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OPTIMAL STRATEGIES IN BIOECONOMIC DIFFERENTIAL GAMES: INSIGHTS FROM CHEBYSHEV TAU METHOD

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Abstract. In the realm of differential games and bioeconomic modeling, where intricate systems and multifaceted interactions abound, we explore the precision and efficiency of the Chebyshev Tau method (CTM). We begin with the Weierstrass Approximation Theorem, employing Chebyshev polynomials to pave the way for solving intricate bioeconomic differential games. Our case study revolves around a three-player bioeconomic differential game, unveiling a unique open-loop Nash equilibrium using Hamiltonians and the FilippovCesari existence theorem. We then transition to numerical implementation, employing CTM to resolve a Three-Point Boundary Value Problem (TPBVP) with varying degrees of approximation.

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1. Introduction

The pursuit of solutions to complex problems often demands the integration of diverse mathematical techniques and numerical methodologies. In the realm of differential games and bioeconomic modeling, where intricate systems and multifaceted interactions abound, the need for precision and efficiency in numerical approaches becomes paramount. This necessity has led to the exploration of innovative methods, such as the Chebyshev Tau method (CTM), which offers a powerful toolset for solving differential equations, including those with nonlinear dynamics and intricate boundary conditions.

Differential game theory, an extension of optimal control theory, explores the complexities of strategic decision-making among multiple control agents, all seeking to maximize individual gains while managing the inevitable conflicts arising from their interplay. This subject has garnered substantial recognition within the realms of management sciences and economics, exerting its impact on diverse domains such as resource administration, and the economics of biological systems. Its applications are evident in seminal works such as [6], which elucidate noncooperative differential games and their real-world applications, spanning areas like marketing, natural resources, and environmental economics. Further investigations, as seen in [12], delve into advertising competition, while [27] explores deterministic and cooperative stochastic differential games, revealing their relevance in resource and environmental economics.

At the core of differential game studies lie equilibrium solutions. While the Nash equilibrium is fundamental in simultaneous games, where players cannot improve outcomes through unilateral deviations [15], differential games introduce an intriguing distinction: closed-loop and open-loop equilibria. In the former, each player's strategy depends on both time and state variables, while the latter prescribes strategies as functions of time and initial states. Finding the best strategy for each player in a differential game can be done by solving a system of three equations. These equations are derived from the essential principle of game theory, and they describe the conditions that must be met for the strategy of each player to be the best possible response to the strategies of the other players [1]. There are several different ways to solve these equations, both analytically and numerically [2].

In practice, numerical solutions become necessary to address differential game complexities due to the scarcity of analytical solutions. This fertile ground has been extensively explored across various contexts. Works such as ([11]-[21]) have examined linear quadratic dynamic games to determine open-loop Nash equilibria, while [17] tackled a nonlinear differential game centered on pollution control. Special cases, like state-dependent Riccati equations, have unveiled the quasi-equilibrium of nonlinear differential games [16], and [28] advanced a dynamic programming approach for zero-sum differential games.

Among the array of numerical techniques, the spectral method stands out as a model of accuracy and efficiency for solving differential equations using orthogonal polynomial series truncations ([3]-[7]). Different spectral methods can be used to solve the system of equations that arises from Pontryagin's maximum principle in differential games [14, 25]. The best method to use depends on the nature of the differential equation and the boundary conditions. This paper presents a new numerical approach that combines Pontryagin's maximum principle with the Tau method to solve this system of equations. The goal of this approach is to find the open-loop Nash equilibrium (OLNE) in the noncooperative differential game with a nonzero-sum.

In this intricate web of mathematical theory, game theory, and numerical analysis, the Chebyshev Tau method emerges as a potent instrument, offering a bridge between abstract concepts and practical solutions. It is within this nexus of theory and application that we navigate the terrain of bioeconomic modeling and differential games, demonstrating the capacity of CTM to illuminate the intricacies of real-world challenges.

2. Problem statement

Definition 2.1. In a three-player noncooperative differential game with nonzero-sum, defined as follows [13]: For each player $i \in \{1, 2, 3\}$, the goal is to maximize their individual performance index $J_i(u_1(\cdot), u_2(\cdot), u_3(\cdot))$ over a finite time horizon $[0, T]$. The performance index is defined as:

$$\max J_i(u_1(\cdot), u_2(\cdot), u_3(\cdot)) = \max_{u_i(\cdot)} \int_0^T L_i(t, x(t), u_1(t), u_2(t), u_3(t))dt + \psi_i(x(T)), \tag{2.1}$$

where:

- $u_1(\cdot)$, $u_2(\cdot)$, and $u_3(\cdot)$ are the controls (strategies) of players respectively,
- function $L_i(t, x(t), u_1(t), u_2(t), u_3(t))$ represents player i's instantaneous payoff, which is influenced by the state $x(t)$ and the control actions of all three players,
- function $\psi_i(x(T))$ is the terminal payoff for player i,
- the state evolves according to the dynamics: $\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), u_3(t))$,
- the initial state is given by $x(0) = x_0$.

