Nonlinear Functional Analysis and Applications
Vol. 29, No. 2 (2024), pp. 487-499
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2024.29.02.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED POINT THEOREMS IN b-MENGER INNER PRODUCT SPACES

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Abstract. The main motivation for this paper is to investigate the fixed point property for nonlinear contraction defined on b-Menger inner product spaces. First, we introduce a b-Menger inner product spaces, then the topological structure is discussed and the probabilistic Pythagorean theorem is given and established. Also we prove the existence and uniqueness of fixed point in these spaces. This result generalizes and improves many previously known results.

1. Introduction and preliminaries

Hilbert spaces are complete normed linear spaces in which the norm is derivable from an inner product satisfying appropriate conditions. Correspondingly, we would expect a theory of probabilistic inner product or probabilistic Hilbert spaces. We recall the classical inner product space notion. The symbol η will denote the null vector of a real vector space.

Definition 1.1. Let Z be a real vector space. An inner product on Z is a function $\langle .,. \rangle$ from $Z \times Z$ into $\mathbb{R}$ such that

(1) $\langle p,p \rangle \geq 0$ for all $p \in Z$, $\langle p,p \rangle = 0$ if and only if $p = \eta$,
(2) $\langle p,q \rangle = \langle q,p \rangle$, for all $p,q \in Z$,
(3) $\langle \alpha p,q \rangle = \alpha \langle p,q \rangle$, for all $p,q \in Z$ and $\alpha \in \mathbb{R}$,
(4) $\langle p+q,r \rangle = \langle p,r \rangle + \langle q,r \rangle$, for all $p,q,r \in Z$.

Then $(Z, \langle .,. \rangle)$ is called an inner product space.

[0]Received September 12, 2023. Revised January 7, 2024. Accepted February 25, 2024.
[0]2020 Mathematics Subject Classification: 54A05, 54A10, 47H10.
[0]Keywords: b-Menger space, inner product, orthogonality, fixed point.

We give some definitions and notions from fixed point theory and probabilistic metric spaces theory that we will use in the sequel. For more details, we refer the reader to [4, 6, 7, 10, 11, 12, 13].

Definition 1.2. A function $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if the following conditions are satisfied for any $\lambda, \mu, \nu, \xi \in [0,1]$:

(1) $T(\mu, 1) = \mu$,
(2) $T(\mu, \nu) = T(\nu, \mu)$,
(3) $T(\mu, \nu) \geq T(\lambda, \xi)$, for $\mu \geq \lambda$, $\nu \geq \xi$,
(4) $T(T(\mu, \nu), \xi) = T(\mu, T(\nu, \xi))$.

If T is a t-norm, then its dual triangular conorm (for short, t-conorm) $T^* : [0,1] \times [0,1] \to [0,1]$ is given by $T^*(\mu, \nu) = 1 - T(1-\mu, 1-\nu)$ for $\mu, \nu \in [0,1]$.

Definition 1.3. A t-norm T is said to be of H-type if the sequence $\{T^n(x)\}$ is equicontinuous at $x = 1$, where $T^1(x) = x$, $T^n(x) = T(T^{n-1}(x))$ for every $n \geq 2$.

A t-norm T is considered continuous if it exhibits continuity at every point $(\mu, \nu) \in [0,1] \times [0,1]$. We mention some usual continuous t-norms:

(1) The minimum t-norm $T_M(\mu, \nu) = min(\mu, \nu)$.
(2) The product t-norm $T_p(\mu, \nu) = \mu\nu$.
(3) The Lukasiewicz t-norm $T_L(\mu, \nu) = max(\mu + \nu - 1, 0)$.

Remark 1.4. We have

(1) $T \leq T_M$ for each t-norm T.
(2) T_M is a t-norm of H-type but there are many t-norms T of H-type with $T \neq T_M$ (see [3]).
(3) The dual t-conorm of T_M is $T^*_M(\mu, \nu) = max(\mu, \nu)$.

Definition 1.5. A function $F : \mathbb{R} \to [0,1]$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{\gamma \in \mathbb{R}} F(\gamma) = 0$ and $\sup_{\gamma \in \mathbb{R}} F(\gamma) = 1$. The class of all distribution functions is denoted by $\mathcal{D}$.

For $F \in \mathcal{D}$, we consider the following limits

$$\ell^- F(\gamma) = \lim_{\gamma' \to \gamma^-} F(\gamma'), \quad \ell^+ F(\gamma) = \lim_{\gamma' \to \gamma^+} F(\gamma').$$

Let $\mathcal{D}^+ = \{F \in \mathcal{D} : F(0) = 0\}$. A simple example of distribution function is the unit step function in $\mathcal{D}^+$

$$\epsilon_a(\gamma) = \begin{cases} 0, & \text{if} \quad \gamma \leq a, \\ 1, & \text{if} \quad \gamma > a. \end{cases}$$

Definition 1.6. ([6]) A b-Menger space is a quadruple (Z, F, T, K), where Z is a nonempty set, F is a function from $Z \times Z$ into $\mathcal{D}^+$, T is a t-norm, $K \geq 1$ is a real number, and the following conditions are satisfied for all $p, q, r \in Z$ and $a, b > 0$

(1) $F_{p,q} = \varepsilon_0 \;\Leftrightarrow\; p = q$,
(2) $F_{p,q} = F_{q,p}$,
(3) $F_{p,q}(K(a+b)) \geq T(F_{p,r}(a), F_{r,q}(b))$.

