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EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS OF VOLTERRA-FREDHOLM SYSTEM INVOLVING CAPUTO DERIVATIVE

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Abstract. In this study, a class of nonlinear boundary fractional Caputo Volterra-Fredholm integro-differential equations (CV-FIDEs) is taken into account. Under specific assumptions about the available data, we firstly demonstrate the existence and uniqueness features of the solution. The Gronwall's inequality, a adequate singular Hölder's inequality, and the fixed point theorem using an a priori estimate procedure. Finally, a case study is provided to highlight the findings.

1. Introduction

Due to its capacity to generalize classical calculus to non-integer orders, fractional calculus has drawn a lot of interest. This mathematical framework is effective for simulating complicated events with non-local and memory-dependent behaviors [3, 4, 5, 6, 7, 10, 9]. A complete bibliography on this topic is available from Miller and Ross [25]. The notion of fractional differential equations (FDEs) has therefore undergone substantial development, the monographs by Lakshmikantham et al. [24], as well as Kilbas et al. [23]

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Any initial value problem and boundary value problem that is transformed into an IE often yields an integro-differential equation (IDE), which may be observed in several scientific models. Both integral and differential operators may be found in many IDE. We must thus look for an effective method for locating analytical solutions to fractional differential equations [13, 20, 29, 30, 31]. In his 1908 thesis, Traian Lalescu became the first scientist to analyses the Volterra integral equations (VIE). Numerous scientific fields, including demography, insurance mathematics, and physics, have numerous uses for VIE.

Due to their usefulness in several engineering and scientific fields, fractional Volterra integro-differential equations (FVIDEs) have recently attracted the attention of many scholars [19, 23, 24, 25, 26, 27, 28]. In particular, using the Caputo fractional derivative for FV-FIDEs, Hamoud [14] provides the required conditions for the existence and uniqueness of solutions for numerous classes of boundary value issues. The existence of impulsive FV-FIDEs in Banach spaces has recently been demonstrated by Hamoud and Ghadle [16] used a combination of the fixed point method, Grownwall's generalized inequality, and Caputo derivative.

In this study, the authors examine an arbitrary border condition, we generalize the result from [11] using the generalized border condition, and we will extend the past studies [2, 29] on fractional boundary value problems for FV-FIDEs of order $\Upsilon \in (2,3)$

$$\begin{cases} {}^cD^{\sigma}\Im(\varsigma) = \Re(\varsigma, \Im(\varsigma), (\Lambda\Im)(\varsigma), (\vartheta\Im)(\varsigma)), \varsigma \in I = [0,\alpha], \sigma \in (\Upsilon-1,\Upsilon), \\ \Im(0) = \Im_0, \Im'(0) = \Im_0^1, \Im''(0) = \Im_0^2, \cdots, \Im^{(\Upsilon-1)}(0) = \Im_0^{\Upsilon-2}, \\ \Im^{(\Upsilon-1)}(\alpha) = \Im_\alpha, \ \Im_\alpha \in \mathtt{C}, \end{cases} \tag{1.1}$$

where ${}^cD^{\sigma}$ is the fractional Caputo derivative of order σ, $\Re : I \times \mathtt{C} \times \mathtt{C} \times \mathtt{C} \to \mathtt{C}$ is a predefined function, and $\Im_0, \Im_0^j (j = 1, 2, \cdots, \Upsilon-2, \Upsilon \geq 4, \Upsilon$ is a integer), and Λ, ϑ are defined by

$$(\Lambda\Im)(\varsigma) = \int_0^{\varsigma} \psi(\varsigma,\wp)\Im(\wp)d\wp, \ (\vartheta\Im)(\varsigma) = \int_0^{\alpha} \psi_1(\varsigma,\wp)\Im(\wp)d\wp,$$

with

$$\varphi_0 = \max\left\{\int_0^{\varsigma} \psi(\varsigma,\wp)d\wp : (\varsigma,\wp) \in I \times I\right\}, \ \varphi_0^1 = \max\left\{\int_0^{\alpha} \psi_1(\varsigma,\wp)d\wp : (\varsigma,\wp) \in I \times I\right\}$$

where $\psi, \psi_1 \in C\left(I \times I, \mathbb{R}^+\right)$.

By combining the fixed point approach, an appropriate singular Gronwall's inequality, and Holder's inequality, several existence and uniqueness conclusions for the FV-FIDE (1.1) be demonstrated.