This game involves the simultaneous optimization of control actions by three players, each aiming to maximize their performance index. The state dynamics and payoffs depend on the choices of all three players, making it a complex interaction. It's noncooperative because players act independently to maximize their objectives, and it is a nonzero-sum since one player's gain doesn't necessarily imply another player's loss.

The objective for each player i is to select their control actions u_i to maximize their respective performance indices while considering the state dynamics, other players' controls, and the terminal payoffs [18].

An open-loop Nash equilibrium (OLNE) for a three-player noncooperative differential game with nonzero-sum is defined as follows:

Definition 2.2. Let's examine a set of functions denoted as $(\varphi_1, \varphi_2, \varphi_3)$ defined over the interval $[0, T]$ and mapping to $\mathbb{R}^3$. Each function φ_i corresponds to one of the three players, labeled as $i = 1, 2, 3$. We refer to this set $(\varphi_1, \varphi_2, \varphi_3)$ as an (OLNE) when, for each player i, there exists an optimal control trajectory u_i that solves the corresponding optimization problem. This optimal control path is determined by the open-loop Nash strategy $u_i = \varphi_i$ [6].

To find the best strategy for each player in a differential game where the players have different goals, we use functions called Hamiltonian functions. These functions are defined as follows [19]:

$$H_i(t, x, u_1, u_2, u_3, \lambda_i) = L_i(t, x, u_1, u_2, u_3) + \lambda_i \cdot f(t, x, u_1, u_2, u_3),$$

where, λ_i represents the costate variable associated with the state variable x for player i.

To simplify notation, we omit the explicit time dependence in the functions x, u_i, and λ_i. Assuming that all functions f, L_i, ψ_i in the optimization problem are continuously differentiable, the first-order necessary conditions for optimality can be obtained using Pontryagins maximum principle.

Based on Pontryagins maximum principle, the set of necessary conditions for the open-loop Nash equilibrium of the nonzero-sum differential game is obtained as follows:

$$\dot{x} = f(t, x, u_1, u_2, u_3), \tag{2.2}$$

$$\dot{\lambda}_i = -\frac{\partial H_i}{\partial x}(t, x, u_1, u_2, u_3, \lambda_i), \tag{2.3}$$

$$\frac{\partial H_i}{\partial u_i}(t, x, u_1, u_2, u_3, \lambda_i) = 0, \tag{2.4}$$

$$x(0) = x_0,$$

$$\lambda_i(T) = \frac{\partial \psi_i}{\partial x}(x(T)).$$

Equation (2.4) can be solved to express u_i, $i = 1, 2, 3$, in terms of x and λ_i, resulting in

$$u_i = \varphi_i(t, x, \lambda_i).$$

Combining this expression with Equations (2.2) and (2.3) gives us a system of differential equations that only depends on the variables t, x and λ_i for $i = 1, 2, 3$. This system of equations is called a three-point boundary value problem (TPBVP):

$$\dot{x} = f(t, x, \varphi_1, \varphi_2, \varphi_3), \tag{2.5}$$

$$\dot{\lambda}_i = -\frac{\partial H_i}{\partial x}(t, x, \varphi_1, \varphi_2, \varphi_3, \lambda_i), \tag{2.6}$$

$$x(0) = x_0, \tag{2.7}$$

$$\lambda_i(T) = \frac{\partial \psi_i}{\partial x}(x(T)), \tag{2.8}$$

where, $\varphi_i = \varphi_i(t, x, \lambda_i)$ for $i = 1, 2, 3$. Typically, this set of TPBVPs exhibits nonlinearity and encompasses divided boundary conditions. Given its intricate nature, discovering a precise analytical resolution for the (OLNE) poses a formidable task. Consequently, the application of appropriate numerical techniques becomes imperative.

3. Tau technique in differential games with nonzero-sum for three players

This section explores how the Tau method can be used to solve the system of three equations that arise from a nonzero-sum differential game with three players. The Tau method works by approximating the function $f(x)$ from $L_w^k(-1, 1)$ with a finite series composed of basis functions:

The approximation of $f(x)$ is expressed as

$$f_N(x) = \sum_{i=0}^{N} f_i T_i(x),$$

where, $T_i(x)$ denotes Chebyshev polynomials and the index i ranges from 0 to N. The coefficients f_i corresponding to $i = 0, 1, \ldots, N$ represent the spectral coefficients [8].