Mbarki and Oubrahim [6, 8, 9] point out that if the t-norm T of a b-Menger space is continuous, then (Z, F, T, L) is a first countable Hausdorff topological space in the topology τ, that is, the family of sets $\{V_p(\gamma) : \gamma > 0\}$ is a base of neighborhoods of point $p \in Z$ for τ, where

$$V_p(\gamma) = \{q \in Z : F_{p,q}(\gamma) > 1 - \gamma\}.$$

Probabilistic normed spaces were introduced by Erstnev [15] in 1963. We define a b-Menger normed space as follows.

Definition 1.7. A b-Menger normed space is a quadruple (Z, F, T, K) where Z is a real vector space, F is a function from Z into $\mathcal{D}^+$, T is a t-norm, $K \geq 1$ is a real number, and the following conditions are satisfied:

(1) $F_p = \varepsilon_0$ if and only if $p = \eta$,
(2) $F_{\alpha p}(\gamma) = F_p(\frac{\gamma}{|\alpha|})$ for all $p \in Z$, $\gamma \in \mathbb{R}$ and $\alpha \neq 0$,
(3) $F_{p+q}(K(a+b)) \geq T\left(F_p(a), F_q(b)\right)$ for all $p, q \in Z$ and $a, b \geq 0$.

It is evident that, if (Z, F, T, K) is a b-Menger normed space, and for each $(p, q) \in Z \times Z$, $\overline{F} : Z \times Z \to \mathcal{D}^+$ is defined by $\overline{F}_{p,q} = F_{p-q}$, then $(Z, \overline{F}, T, K)$ is a b-Menger space.

Many important applications of the theory of probabilistic metric spaces can be found in approximation of random signals, integral equations, stochastic optimizations and quantum particle physics (see [2],[5]).

2. b-Menger inner product spaces

The first attempt to develop a theory of probabilistic inner product spaces was made by Senechal [14] in 1965. After then a simplified form of this definition was given by Zhang in [16]. Inspired by the work of these researchers, we introduce a b-Menger inner product space as follows.

Definition 2.1. A b-Menger inner product space is a quadruple (Z, F, T, K) where Z is a real vector space, F is a function from $Z \times Z$ into $\mathcal{D}$, T is a t-norm, $K \geq 1$ is a real number, and the following conditions are satisfied for all $p, q, r \in Z$ and $a, b > 0$

(1) $F_{p,p}(0) = 0$, $F_{p,p} = \epsilon_0$ if and only if $p = \eta$,

(2) $F_{p,q} = F_{q,p}$,
(3) For any real number $\alpha \in \mathbb{R}$,

$$\begin{cases} F_{\alpha p,q}(\gamma) = F_{p,q}(\frac{\gamma}{\alpha}) \text{if} \alpha > 0, \\ F_{\alpha p,q} = \epsilon_0 \text{if} \alpha = 0, \\ F_{\alpha p,q}(\gamma) = 1 - \ell^+ F_{p,q}(\frac{\gamma}{\alpha}) \text{if} \alpha < 0, \end{cases}$$

(4) $F_{p+q,r}(K(a+b)) \geq T(F_{p,r}(a), F_{q,r}(b))$,
(5) $F_{p,q}(ab) \geq T(F_{p,p}(a^2), F_{q,q}(b^2))$.

Every classical real inner product space is a b-Menger inner product space. Moreover, we have the following example.

Example 2.2. Let $(Z, \langle ., . \rangle)$ be a real inner product space, T be a t-norm and consider the mapping $F_{p,q}(\gamma) = \epsilon_{K\langle p,q \rangle}(\gamma)$ with $K \geq 1$. Then (Z, F, T, K) is a b-Menger inner product space.

We verify that all conditions of Definition 2.1 are satisfied. It is easy to check the conditions (1) and (2).

