex subsets in H, we have $p \in C_1 \cap C_2$. Next we prove that

$$p \in \Gamma := F(W)\bigcap\Omega\bigcap F(\mathcal{S}).$$

Indeed, from Lemma 2.8 and (3.16), we can obtain that $p \in F(\mathcal{S})$. Next we show that $p \in \Omega$. Since $y_n = S_rx_n$, we derive

$$\Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle \geq 0, \quad \forall x \in H.$$

From the monotonicity of Θ, we have

$$\frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle + \varphi(x) - \varphi(y_n) \geq -\Theta(y_n, x) \geq \Theta(x, y_n),$$

and hence

$$\langle\frac{K'(y_{n_i}) - K'(x_{n_i})}{r}, \eta(x, y_{n_i})\rangle + \varphi(x) - \varphi(y_{n_i}) \geq \Theta(x, y_{n_i}).$$

By conditions (i)-(c), (ii), and $y_{n_i} \rightharpoonup p$, we have

$$\Theta(x, p) + \varphi(p) - \varphi(x) \leq 0, \quad \forall x \in H.$$

For $0 < t \leq 1$, let $x_t = tx + (1-t)p$. It implies that

$$\Theta(x_t, p) + \varphi(p) - \varphi(x_t) \leq 0.$$

From the convexity of equilibrium bifunction $\Theta(x,y)$ in the second variable y, we have

$$\begin{aligned} 0 &= \Theta(x_t,x_t)+\varphi(x_t)-\varphi(x_t) \\ &\leq t\Theta(x_t,x)+(1-t)\Theta(x_t,p)+t\varphi(x)+(1-t)\varphi(p)-\varphi(x_t) \\ &\leq t[\Theta(x_t,x)+\varphi(x)-\varphi(x_t)], \end{aligned}$$

and hence $\Theta(x_t,x)+\varphi(x)-\varphi(x_t)\geq 0$. This implies that

$$\Theta(p,x)+\varphi(x)-\varphi(p)\geq 0, \quad \forall x\in H.$$

Therefore, $p\in\Omega$. Finally we prove $p\in F(W)=\bigcap_{n=1}^{\infty}F(W_n)$, where $F(W_n)=\bigcap_{n=1}^{\infty}F(T_i)$, $n\geq 1$ and $F(W_{n+1})\subset F(W_n)$. In fact, if $p\notin F(W)$, then exists a positive integer m such that $p\notin F(T_m)$, and so $p\notin\bigcap_{i=1}^{m}F(T_i)$. Hence for any $n\geq m, p\notin\bigcap_{i=1}^{n}F(T_i)=F(W_n)$, *i.e.*, $p\notin W_np$. This together with $p=S(s)p$, $\forall s\geq 0$ shows $p=S(s)p\neq S(s)P_{C_2}W_np$, $\forall s\geq 0$. Therefore $p\neq S_nP_{C_2}W_np$, $\forall n\geq m$. By Opial's condition and (3.19), we obtain that

$$\begin{aligned} &\liminf_{i\to\infty}\|y_{n_i}-p\| \\ <\ &\liminf_{i\to\infty}\|y_{n_i}-S_{n_i}P_{C_2}W_{n_i}p\| \\ \leq\ &\liminf_{i\to\infty}\|y_{n_i}-S_{n_i}P_{C_2}W_{n_i}y_{n_i}\|+\liminf_{i\to\infty}\|S_{n_i}P_{C_2}W_{n_i}y_{n_i}-S_{n_i}P_{C_2}W_{n_i}p\| \\ \leq\ &\liminf_{i\to\infty}\|y_{n_i}-p\|. \end{aligned}$$

This is a contradiction. Hence we get $p\in\Gamma$. The conclusion $p\in F(W)\bigcap\Omega\bigcap F(\mathcal{S})$ is proved. Therefore, from Lemma 2.11, (3.18) and (3.21), we have

$$\begin{aligned} &\limsup_{n\to\infty}\langle u+[\gamma f-(I+\mu A)]x^*,x_n-x^*\rangle \\ =\ &\limsup_{n\to\infty}\langle u+[\gamma f-(I+\mu A)]x^*,y_n-x^*\rangle \\ =\ &\lim_{i\to\infty}\langle u+[\gamma f-(I+\mu A)]x^*,y_{n_i}-x^*\rangle \\ =\ &\langle u+[\gamma f-(I+\mu A)]x^*,p-x^*\rangle \\ \leq\ &0. \end{aligned}$$

Step 5. We prove that $\{x_n\}$ and $\{y_n\}$ converge strongly to x^*.