Definition 3.1. The Chebyshev polynomials, denoted as $T_n(x)$ with $n = 0, 1, 2, \ldots$, assume a distinctive role as the eigenfunctions within the context of the singular Sturm-Liouville problem, given by:

$$(1 - x^2)T_n''(x) - 2xT_n'(x) + n(n+1)T_n(x) = 0.$$

These polynomials showcase the property of orthogonality across the interval $[-1,1]$, particularly concerning the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. Their significance is further underscored by the adherence to a recurring pattern defined as follows:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

This recurrence formula is applicable for $n = 1, 2, \ldots$, and the sequence begins with $T_0(x) = 1$ and $T_1(x) = x$.

Theorem 3.2. *Let $f(x) \in H_w^k(-1,1)$ (Sobolev space), where*

$$f_N(x) = \sum_{i=0}^{N} f_i T_i(x)$$

represents the optimal approximation of $f(x)$ in the L_w^2 norm. Then,

$$\|f(x) - f_N(x)\|_{L_w^2[-1,1]} \leq C_0 N^{-k} \|f(x)\|_{H_w^k(-1,1)},$$

where C_0 is a positive constant that depends on the norm we choose, but not on the function $f(x)$ or the number of terms N.

Proof. We commence by considering the optimal approximation

$$f_N(x) = \sum_{i=0}^{N} f_i T_i(x)$$

of $f(x)$ within the L_w^2 norm. We define the approximation error as $e_N(x) = f(x) - f_N(x)$.

Utilizing the triangle inequality, we estimate this error. A crucial observation is the orthogonality of the error $e_N(x)$ to all Chebyshev polynomials $T_i(x)$ for $i = 0, 1, \ldots, N$.

By employing the properties of Chebyshev polynomials, we evaluate the L_w^2 norm of $f_N(x)$. Combining these steps, we establish the desired inequality: the norm of the error $e_N(x)$ in $L_w^2[-1,1]$ is bounded by

$$C_0 N^{-k} \|f(x)\|_{H_w^k(-1,1)},$$

where C_0 is a positive constant that depends on the norm we choose, but not on the function $f(x)$ or the number of terms N. □

Taking into account the principles stated in Theorem 3.2, it has been established that Chebyshev's polynomial approximations display a rate of N^{-k}. The fundamental ideas of the method introduced in this context and the theoretical investigation of its convergence are derived from the well-established Weierstrass approximation theorem.

Theorem 3.3. *Consider a function f belonging to the space $L^2_w[-1,1]$, and let N be a natural number. In this context, there exists a special polynomial $f^*_N \in P_N$, where P_N comprises all polynomials with a degree of no more than N that meet the following condition:*

For any polynomial f_N within P_N, the difference between f and f_N is minimized by f^*_N, yielding the smallest L^2_w norm difference:

$$\|f - f^*_N\|_w = \inf_{f_N \in P_N} \|f - f_N\|_w,$$

where, the polynomial $f^*_N(x)$ is uniquely defined as a combination of Chebyshev polynomials $T_k(x)$ and coefficients f_k, which are determined by the L^2_w orthogonality and the weight-adjusted norms.

Proof. Begin with the assumption that f is an element of $L^2_w[-1,1]$, and N is a natural number. Define P_N as the collection of all polynomials with degrees not exceeding N. Introduce $e_N(x)$ as the difference between $f(x)$ and $f^*_N(x)$, where $f^*_N(x)$ is the polynomial that optimally approximates f within P_N. Apply the triangle inequality to bound the L^2_w norm of $e_N(x)$:

$$\|e_N(x)\|_{L^2_w[-1,1]} \leq \|f(x)\|_{L^2_w[-1,1]} + \|f^*_N(x)\|_{L^2_w[-1,1]},$$

exploit the inherent orthogonality of Chebyshev polynomials to demonstrate that $e_N(x)$ is orthogonal to all $T_k(x)$ for $k = 0, 1, \ldots, N$. Employ the L^2_w orthogonality and the adapted weight function $w(x)$ to compute the coefficients f_k and construct the polynomial $f^*_N(x)$. Consolidate the aforementioned steps to affirm that $f^*_N(x)$ effectively minimizes the L^2_w norm difference:

$$\|f - f^*_N\|_w = \inf_{f_N \in P_N} \|f - f_N\|_w.$$

□

This proof underscores the specific characteristics of Chebyshev polynomials, their orthogonality, and their aptitude for approximating functions in the context of L^2_w norms. The unique property of $f^*_N(x)$ in minimizing the approximation error is substantiated within this theorem.

Incorporating Chebyshev polynomials over the interval $[0, T]$ requires a domain transformation, achieved by the variable substitution:

$$x = \frac{2t}{T} - 1.$$

3.1. **Solving Three-Point Boundary Value Problems.** For solving the Three-Point Boundary Value Problems (TPBVPs), we approximate the solutions x and λ_i (where $i = 1, 2, 3$) using a linear combination of adjusted

Chebyshev polynomials, expressed as:

$$x \approx x_N = \sum_{i=0}^{N} a_i T_i^*, \tag{3.1}$$

$$\lambda_1 \approx \lambda_{1N} = \sum_{i=0}^{N} b_i T_i^*, \tag{3.2}$$

$$\lambda_2 \approx \lambda_{2N} = \sum_{i=0}^{N} c_i T_i^*, \tag{3.3}$$

$$\lambda_3 \approx \lambda_{3N} = \sum_{i=0}^{N} d_i T_i^*, \tag{3.4}$$

where a_i, b_i, c_i, and d_i represent coefficients that are yet to be determined.