For each $\alpha \in \mathbb{R}$, if $\alpha > 0$, then

$$F_{\alpha p,q}(\gamma) = \epsilon_{K\langle \alpha p,q \rangle}(\gamma) = \epsilon_{K\langle p,q \rangle}(\frac{\gamma}{\alpha}) = F_{p,q}(\frac{\gamma}{\alpha}).$$

If $\alpha = 0$, then

$$F_{\alpha p,q}(\gamma) = \epsilon_{K\langle \alpha p,q \rangle}(\gamma) = \epsilon_{K\langle \eta,q \rangle}(\gamma) = \epsilon_0(\gamma),$$

and if $\alpha < 0$,

$$\begin{aligned} F_{\alpha p,q}(\gamma) &= \epsilon_{K\langle \alpha p,q \rangle}(\gamma) = \epsilon_{-K\langle p,q \rangle}(-\frac{\gamma}{\alpha}) \\ &= 1 - \ell^+ \epsilon_{K\langle p,q \rangle}(\frac{\gamma}{\alpha}) = 1 - \ell^+ F_{p,q}(\frac{\gamma}{\alpha}). \end{aligned}$$

Then the condition (3) is satisfied. Let $a, b > 0$ and $p, q, r \in Z$. If $a + b > \langle p+q, r \rangle$, then it is evident that

$$F_{p+q,r}(K(a+b)) = \epsilon_{K\langle p+q,r \rangle}(K(a+b)) = 1 \geq T(F_{p,r}(a), F_{q,r}(b)).$$

If $a+b \leq \langle p+q, r \rangle = \langle p, r \rangle + \langle q, r \rangle$, then we have either $a \leq \langle p, r \rangle$ or $b \leq \langle q, r \rangle$, and so

$$\begin{aligned} F_{p+q,r}(K(a+b)) &= \epsilon_{K\langle p+q,r \rangle}(K(a+b)) = 0 \\ &= T\left(\epsilon_{K\langle p,r \rangle}(a), \epsilon_{K\langle q,r \rangle}(b)\right) \\ &= T(F_{p,r}(a), F_{q,r}(b)). \end{aligned}$$

This means that the condition (4) is satisfied.

Let $a, b > 0$ and $p, q \in Z$. If $ab > K\langle p, q \rangle$, then it is evident that

$$F_{p,q}(ab) = \epsilon_{K\langle p,q \rangle}(ab) = 1 \geq T(F_{p,p}(a^2), F_{q,q}(b^2)).$$

If $ab \leq K\langle p,q\rangle$, then from $\langle p,q\rangle \leq \sqrt{\langle p,p\rangle\langle q,q\rangle}$ we have either $a^2 \leq K\langle p,p\rangle$ or $b^2 \leq K\langle q,q\rangle$, and so

$$F_{p,q}(ab) = \epsilon_{K\langle p,q\rangle}(ab) = 0 = T\left(\epsilon_{K\langle p,p\rangle}(a^2), \epsilon_{K\langle q,q\rangle}(b^2)\right) = T\left(F_{p,p}(a^2), F_{q,q}(b^2)\right).$$

Hence the condition (5) is also satisfied.

Using the above definition, we study the topological structure and some elementary properties for b-Menger inner product spaces.

Lemma 2.3. *Let (Z,F,T,K) be a b-Menger inner product space. Then*

(1) *$F_{\eta,p}(\gamma) = \epsilon_0(\gamma)$ for all $p \in Z$ and $\gamma \in \mathbb{R}$,*
(2) *$F_{r,p+q}(K(a+b)) \geq T(F_{r,p}(a), F_{r,q}(b))$ for all $p,q,r \in Z$ and $a,b > 0$,*
(3) *$F_{p,q}(ab) \geq T(F_{p,p}(a^2), F_{q,q}(b^2))$ for all $p,q,r \in Z$ and $a,b < 0$,*
(4) *$F_{-p,q}(\gamma) = 1 - \ell^+F_{p,q}(-\gamma)$ and $\ell^+F_{-p,q}(\gamma) = 1 - F_{p,q}(-\gamma)$, for all $p,q \in Z$ and $\gamma \in \mathbb{R}$,*
(5) *$F_{-p,-q}(\gamma) = F_{p,q}(\gamma)$ and $F_{-p,q}(\gamma) = F_{p,-q}(\gamma)$, for all $p,q \in Z$ and $\gamma \in \mathbb{R}$,*
(6) *$F_{-p,p}(\gamma) = 1$ for all $p \in Z$ and $\gamma > 0$,*
(7) *$F_{p-q,q}(K\gamma) \geq F_{p,q}(\gamma)$ for all $p,q \in Z$ and $\gamma > 0$,*
(8) *$F_{p+q,r}(K(a+b)) \leq T^*(F_{p,r}(a), F_{q,r}(b))$ for all $p,q,r \in Z$ and $a,b < 0$, where T is continuous.*