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ =\ &\|\alpha_n(u+\gamma f(y_n)-\overline{A}x^*)+\beta_n(x_n-x^*) \\ &+((1-\beta_n)I-\alpha_n\overline{A})(S_nP_{C_2}W_ny_n-x^*)\|^2 \\ \leq\ &\|\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n\overline{A})(S_nP_{C_2}W_ny_n-x^*)\|^2 \\ &+2\alpha_n\langle u+\gamma f(y_n)-\overline{A}x^*,x_{n+1}-x^*\rangle \\ \leq\ &[\beta_n\|x_n-x^*\|+((1-\beta_n-\alpha_n(1+\mu)\overline{\gamma})\|x_n-x^*\|]^2 \\ &+2\alpha_n\langle u+\gamma f(x^*)-\overline{A}x^*\rangle+2\alpha_n\langle\gamma f(y_n)-\gamma f(x^*),x_{n+1}-x^*\rangle \\ \leq\ &((1-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-x^*\|^2+2\alpha_n\gamma\xi\|x_n-x^*\|\|x_{n+1}-x^*\| \\ &+2\alpha_n\langle u+\gamma f(x^*)-\overline{A}x^*,x_{n+1}-x^*\rangle \\ \leq\ &((1-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-x^*\|^2+\alpha_n\gamma\xi\{\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\} \\ &+2\alpha_n\langle u+\gamma f(x^*)-\overline{A}x^*,x_{n+1}-x^*\rangle. \end{aligned}$$

Hence

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ \leq\ & \tfrac{(1-\alpha_n(1+\mu)\overline{\gamma})^2+\alpha_n\gamma\xi}{1-\alpha_n\gamma\xi}\|x_n - x^*\|^2 + \tfrac{2\alpha_n}{1-\alpha_n\gamma\xi}\langle u + \gamma f(x^*) - \overline{A}x^*, x_{n+1} - x^*\rangle \\ =\ & [1 - \tfrac{2((1+\mu)\overline{\gamma}-\gamma\xi)\alpha_n}{1-\alpha_n\gamma\xi}]\|x_n - x^*\|^2 + \tfrac{((1+\mu)\alpha_n\overline{\gamma_n})^2}{1-\alpha_n\gamma\xi}\|x_n - x^*\|^2 \\ & +\tfrac{2\alpha_n}{1-\alpha_n\gamma\xi}\langle u + \gamma f(x^*) - \overline{A}x^*, x_{n+1} - x^*\rangle \\ \leq\ & [1 - \tfrac{2((1+\mu)\overline{\gamma}-\gamma\xi)\alpha_n}{1-\alpha_n\gamma\xi}]\|x_n - x^*\|^2 + \tfrac{2((1+\mu)\overline{\gamma}-\gamma\xi)\alpha_n}{1-\alpha_n\gamma\xi} \\ & \times\{\tfrac{(\alpha_n(1+\mu)\gamma^2)M_4}{2((1+\mu)\overline{\gamma}-\gamma\xi)} + \tfrac{1}{(1+\mu)\overline{\gamma}-\gamma\xi}\langle u + \gamma f(x^*) - \overline{A}x^*, x_{n+1} - x^*\rangle\}, \end{aligned}$$

where $M_4 = sup\{\|x_n - x^*\| : n \geq 1\|\}$.
Taking $a_n = \|x_n - x^*\|^2, \gamma_n = \frac{2((1+\mu)\overline{\gamma}-\gamma\xi)\alpha_n}{1-\alpha_n\gamma\xi}$ and $\delta_n = \frac{2((1+\mu)\overline{\gamma}-\gamma\xi)\alpha_n}{1-\alpha_n\gamma\xi} \times \{\frac{(\alpha_n(1+\mu)\gamma^2)M_4)}{2((1+\mu)\overline{\gamma}-\gamma\xi)} + \frac{1}{(1+\mu)\overline{\gamma}-\gamma\xi}\langle u + \gamma f(x^*) - \overline{A}x^*, x_{n+1} - x^*\rangle\}$, by the assumption of Theorem 3.1, we can see all the conditions in Lemma 2.6 are satisfied. So the sequence $x_n \to x^* \in \Gamma$. This completes the proof. □

In the case that $C_1 = C_2 = C$, we have the following result.