The adjusted Chebyshev polynomial is given by

$$T_i^* = T_i\left(\frac{2t}{T} - 1\right),$$

with T_i being the Chebyshev polynomial over the interval $[0, T]$.

The first derivatives of x and λ_i (where $i = 1, 2, 3$) can be approximated as:

$$\dot{x} \approx \dot{x}_N = \frac{2}{T}\sum_{i=0}^{N} a_i {T_i^*}', \tag{3.5}$$

$$\dot{\lambda}_1 \approx \dot{\lambda}_{1N} = \frac{2}{T}\sum_{i=0}^{N} b_i {T_i^*}', \tag{3.6}$$

$$\dot{\lambda}_2 \approx \dot{\lambda}_{2N} = \frac{2}{T}\sum_{i=0}^{N} c_i {T_i^*}', \tag{3.7}$$

$$\dot{\lambda}_3 \approx \dot{\lambda}_{3N} = \frac{2}{T}\sum_{i=0}^{N} d_i {T_i^*}'. \tag{3.8}$$

These expressions can be represented in vector form as:

$$x \approx x_N = A^T T^*, \tag{3.9}$$

$$\lambda_1 \approx \lambda_{1N} = B^T T^*, \tag{3.10}$$

$$\lambda_2 \approx \lambda_{2N} = C^T T^*, \tag{3.11}$$

$$\lambda_3 \approx \lambda_{3N} = D^T T^*, \tag{3.12}$$

$$\dot{x} \approx \dot{x}_N = A^T S, \tag{3.13}$$

$$\dot{\lambda}_1 \approx \dot{\lambda}_{1N} = B^T S, \tag{3.14}$$

$$\dot{\lambda}_2 \approx \dot{\lambda}_{2N} = C^T S, \tag{3.15}$$

$$\dot{\lambda}_3 \approx \dot{\lambda}_{3N} = D^T S, \tag{3.16}$$

where, $A^T = [a_0, \dots, a_N]$, $B^T = [b_0, \dots, b_N]$, $C^T = [c_0, \dots, c_N]$, $D^T = [d_0, \dots, d_N]$, $T^* = [t_0^*, \dots, t_N^*]^T$, $S = \frac{2}{T}[t_0^{*'}, \dots, t_N^{*'}]$.

The application of the Tau method involves substituting equations (3.9) (3.16) into the given differential equations (2.5) and (2.6) to formulate residuals:

$$\begin{aligned} R_1 &= \dot{x}_N - f(t, x_N, \varphi_{1N}, \varphi_{2N}, \varphi_{3N}), \\ R_2 &= \dot{\lambda}_{1N} + \frac{\partial H_1}{\partial x_N}(t, x_N, \varphi_{1N}, \varphi_{2N}, \varphi_{3N}, \lambda_{1N}), \\ R_3 &= \dot{\lambda}_{2N} + \frac{\partial H_2}{\partial x_N}(t, x_N, \varphi_{1N}, \varphi_{2N}, \varphi_{3N}, \lambda_{2N}), \\ R_4 &= \dot{\lambda}_{3N} + \frac{\partial H_3}{\partial x_N}(t, x_N, \varphi_{1N}, \varphi_{2N}, \varphi_{3N}, \lambda_{3N}). \end{aligned}$$

When you multiply these discrepancies by T_i^*, perform integration over the interval $[0, T]$, and equate the result to zero, it results in the subsequent set of algebraic equations:

$$\begin{cases} \int_0^T R_1 T_i^* \, dt = 0, \\ \int_0^T R_2 T_i^* \, dt = 0, \\ \int_0^T R_3 T_i^* \, dt = 0, \\ \int_0^T R_4 T_i^* \, dt = 0, \\ x_N(0) = x_0, \\ \lambda_{jN}(T) = \frac{\partial \psi_j(x_N(T))}{\partial x_N}, \quad j = 1, 2, 3, \end{cases}$$

this system of equations aids in determining the coefficients a_i, b_i, c_i, and d_i within the vectors A, B, C, and D.

4. Case study: Bioeconomic differential game with three players

This section uses a bioeconomic model to demonstrate the precision and effectiveness of the Chebyshev Tau method (CTM). In this model, three firms harvest a shared natural renewable resource, such as a fishery. This bioeconomic model was chosen because its three-point boundary value problems (TPBVPs) are significantly more nonlinear than those of many other economic models, such as Sorger's competitive advertising model [20]. This nonlinearity makes it a good example of the accuracy and efficiency of the CTM. The objective of the firms is to maximize their profits over a fixed time horizon $[0, T]$. The profit of each firm depends on the amount of the resource they harvest and the price of the resource [4].