Proof. (1) It follows from assertion (3) of Definition 2.1.
(2) It follows from conditions (2) and (4) of Definition 2.1.
(3) It follows from (5) of Definition 2.1.
(4) It follows from (3) of Definition 2.1.
(5) From (4) we deduce that the assertion holds.
(6) Let $\gamma > 0$ and $p \in Z$. From (1) of Definition 2.1 it follows that $F_{p,p}(-\gamma) \leq F_{p,p}(0) = 0$, which implies that $\ell^+F_{p,p}(-\gamma) = 0$. Thus, from (3) of Definition 2.1 we obtain

$$F_{-p,p}(\gamma) = 1 - \ell^+F_{p,p}(-\gamma) = 1.$$

(7) Let $\gamma > 0$ and $p,q \in Z$. For any $\epsilon \in (0,\gamma)$, from (4) of Definition 2.1 and Lemma 2.3 (6) we obtain

$$\begin{aligned} F_{p-q,q}(K\gamma) &= F_{p-q,q}(K(\gamma-\epsilon+\epsilon) \\ &\geq T(F_{p,q}(\gamma-\epsilon), F_{-q,q}(\epsilon)) \\ &= T(F_{p,q}(\gamma-\epsilon), 1). \end{aligned}$$

Then

$$F_{p-q,q}(K\gamma) \geq F_{p,q}(\gamma-\epsilon). \tag{2.1}$$

Thus, letting $\epsilon \to 0^+$ in (2.1), we obtain the inequality by the left-continuity of $F_{p,q}$.

(8) Let $a, b < 0$ and $p, q, r \in Z$. From (4) of Definition 2.1 it follows that

$$F_{p+q,-r}(K(-a-b)) \geq T(F_{p,-q}(-a), F_{q,-r}(-b)). \tag{2.2}$$

Since T is continuous, from (2.2) we have

$$\ell^+ F_{p+q,-r}(K(-a-b)) \geq T(\ell^+ F_{p,-q}(-a), \ell^+ F_{q,-r}(-b)). \tag{2.3}$$

Thus, by Lemma 2.3 (4) and (2.3) we get

$$\begin{aligned} F_{p+q,r}(K(a+b)) &= 1 - \ell^+ F_{p+q,-r}(K(-a-b)) \\ &\leq 1 - T(\ell^+ F_{p,-r}(-a), \ell^+ F_{q,-r}(-b)) \\ &= 1 - T(1 - F_{p,r}(a), 1 - F_{q,r}(b)) \\ &= T^*(F_{p,r}(a), F_{q,r}(b)). \end{aligned}$$

□

In the next theorem we show that every b-Menger inner product space is a b-Menger normed space.

Theorem 2.4. *Let (Z, F, T, K) be a b-Menger inner product space. Define $\overline{F} : Z \to \mathcal{D}^+$ by*

$$\overline{F}_p(K\gamma) = \begin{cases} 0 & \text{if} \quad \gamma \leq 0, \\ F_{p,p}(\gamma^2) & \text{if} \quad \gamma > 0. \end{cases}$$

If $T = T_M$, then $(Z, \overline{F}, T_M, K)$ is a b-Menger normed space.

Proof. We verify that all conditions of Definition 1.7 are satisfied. It is clear that the conditions (1) and (2) follow from conditions (1) and (2) of Definition 2.1 respectively. For any $p, q \in Z$ and $a, b > 0$, setting, $u = F_{p,p}(a^2)$, $v = F_{p,q}(ab)$ and $w = F_{q,q}(b^2)$, by the condition (5) of Definition 2.1 we have $v \geq T_M(u, w)$. Moreover, by the condition (4) of Definition 2.1 we obtain

$$\begin{aligned} \overline{F}_{p+q}(K(a+b)) &= F_{p+q,p+q}(K(a+b)^2) \\ &\geq T_M(T_M(u, v), T_M(v, w)) \\ &= T_M(T_M(u, w), v) \\ &\geq T_M(T_M(u, w), T_M(u, w)) \\ &= T_M(u, w) \\ &= T_M(\overline{F}_p(a), \overline{F}_q(b)). \end{aligned}$$

Then the condition (3) holds. □

Theorem 2.5. *Let (Z, F, T, K) be a b-Menger inner product space with continuous t-norm T. Then it is first countable Hausdorff topological vector space* *with the topology τ_F whose neighborhood base of origin η is $\{V(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1]\}$, where*

$$V(\epsilon,\lambda) = \{p \in Z : F_{p,p}(\epsilon^2) > 1-\lambda\}$$

Proof. Using the structure of topological vector space given in [1], we can verify the following assertions.

(a) If $L_1 = V(\epsilon_1,\lambda_1)$ and $L_2 = V(\epsilon_2,\lambda_2)$, there is $L_0 = V(\epsilon_0,\lambda_0)$ such that $L_0 \subset L_1 \cap L_2$.
(b) If $L = V(\epsilon,\lambda)$, there is $L' = V(\epsilon',\lambda')$ such that $L' + L' \subset L$.
(c) If $L = V(\epsilon,\lambda)$ and $\alpha \in \mathbb{R}$ with $|\alpha| \le 1$, it holds $\alpha L \subset L$.
(d) If $L = V(\epsilon,\lambda)$ and each $p \in Z$, there is $\mu > 0$ such that $\mu p \in L$.
(e) If $p \in Z$ and $p \neq \eta$, there exist $\epsilon_0 > 0$ and $\lambda_0 \in (0,1]$ such that $p \notin V(\epsilon_0,\lambda_0)$. Let $\epsilon_n = \lambda_n = \frac{1}{n}$. The sets $\{V(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1]\}$ and $\{V(\epsilon_n,\lambda_n) : n \in \mathbb{N}\}$ are equivalent. Since $\{V(\epsilon_n,\lambda_n) : n \in \mathbb{N}\}$ is countable, τ_F is first countable.