Corollary 3.1. *Let C be two nonempty closed convex subsets of a real Hilbert space H. Let $\varphi : C \to \mathbb{R}$ be a lower semicontinous and convex function, $f : H \to H$ be a contractive mapping with a contractive constant $\xi \in (0,1)$ and let $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions* (H1)-(H3). *Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k-$strict pseudo-contractions and $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by* (2.1). *For each $n \geq 1$, let $W_n : C \to C$ be the $W-$mapping define by* (2.2). *Let $\mathcal{S}$=$\{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semi-group on C. Let $\mu > 0$, $\gamma > 0$, $r > 0$ be three constants and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < (1 + \mu\overline{\gamma})/\xi$. For given $x_0 \in H$ arbitrarily and fixed $u \in H$, $\{x_n\}, \{y_n\}$ are the sequences generated by*

$$\begin{cases} \Theta(y_n, x) + \varphi(x) - \varphi(y_n) + \frac{1}{r}\langle K'(y_n) - K'(x_n), \eta(x, y_n)\rangle, \ \forall x \in C, \\ x_{n+1} = \alpha_n(u + \gamma f(y_n)) + \beta_n x_n \\ \qquad +((1 - \beta_n)I - \alpha_n(I + \mu A))\frac{1}{t_n}\int_0^{t_n} S(s)W_n y_n ds, \ \ \forall n \geq 1, \end{cases} \tag{3.22}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $(0,1)$ and $\{t_n\}$ is a sequence in $(0,\infty)$. If the following conditions are satisfied:

(i) *$\eta : C \times C \to H$ is $\lambda-$Lipschitz continuous such that*
 (a) *$\eta(x, y) + \eta(y, x) = 0, \quad \forall x, y \in C;$*
 (b) *$\eta(\cdot, \cdot)$ is affine in the first variable;*

(c) *for each fixed $x \in C$, the mapping $y \mapsto \eta(x,y)$ is sequentially continuous from the weak topology to the weak topology;*

(ii) *$K : C \to \mathbb{R}$ is η-strongly convex with constant $\mu > 0$, and its derivative K' is not only sequentially continuous from the weak topology to the strong topology but also Lipschitz continuous with a Lipschitz constant $\nu > 0$ and $\mu \geq \lambda\nu$;*

(iii) *for each $x \in C$, there exists a bounded subset $D_x \subset C$ and a point $z_x \in C$ such that for any $y \in C_1 \setminus D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

(iv) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty} \beta \leq \limsup_{n\to\infty} \beta_n < 1$;

(vi) $\lim_{n\to\infty} t_n = \infty$ *with bounded* $\sup_{n\geq 1} |t_n - t_{n+1}|$;

(vii) $\Gamma := F(W) \bigcap \Omega \bigcap F(\mathcal{S}) \neq \emptyset,\ F(W) = \bigcap_{i=1}^{\infty} F(T_i)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (3.22) converge strongly to $x^* \in \Gamma$ which solves the following optimization problem (OP).

$$\min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x)$$

provided S_r is firmly nonexpansive, i.e.,

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle, \quad \forall x, y \in C.$$

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $f : H \to H$ be a contractive mapping with a contractive constant $\xi \in (0,1)$. Let $\{V_i : C \to C\}_{i=1}^{\infty}$ be a countable family of uniformly $k-$strict pseudo-contractions and $\{T_i : C \to C\}_{i=1}^{\infty}$ be the countable family of nonexpansive mappings defined by (2.1). For each $n \geq 1$, let $W_n : C \to C$ be the $W-$mapping define by (2.2). Let $\mathcal{S}$=$\{S(s) : 0 \leq s < \infty\}$ be a nonexpansive semi-group on C. Let $\mu > 0$, $\gamma > 0$ be two constants and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < (1 + \mu\overline{\gamma})/\xi$. For given $x_0 \in H$ arbitrarily and fixed $u \in H$, $\{x_n\}$ are the sequence generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n(u + \gamma f(x_n)) + \beta_n x_n \\ &\quad + ((1 - \beta_n)I - \alpha_n(I + \mu A))\frac{1}{t_n}\int_0^{t_n} S(s)W_n x_n ds, \quad \forall n \geq 1, \end{aligned} \tag{3.23}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in (0,1) *and $\{t_n\}$ is a sequence in $(0, \infty)$. If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\Gamma := F(W) \bigcap F(\mathcal{S}) \neq \emptyset,\ F(W) = \bigcap_{i=1}^{\infty} F(T_i)$;
(iv) $\lim_{n\to\infty} t_n = \infty$ *with bounded* $\sup_{n\geq 1} |t_n - t_{n+1}|$.

Then the sequence $\{x_n\}$ *defined by* (3.23) *converge strongly to* $x^* \in \Gamma$ *which solves the following optimization problem (OP)*

$$\min_{x\in\Gamma} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x).$$

Proof. Put $\Theta(x, y) = 0$, $\varphi(x) = 0$ for all $x, y \in H$ and $r = 1$. Take $K(x) = \frac{\|x\|^2}{2}$ for all $x, y \in H$. Then we get $y_n = x_n$ in Corollary 3.2. Then the conclusion follows. □

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