The following state equation and initial condition describe how the renewable resource population changes over time in its natural habitat [4]:

$$\dot{x}(t) = F(x(t)) - q_1 x(t)u_1(t) - q_2 x(t)u_2(t) - q_3 x(t)u_3(t), \; x(0) = x_0.$$

In the equation previously outlined, the continuous function denoted as $F(.) : \mathbb{R} \to \mathbb{R}$ delineates the fundamental growth pattern of the sustainable resource. This function adheres to the framework of a logistic growth model, characterized by the equation

$$F(x(t)) = rx(t)t(1 - \frac{x(t)}{k}),$$

where r signifies the inherent growth rate, k symbolizes the capacity threshold, $x(t) > 0$ signifies the resource population at time t, and $u_1(t) \geq 0$, $u_2(t) \geq 0$, and $u_3(t) \geq 0$ correspond to the exploitation endeavors of the three entities at time t. The constants $q_1 > 0$, $q_2 > 0$, and $q_3 > 0$ represent the coefficients of resource availability. An account of the remuneration for each entity over the timeframe $[0, T]$ is articulated as follows:

$$J_1(u_1(.), u_2(.), u_3(.)) = \int_0^T \left(\pi_1 q_1 x(t)u_1(t) - \frac{1}{2}u_1^2(t) \right) dt \qquad \text{for firm 1},$$

$$J_2(u_1(.), u_2(.), u_3(.)) = \int_0^T \left(\pi_2 q_2 x(t)u_2(t) - \frac{1}{2}u_2^2(t) \right) dt \qquad \text{for firm 2},$$

$$J_3(u_1(.), u_2(.), u_3(.)) = \int_0^T \left(\pi_3 q_3 x(t)u_3(t) - \frac{1}{2}u_3^2(t) \right) dt \qquad \text{for firm 3},$$

let π_1, π_2, and π_3 stand as unchanging values, symbolizing the unit cost of the renewable natural resource associated with each of the three firms. The expressions $\frac{1}{2}u_1^2$, $\frac{1}{2}u_2^2$, and $\frac{1}{2}u_3^2$ signify the expenditure linked to harvesting efforts u_1, u_2, and u_3, as per reference [4]. To ascertain the Nash equilibrium within this bioeconomic competition, we present the Hamiltonian function for each enterprise as follows:

$$H_1(t, x, u_1, u_2, u_3, \lambda_1) = \pi_1 q_1 x u_1 - \frac{1}{2}u_1^2 + \lambda_1 \left(F(x) - q_1 x u_1 - q_2 x u_2 - q_3 x u_3 \right),$$

$$H_2(t, x, u_1, u_2, u_3, \lambda_2) = \pi_2 q_2 x u_2 - \frac{1}{2}u_2^2 + \lambda_2 \left(F(x) - q_1 x u_1 - q_2 x u_2 - q_3 x u_3 \right),$$

$$H_3(t, x, u_1, u_2, u_3, \lambda_3) = \pi_3 q_3 x u_3 - \frac{1}{2}u_3^2 + \lambda_3 \left(F(x) - q_1 x u_1 - q_2 x u_2 - q_3 x u_3 \right).$$

By minimizing H_1, H_2, and H_3 with respect to u_1, u_2, and u_3 respectively, we determine the (OLNE) for each firm:

$$\frac{\partial H_1}{\partial u_1} = 0 \Rightarrow \pi_1 q_1 x - u_1 - \lambda_1 q_1 x = 0 \Rightarrow u_1 = q_1 x(\pi_1 - \lambda_1), \tag{4.1}$$

$$\frac{\partial H_2}{\partial u_2} = 0 \Rightarrow \pi_2 q_2 x - u_2 - \lambda_2 q_2 x = 0 \Rightarrow u_2 = q_2 x(\pi_2 - \lambda_2), \tag{4.2}$$

$$\frac{\partial H_3}{\partial u_3} = 0 \Rightarrow \pi_3 q_3 x - u_3 - \lambda_3 q_3 x = 0 \Rightarrow u_3 = q_3 x(\pi_3 - \lambda_3). \tag{4.3}$$

The adjoint dynamics of each player are defined as:

$$\dot{\lambda}_1 = -\frac{\partial H_1}{\partial x} = -\pi_1 q_1 u_1 - \lambda_1 \dot{F}(x) + \lambda_1 q_1 u_1 + \lambda_1 q_2 u_2 + \lambda_1 q_3 u_3, \tag{4.4}$$

$$\dot{\lambda}_2 = -\frac{\partial H_2}{\partial x} = -\pi_2 q_2 u_2 - \lambda_2 \dot{F}(x) + \lambda_2 q_1 u_1 + \lambda_2 q_2 u_2 + \lambda_2 q_3 u_3, \tag{4.5}$$

$$\dot{\lambda}_3 = -\frac{\partial H_3}{\partial x} = -\pi_3 q_3 u_3 - \lambda_3 \dot{F}(x) + \lambda_3 q_1 u_1 + \lambda_3 q_2 u_2 + \lambda_3 q_3 u_3. \tag{4.6}$$

Substituting Equations (4.1), (4.2), and (4.3) into Equations (4.4), (4.5), and (4.6) respectively, we obtain the adjoint dynamics for each player.