□

3. Orthogonality

In this section we study the orthogonality of two vectors in a b-Menger inner product space.

Definition 3.1. Let (Z,F,T,K) be a b-Menger inner product space. Two vectors $p,q \in Z$ are orthogonal if $F_{p,q} = \epsilon_0$ and we write $p \perp q$.

Lemma 3.2. *Let (Z,F,T,K) be a b-Menger inner product space. Then the following assertions hold:*

(1) *$p \perp \eta$, for all $p \in Z$,*
(2) *If $p \perp q$, then $q \perp p$,*
(3) *If $p \perp p$, then $p = \eta$,*
(4) *If $p \perp q$, then for any $\alpha \in \mathbb{R}$, $p \perp \alpha q$,*
(5) *If T is a continuous t-norm, $p \perp q$ and $p \perp r$, then $p \perp (q+r)$.*

Proof. The properties (1)-(4) follow immediately from the conditions (1), (2) and (3) of Definition 2.1. We prove the property (5). Let $\gamma > 0$, from Lemma 2.3 (2) we get

$$F_{p,q+r}(\gamma) \ge T(F_{p,q}(\frac{\gamma}{2K}), F_{p,r}(\frac{\gamma}{2K})) = T(1,1) = 1.$$

Let $\gamma < 0$, in view of Lemma 2.3 (8) we obtain

$$F_{p,q+r}(\gamma) \le T^*(F_{p,q}(\frac{\gamma}{2K}), F_{p,r}(\frac{\gamma}{2K})) = T^*(0,0) = 0.$$

Hence, by the left-continuity of $F_{p,q+r}$, we have $F_{p,q+r}(\gamma) = \epsilon_0(\gamma)$ for every $\gamma \in \mathbb{R}$, then $p \perp (q+r)$. □

Theorem 3.3. *Let* (Z, F, T, K) *be a b-Menger inner product space. Consider* $p, q \in Z$ *such that* $p \perp q$.

(1) *Let* $\alpha \in \mathbb{R}$. *Then*

$$F_{p+\alpha p,q}(\frac{\gamma}{K}) \le F_{p,p}(\gamma) \le F_{p+\alpha q,p}(K\gamma)$$

and

$$F_{q+\alpha p,q}(\frac{\gamma}{K}) \le F_{q,q}(\gamma) \le F_{q+\alpha p,q}(K\gamma),$$

for all $\gamma \in \mathbb{R}$.

(2) *For all* $\gamma > 0$,

$$\begin{aligned} T_M(F_{p,p}(K^2\gamma), F_{q,q}(K^2\gamma)) &\ge F_{p\pm q,p\pm q}(\gamma) \\ &\ge \sup_{\gamma=K^2(a+b),a>0,b>0} T(F_{p,p}(a), F_{q,q}(b)). \end{aligned} \tag{3.1}$$

Proof. The assertion (2) is a Pythagorean theorem in b-Menger inner product space.

(1) We show the first inequality, the second inequality is proved similarly. Let $\gamma > 0$ and $\mu \in (0, \gamma)$. Since $F_{\alpha q,p}(\gamma) = 1$ from Lemma 3.2 (4), by condition (4) of Definition 2.1 we obtain

$$F_{p+\alpha q,p}(K\gamma) \ge T(F_{p,p}(\gamma - \mu), F_{\alpha q,p}(\mu)) = F_{p,p}(\gamma - \mu).$$

We conclude that $F_{p+\alpha q,p}(K\gamma) \ge F_{p,p}(\gamma)$ by taking $\mu \to 0^+$. On the other hand, since $F_{-\alpha q,p}(\mu) = 1$, by condition (4) of Definition 2.1 we get

$$\begin{aligned} F_{p,p}(K\gamma) &= F_{p+\alpha q-\alpha q,p}(K\gamma) \\ &\ge T(F_{p+\alpha q,p}(\gamma - \mu), F_{-\alpha q,p}(\mu)) \\ &= F_{p+\alpha q,p}(\gamma - \mu). \end{aligned}$$

We deduce that $F_{p+\alpha q,p}(\gamma) \le F_{p,p}(K\gamma)$ by taking $\mu \to 0^+$. Let $\gamma < 0$. Then using condition (1) of Definition 2.1 we obtain $F_{p,p}(\gamma) = 0$. From Lemma 2.3 (8) we get

$$F_{p+\alpha q,p}(t) \le T^*(F_{p,p}(\frac{\gamma}{2K}), F_{\alpha q,p}(\frac{\gamma}{2K}) = T^*(0,0) = 0.$$

Taking $\gamma \to 0^-$, we have $F_{p+\alpha q,p}(0) = 0$ by the left-continuity of $F_{p+\alpha q,p}$, the inequality holds.

(2) Let $\epsilon \in (0, \gamma)$. By Lemma 2.3 (6) and condition (4) of Definition 2.1 we obtain

$$F_{p+q,-q}(\epsilon) \ge T\left(F_{p,-q}(\frac{\epsilon}{2}), F_{q,-q}(\frac{\epsilon}{2})\right) = T(1,1) = 1.$$

Thus, from Theorem 3.3 (1), using condition (4) of Definition 2.1 we obtain

$$\begin{aligned} F_{p,p}(K^2\gamma) &\geq F_{p+q,p}(K\gamma) = F_{p+q,p+q-q}(K(\gamma-\epsilon+\epsilon)) \\ &\geq T\left(F_{p+q,p+q}(\gamma-\epsilon), F_{p+q,-q}(\epsilon)\right) = F_{p+q,p+q}(\gamma-\epsilon). \end{aligned}$$

We deduce that $F_{p,p}(K^2\gamma) \geq F_{p+q,p+q}(\gamma)$ by letting $\epsilon \to 0^+$. Likewise, $F_{q,q}(K^2\gamma) \geq F_{p+q,q+p}(\gamma)$. For each $a, b > 0$ with $K^2(a+b) = \gamma$, using Theorem 3.3 (1), from condition (4) of Definition 2.1 we get

$$\begin{aligned} F_{p+q,p+q}(\gamma) &= F_{p+q,p+q}(K^2(a+b)) \\ &\geq T\left(F_{p+q,p}(Ka), F_{p+q,q}(Kb)\right) \\ &= T(F_{p,p}(a), F_{q,q}(b)). \end{aligned}$$

Then we conclude that

$$\begin{aligned} T_M(F_{p,p}(K^2\gamma), F_{q,q}(K^2\gamma)) &\geq F_{p+q,p+q}(\gamma) \\ &\geq \sup_{\gamma=K^2(a+b),a>0,b>0} T(F_{p,p}(a), F_{q,q}(b)). \end{aligned} \tag{3.2}$$

Replacing $-q$ by q in (3.2), by Lemma 2.3 (5) we obtain

$$\begin{aligned} T_M(F_{p,p}(K^2\gamma), F_{q,q}(K^2\gamma)) &\geq F_{p-q,p-q}(\gamma) \\ &\geq \sup_{\gamma=K^2(a+b),a>0,b>0} T(F_{p,p}(a), F_{q,q}(b)). \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3), we deduce that (3.1) holds.

□

4. Fixed point theorem in b-Menger inner product spaces

Now we can state and prove the main fixed point theorem of this paper. We need the following concept.

Definition 4.1. Let (Z, F, T, K) be a b-Menger inner product space with T continuous.

(1) A sequence $\{z_n\}$ in Z converges to $z \in Z$, if for each $\epsilon > 0$ and $\lambda \in (0, 1]$ there is $N \in \mathbb{N}$ such that $F_{z_n-z,z_n-z}(\epsilon^2) > 1-\lambda$ for all $n > N$ (equivalently $\lim_{n\to\infty} F_{z_n-z,z_n-z}(\gamma) = 1$ for all $\gamma > 0$).

(2) $\{z_n\}$ is a Cauchy sequence if for each $\epsilon > 0$ and $\lambda \in (0, 1]$ there is $N \in \mathbb{N}$ such that $F_{z_n-z_{n+j},z_n-z_{n+j}}(\epsilon^2) > 1-\lambda$, whenever $n > N$ and $j \in \mathbb{N}$.

(3) (Z, F, T, K) is said to be complete, if each Cauchy sequence in Z converges to some point in Z.

Theorem 4.2. *Let (Z, F, T, K) be a complete b-Menger inner product space such that T is of H-type. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a function such that $\varphi(\gamma) > \gamma$ and $\lim_{n\to\infty} \varphi^n(\gamma) = +\infty$ for all $\gamma > 0$, where $\varphi^n(\gamma)$ is the nth iteration of $\varphi(\gamma)$. Let $A, B : Z \to Z$ be two mappings such that*

$$F_{Bp-Aq,r}(\gamma) \geq F_{p-q,r}(K\varphi(\gamma)) \quad for\ p, q, r \in Z\ and\ \gamma > 0. \tag{4.1}$$

Then A and B have a unique common fixed point in Z.