The system of TPBVPs for this bioeconomic game is expressed as:

$$\dot{x} = F(x) - \frac{1}{2} q_1 x^2 (\pi_1 - \lambda_1) - \frac{1}{2} q_2 x^2 (\pi_2 - \lambda_2) - \frac{1}{2} q_3 x^2 (\pi_3 - \lambda_3), \tag{4.7}$$

$$\begin{aligned} \dot{\lambda}_1 = & -\pi_1 q_1^2 x(\pi_1 - \lambda_1) - \lambda_1 \dot{F}(x) + \lambda_1 q_1^2 x(\pi_1 - \lambda_1) \\ & + \lambda_1 q_2^2 x(\pi_2 - \lambda_2) + \lambda_1 q_3^2 x(\pi_3 - \lambda_3), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \dot{\lambda}_2 = & -\pi_2 q_2^2 x(\pi_2 - \lambda_2) - \lambda_2 \dot{F}(x) + \lambda_2 q_1^2 x(\pi_1 - \lambda_1) \\ & + \lambda_2 q_2^2 x(\pi_2 - \lambda_2) + \lambda_2 q_3^2 x(\pi_3 - \lambda_3), \end{aligned} \tag{4.9}$$

$$\begin{aligned} \dot{\lambda}_3 = & -\pi_3 q_3^2 x(\pi_3 - \lambda_3) - \lambda_3 \dot{F}(x) + \lambda_3 q_1^2 x(\pi_1 - \lambda_1) \\ & + \lambda_3 q_2^2 x(\pi_2 - \lambda_2) + \lambda_3 q_3^2 x(\pi_3 - \lambda_3), \end{aligned} \tag{4.10}$$

$$x(0) = x_0, \tag{4.11}$$

$$\lambda_1(T) = 0, \lambda_2(T) = 0, \lambda_3(T) = 0. \tag{4.12}$$

Supposing that the singular resolution of Formula (4.7) with the initial state stipulated in Formula (4.11) is designated as y, and signifying the individual resolutions of Formulas (4.8),(4.9) and (4.10), with end conditions illustrated in Formula (4.12), as γ_1, γ_2, and γ_3 respectively, the ensuing theorem characterizes the unique (OLNE) for this extended bioeconomic game.

The upcoming theorem describes the distinctive (OLNE) found within the bioeconomic game that has been introduced.

Theorem 4.1. *The best strategy for each player in the three-player differential game, given the strategies of the other players, is as follows:*

$$u_1 = q_1 y(\pi_1 - \lambda_1), \tag{4.13}$$

$$u_2 = q_2 y(\pi_2 - \lambda_2), \tag{4.14}$$

$$u_3 = q_3 y(\pi_3 - \lambda_3). \tag{4.15}$$

Proof. Given the controls $v_i \geq 0$ for $i = 1, 2, 3$, we address the subsequent optimal control scenarios:

(1) $\max_{u_1 \geq 0} J_1(u_1(.), v_2(.), v_3(.)) = \int_0^T \left(\pi_1 q_1 x u_1 - \frac{1}{2}u_1^2\right) dt$
such that,. $\dot{x} = F(x) - q_1 x u_1 - q_2 x v_2 - q_3 x v_3, \quad x(0) = x_0,$

(2) $\max_{u_2 \geq 0} J_2(v_1(.), u_2(.), v_3(.)) = \int_0^T \left(\pi_2 q_2 x u_2 - \frac{1}{2}u_2^2\right) dt$
such that, $\dot{x} = F(x) - q_1 x v_1 - q_2 x u_2 - q_3 x v_3, \quad x(0) = x_0,$

(3) $\max_{u_3 \geq 0} J_3(v_1(.), v_2(.), u_3(.)) = \int_0^T \left(\pi_3 q_3 x u_3 - \frac{1}{2}u_3^2\right) dt$
such that. $\dot{x} = F(x) - q_1 x v_1 - q_2 x v_2 - q_3 x u_3, \quad x(0) = x_0.$

These scenarios involve linear dynamics concerning the control variables u_i for $i = 1, 2, 3$, and the integrand of the performance index J_i for $i = 1, 2, 3$ exhibits concavity with respect to u_i, given that $\frac{\partial^2 J_i}{\partial u_i^2} = -1 < 0$, $i = 1, 2, 3$.