Proof. From Lemma 2.3 (5), by (4.1) we obtain

$$F_{Ap-Bq,r}(\gamma) = F_{Bq-Ap,-r}(\gamma) \geq F_{q-p,-r}(K\varphi(\gamma)) = F_{p-q,r}(K\varphi(\gamma)) \tag{4.2}$$

for $p, q, r \in Z$ and $\gamma > 0$. Let $p_0 \in Z$ be arbitrary, and we consider the sequence $\{p_n\}$ defined by $p_{2i-1} = Ap_{2i-2}$ and $p_{2i} = Bp_{2i-1}$ for $i \in \mathbb{N}$. Let $\gamma > 0$. For any $r \in Z$, in view of (4.1) and (4.2) we get

$$\begin{aligned} F_{p_{2i}-p_{2i+1},r}(\gamma) &= F_{Bp_{2i-1}-Ap_{2i},r}(\gamma) \\ &\geq F_{p_{2i-1}-p_{2i},r}(K\varphi(\gamma)) \\ &= F_{Ap_{2i-2}-Ap_{2i-1},r}(\varphi(t)) \\ &\geq F_{p_{2i-2}-p_{2i-1},r}(K^2\varphi^2(\gamma)) \\ &\vdots \\ &\geq F_{p_0-p_1,r}(K^{2i}\varphi^{2i}(\gamma)) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} F_{p_{2i-1}-p_{2i},r}(\gamma) &= F_{Ap_{2i-2}-Bp_{2i-1},r}(\gamma) \\ &\geq F_{p_{2i-2}-p_{2i-1},r}(K\varphi(\gamma)) \\ &= F_{Bp_{2i-3}-Ap_{2i-2},r}(\varphi(t)) \\ &\geq F_{p_{2i-3}-p_{2i-2},r}(K^2\varphi^2(\gamma)) \\ &\vdots \\ &\geq F_{p_0-p_1,r}(K^{2i-1}\varphi^{2i-1}(\gamma)). \end{aligned} \tag{4.4}$$

Using (4.3) and (4.4) it follows that

$$F_{p_n-p_{n+1},r}(\gamma) \geq F_{p_0-p_1,r}(K^n\varphi^n(\gamma)) \quad for\ all\ n \in \mathbb{N}. \tag{4.5}$$

Hence, by (4.5) and condition (2) of Definition 2.1 we have

$$\begin{aligned} F_{p_n-p_{n+1},p_n-p_{n+1}}(\gamma) &\geq F_{p_0-p_1,p_n-p_{n+1}}(K^n\varphi^n(\gamma)) \\ &= F_{p_n-p_{n+1},p_0-p_1}(K^n\varphi^n(\gamma)) \\ &\geq F_{p_0-p_1,p_0-p_1}(K^{2n}\varphi^{2n}(\gamma)). \end{aligned} \tag{4.6}$$

Since $\lim_{n\to\infty} K^{2n}\varphi^n(\gamma) = +\infty$, then $\lim_{n\to\infty} F_{p_0-p_1,p_0-p_1}(K^{2n}\varphi^{2n}(\gamma)) = 1$. By (4.6) we conclude that for all $\gamma > 0$,

$$F_{p_n-p_{n+1},p_n-p_{n+1}}(\gamma) \to 1 \ \ as \ \ n \to \infty. \tag{4.7}$$

Next, let $r \in Z$ and $n \in \mathbb{N}$. We shall apply induction to show that, for any $j \in \mathbb{N}$,

$$F_{p_{n+1}-p_{n+j+1},r}(\gamma) \geq T^j(F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma)). \tag{4.8}$$

This is obvious for $j = 1$, since from (4.1) and (4.2) we get

$$F_{p_{n+1}-p_{n+2},r}(\gamma) \geq F_{p_n-p_{n+1},r}(K\varphi(\gamma)) \geq F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma).$$

Next, suppose that (4.8) is true for $j = k$. in view of (4.1) and (4.2), condition (4) of Definition 2.1, the monotonicity of T and by induction hypothesis, we obtain

$$\begin{aligned} F_{p_{n+1}-p_{n+k+2},r}(\gamma) &\geq F_{p_n-p_{n+k+1},r}(K\varphi(\gamma)) \\ &= F_{p_n-p_{n+1}+p_{n+1}-p_{n+k+1},r}(K(\varphi(\gamma) - \gamma + \gamma)) \\ &\geq T\left(F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma), F_{p_{n+1}-p_{n+k+1},r}(\gamma)\right) \\ &\geq T\left(F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma), T^k F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma))\right) \\ &= T^{k+1}(F_{p_n-p_{n+1},r}(\varphi(\gamma) - \gamma)), \end{aligned}$$

which completes the induction. Hence, by (4.8) and assertion (2) of Definition 2.1, we obtain

$$\begin{aligned} F_{p_{n+1}-p_{n+j+1},p_{n+1}-p_{n+j+1}}(\gamma) &\geq T^j\left(F_{p_n-p_{n+1},p_{n+1}-p_{n+j+1}}(\varphi(\gamma) - \gamma)\right) \\ &= T^j\left(F_{p_{n+1}-p_{n+j+1},p_n-p_{n+1}}(\varphi(\gamma) - \gamma)\right) \\ &\geq T^{2j}\left[F_{p_n-p_{n+1},p_n-p_{n+1}}\left(\varphi(\varphi(\gamma){-}\gamma){-}(\varphi(\gamma){-}\gamma)\right)\right] \end{aligned} \tag{4.9}$$

for all $n, j \in \mathbb{N}$.

Fix $\epsilon > 0$ and $\delta \in (0, 1]$. Since T is a t-norm of H-type, there exists $\lambda > 0$ such that

$$T^{2j}(x) > 1 - \lambda, \quad for\ all\ x \in (1 - \lambda, 1]\ and\ j \in \mathbb{N}. \tag{4.10}$$

We have $\varphi(\gamma) > \gamma$ for all $\gamma > 0$, then $\varphi(\varphi(\gamma) - \gamma) - (\varphi(\gamma) - \gamma) > 0$. From (4.7), so $\lim_{n\to\infty} F_{p_n-p_{n+1},p_n-p_{n+1}}(\varphi(\varphi(\gamma) - \gamma) - (\varphi(\gamma) - \gamma)) = 1$ for all $\gamma > 0$. Then, there exists $N \in \mathbb{N}$ such that

$$F_{p_n-p_{n+1},p_n-p_{n+1}}\left(\varphi(\varphi(\epsilon^2) - \epsilon^2) - (\varphi(\epsilon^2) - \epsilon^2)\right) > 1 - \lambda$$

for all $n \geq N$. Thus, by (4.9) and (4.10) we obtain

$$F_{p_{n+1}-p_{n+j+1},p_{n+1}-p_{n+j+1}}(\epsilon^2) > 1 - \lambda$$

for all $n \geq N$ and $j \in \mathbb{N}$. We conclude that $\{p_n\}$ is a Cauchy sequence. Since Z is complete, there exists $v \in Z$ such that $\lim_{n\to\infty} p_n = v$, that is,

$$\lim_{n\to\infty} F_{p_n - v, p_n - v}(\gamma) = 1 \quad for\ all\ \gamma > 0. \tag{4.11}$$

Now we will show that v is a common fixed point of A and B. By (4.1), (4.2), assertion (2) of Definition 2.1 and (4.11), it follows that

$$\begin{aligned} F_{p_{2i}-Av, p_{2i}-Av}(\gamma) &= F_{Bp_{2i-1}-Av, Bp_{2i-1}-Av}(\gamma) \\ &\geq F_{p_{2i-1}-v, p_{2i-1}-v}(K^2\varphi^2(\gamma)) \\ &\to 1 \ \ as \ \ i \to \infty \end{aligned}$$

and

$$\begin{aligned} F_{p_{2i+1}-Bv, p_{2i+1}-Bv}(\gamma) &= F_{Ap_{2i}-Bv, Ap_{2i}-Bv}(\gamma) \\ &\geq F_{p_{2i}-v, p_{2i}-v}(K^2\varphi^2(\gamma)) \\ &\to 1 \ \ as \ \ i \to \infty, \end{aligned}$$

which means that $\lim_{i\to\infty} p_{2i} = Av$ and $\lim_{i\to\infty} p_{2i+1} = Av$. By Theorem 2.5, (Z, F, T, K) is a Hausdorff space and by (4.11) we get $Av = v = Bv$.

To prove uniqueness, suppose that there exists another common fixed point w in Z of A and B. For each $\gamma > 0$, by (4.1) we get inductively

$$\begin{aligned} F_{v-w, v-w}(\gamma) &= F_{Bv-Aw, Bv-Aw}(\gamma) \\ &\geq F_{v-w, Bv-Aw}(K\varphi(\gamma)) \\ &= F_{Bv-Aw, v-w}(\varphi(\gamma)) \\ &\geq F_{v-w, v-w}(K^2\varphi^2(\gamma)) \\ &\geq F_{v-w, v-w}(K^4\varphi^4(\gamma)) \\ &\vdots \\ &\geq F_{v-w, v-w}(K^{2n}\varphi^{2n}(\gamma)). \end{aligned} \tag{4.12}$$

Then in view of (4.12) we have $F_{v-w,v-w}(\gamma) = 1$ because $\lim_{n\to\infty} \varphi^{2n}(\gamma) = +\infty$, then $v = w$. This completes the proof. □

Corollary 4.3. *Let (Z, F, T, K) be a complete b-Menger inner product space such that T is of H-type. Let $s \in (0, \frac{1}{K})$ and $A : Z \to Z$ be a mapping such that*

$$F_{Ap-Aq,r}(\gamma) \geq F_{p-q,r}(\frac{\gamma}{s}) \quad for\ p, q, r \in Z\ and\ \gamma > 0.$$

Then A has a unique fixed point in Z.

Proof. We take $\varphi(\gamma) = \frac{\gamma}{s}$ and $A = B$ in the Theorem 4.2. □

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