Henceforth, the presence and singular characteristics stipulated by the FilippovCesari theorem of existence [5] remain valid in these optimal control situations. Derived from this examination, it becomes undeniably clear that solutions meeting these criteria are precisely governed by Equations (4.13), (4.14) and (4.15). Consequently, the distinct (OLNE) in the aforementioned differential game involving three participants is securely confirmed. □

The ensemble of Three-Point Boundary Value Problems (TPBVPs) delineated by Equations (4.7)(4.12) constitutes a collection of nonlinear differential equations characterized by split boundary values and generally evades a closed-form analytical solution. To tackle this scenario numerically, employing the methodology introduced earlier, we adopt a set of standard parameter values as follows:

$$x_0 = 0.1, \quad q_1 = q_2 = q_3 = 1, \quad \pi_1 = 2, \quad \pi_2 = 1.5, \quad \pi_3 = 1.8,$$

$$r = 0.1, \quad k = 100, \quad T = 1.$$

Consequently, the amenable numerical representation of the TPBVP system, involving three players, can be articulated as:

$$
\begin{cases}
\dot{x} = 0.1x - 3.501x^2 + x^2\lambda_1 + x^2\lambda_2 + x^2\lambda_3, \\
\dot{\lambda}_1 = -4x - 0.1\lambda_1 + 5.502x\lambda_1 - x\lambda_1^2 - x\lambda_1\lambda_2 - x\lambda_1\lambda_3, \\
\dot{\lambda}_2 = -2.25x - 0.1\lambda_2 + 5.002x\lambda_2 - x\lambda_2^2 - x\lambda_1\lambda_2 - x\lambda_2\lambda_3, \\
\dot{\lambda}_3 = -2.9x - 0.1\lambda_3 + 5.732x\lambda_3 - x\lambda_3^2 - x\lambda_1\lambda_3 - x\lambda_2\lambda_3, \\
x(0) = 0.1, \\
\lambda_1(1) = 0, \lambda_2(1) = 0, \lambda_3(1) = 0.
\end{cases}
$$

In order to confront this intricate TPBVP system, we consider the subsequent approximations for x, λ_1, and λ_2:

$$
\begin{cases}
x & \approx x_N = \sum_{i=0}^{N} a_i T_i^* = A^T T^* \\
\lambda_1 & \approx \lambda_{1N} = \sum_{i=0}^{N} b_i T_i^* = B^T T^*, \\
\lambda_2 & \approx \lambda_{2N} = \sum_{i=0}^{N} c_i T_i^* = C^T T^*, \\
\lambda_3 & \approx \lambda_{3N} = \sum_{i=0}^{N} d_i T_i^* = D^T T^*,
\end{cases}
$$

where $A^T = [a_0, \dots, a_N]$, $B^T = [b_0, \dots, b_N]$, $C^T = [c_0, \dots, c_N]$, and $D^T = [d_0, \dots, d_N]$ are unknown vectors, and $T^* = [t_0^*, \dots, t_N^*]^T$ signifies the vector of shifted Chebyshev Polynomials.

Substituting these approximations into the TPBVP system's equations yields the ensuing residual expressions:

$$
\begin{aligned}
R_1 = \frac{2}{T}\sum_{i=0}^{N} a_i T_i^{*'} - 0.1\sum_{i=0}^{N} a_i T_i^* + 3.501(\sum_{i=0}^{N} a_i T_i^*)^2 - (\sum_{i=0}^{N} a_i T_i^*)^2 \sum_{i=0}^{N} b_i T_i^* \\
- (\sum_{i=0}^{N} a_i T_i^*)^2 \sum_{i=0}^{N} c_i T_i^* - (\sum_{i=0}^{N} a_i T_i^*)^2 \sum_{i=0}^{N} d_i T_i^*,
\end{aligned}
$$

$$
\begin{aligned}
R_2 = \frac{2}{T}\sum_{i=0}^{N} b_i T_i^{*'} + 4\sum_{i=0}^{N} a_i T_i^* + 0.1\sum_{i=0}^{N} b_i T_i^* - 5.502\sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \\
+ \sum_{i=0}^{N} a_i T_i^* (\sum_{i=0}^{N} b_i T_i^*)^2 + \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \sum_{i=0}^{N} c_i T_i^* \\
- \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \sum_{i=0}^{N} d_i T_i^*,
\end{aligned}
$$

$$R_3 = \frac{2}{T}\sum_{i=0}^{N} c_i T_i^{*'} + 2.25\sum_{i=0}^{N} a_i T_i^* + 0.1\sum_{i=0}^{N} c_i T_i^* - 5.002\sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} c_i T_i^*$$
$$+ \sum_{i=0}^{N} a_i T_i^* (\sum_{i=0}^{N} c_i T_i^*)^2 + \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \sum_{i=0}^{N} c_i T_i^*$$
$$- \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \sum_{i=0}^{N} d_i T_i^*,$$

$$R_4 = \frac{2}{T}\sum_{i=0}^{N} d_i T_i^{*'} + 2.9\sum_{i=0}^{N} a_i T_i^* + 0.1\sum_{i=0}^{N} d_i T_i^* - 5.732\sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} d_i T_i^*$$
$$+ \sum_{i=0}^{N} a_i T_i^* (\sum_{i=0}^{N} d_i T_i^*)^2 + \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} b_i T_i^* \sum_{i=0}^{N} d_i T_i^*$$
$$- \sum_{i=0}^{N} a_i T_i^* \sum_{i=0}^{N} c_i T_i^* \sum_{i=0}^{N} d_i T_i^*.$$

The numerical outcomes for the optimal payoff functions J_1, J_2, and J_3 with varying N values are presented in the following tables.The graphs of approximate solutions for (OLNE) for N = 14 are given in Figure (1).

Table 1. Optimal payoff function J_1 for the three-player illustration with CTM.

N	$J_{1\text{CTM}}$
5	0.0163802101695222542167046941782224
7	0.016380209075360448134112122738199
9	0.016380209069981763839152784671981
11	0.016380209069971537054880540997263

Table 2. Optimal payoff function J_2 for the three-player illustration with CTM.

N	$J_{2\text{CTM}}$
5	0.0092479570970383153611687730098446
7	0.0092479570969001533684413638355018
9	0.0092479570969023516184877824151884
11	0.0092479570969023099745970910988349

TABLE 3. Optimal payoff function J_3 for the three-player illustration with CTM.

N	$J_{3\text{CTM}}$
5	0.012789302542628878759568960346686
7	0.012789302542421004237271735456236
9	0.012789302542423635731872013086763
11	0.012789302542423608113133288430303

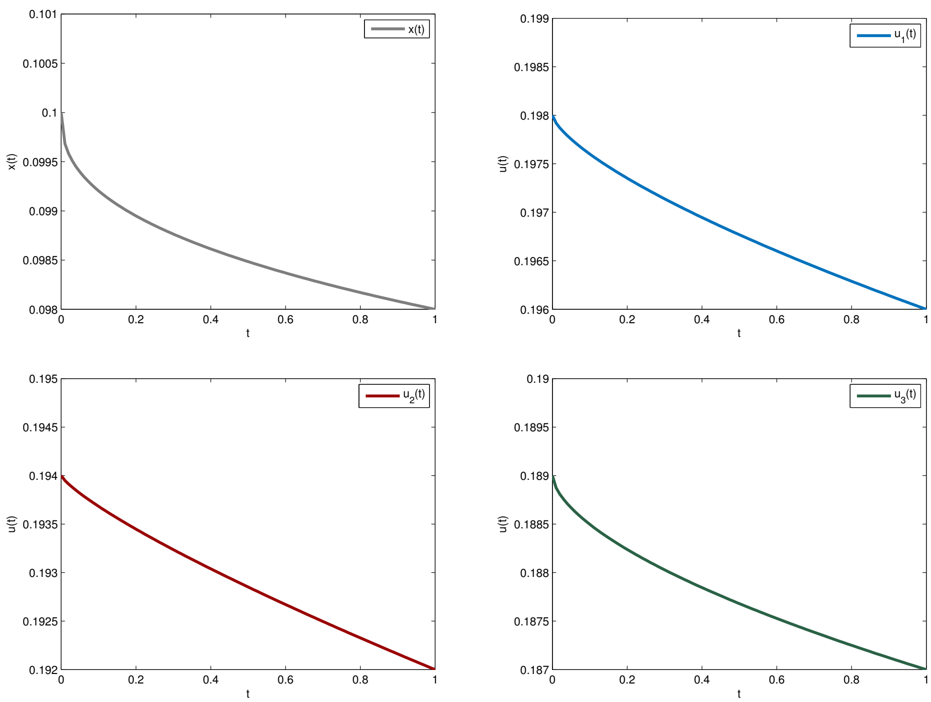

FIGURE 1. Plots of approximate (OLNE) for illustrative example when N = 14.

5. CONCLUSION

In conclusion, this research demonstrates the efficacy of the Chebyshev Tau Method (CTM) in solving complex bioeconomic differential games with open-loop Nash equilibria. By applying CTM to challenging scenarios like the three-player fishery model, we unveil a powerful numerical approach for addressing intricate systems that lack closed-form solutions. The success of this method highlights its potential in various fields, opening new horizons for tackling nonlinear dynamics and decision-making processes in economics, ecology, and beyond. Further exploration of CTM and its applications promises to advance our understanding of complex systems and enhance decision support in real-world scenarios.

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