2. Auxiliary results

In recent years, both conceptually and practically, a lot of emphasis has been paid to the study of these issues. Without attempting to be thorough, the following recent papers on this issue will be addressed in [7, 15, 17, 18, 22]. A lot of writers utilize the following conditions: The nonlinear term $\Re$ must meet the mathematical condition $\mathrm{F}(\Im)$ that functions exist for there to be solutions.

We'll now go through the vocabulary, notation, and early results that will be applied to the whole task. Every continuous function in the Banach space I into $\complement$ are denoted by the notation $C(I,\complement)$ with the norm $\|\Im\|_\infty := \sup\{\|\Im(\varsigma)\| : \varsigma \in I\}$. Regarding measurable functions $\mu : I \to \mathbb{R}$, the norm given by $1 \leq \omega < \infty$,

$$\|\mu\|_{L^\omega(I,\mathbb{R})} = \left(\int_I |\mu(\varsigma)|^\omega dt\right)^{\frac{1}{\omega}}.$$

We define $L^\omega(I,\mathbb{R})$ all measurable Lebesgue functions μ in Banach space with $\|\mu\|_{L^\omega(I,\mathbb{R})} < \infty$.

The following fundamental definitions and characteristics of fractional calculus theory are necessary in order to understand the content of this article, see [23], for example, for further details.

Definition 2.1. ([21, 26]) The fractional Caputo derivative of order σ of χ for an appropriate function χ define on $[s,d]$ is given by

$$\left({}^cD^\sigma_{s+}\chi\right)(\varsigma) = \frac{1}{\Gamma(\Upsilon-\sigma)}\int_s^\varsigma (\varsigma-\wp)^{\Upsilon-\sigma-1}\chi^{(\Upsilon)}(\wp)d\wp.$$

Lemma 2.2. ([1]) *Let $\varpi : I \to \complement$ be continuous. A function $\Im \in C(I,\complement)$ is a solution of the following equation*

$$\begin{aligned}\Im(\varsigma) = &\frac{1}{\Gamma(\sigma)}\int_0^\varsigma (\varsigma-\wp)^{\sigma-1}\varpi(\wp)d\wp \\ &- \frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^\alpha (\alpha-\wp)^{\sigma-\Upsilon}\varpi(\wp)d\wp \\ &+ \Im_0 + \Im_0^1\varsigma + \frac{\Im_0^2}{2!}\varsigma^2 + \cdots + \frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2} + \frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1}\end{aligned}$$

if and only if $\Im$ is a solution of the following equation

$$\begin{cases} {}^cD^\sigma\Im(\varsigma) = \varpi(\varsigma), \varsigma \in I = [0,\alpha], \sigma \in (\Upsilon-1,\Upsilon), \\ \Im(0) = \Im_0, \Im'(0) = \Im_0^1, \Im''(0) = \Im_0^2, \cdots, \Im^{(\Upsilon-1)}(0) = \Im_0^{\Upsilon-2}, \\ \Im^{(\Upsilon-1)}(\alpha) = \Im_\alpha. \end{cases} \tag{2.1}$$

Lemma 2.3. ([1]) *Let* $\Re : I\times\mathtt{C}\times\mathtt{C}\times\mathtt{C}\to\mathtt{C}$ *be continuous function.* $\Im\in C(I,\mathtt{C})$ *is a solution of the following equation*

$$\begin{aligned}\Im(\varsigma) &= \frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\quad - \frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\quad + \Im_0+\Im_0^1\varsigma+\frac{\Im_0^2}{2!}\varsigma^2+\cdots+\frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2}+\frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1} \qquad (2.2)\end{aligned}$$

if and only if $\Im$ *is a solution of the FV-FIDE* (1.1).

Lemma 2.4. ([26]) *Let* $\Xi : I\to\mathtt{C}$ *be a measurable function. If* $\|\Xi\|$ *is Lebesgue integrable, then* Ξ *is Bochner integrable.*

3. Main results

Before expressing and demonstrating the key findings, we first introduce the following hypothesis. The foundation of our initial finding is the Banach contraction principle. We make the following presumptions:

(E1) $\Re : I\times\mathtt{C}\times\mathtt{C}\times\mathtt{C}\to\mathtt{C}$ is measurable with respect to ς on I.

(E2) There exist real-valued functions $m_1(\varsigma)$, $m_2(\varsigma), m_3(\varsigma)\in L^{\frac{1}{\sigma_1}}(I,\mathtt{C})$ and a constant $\sigma_1\in(0,\sigma-\Upsilon+1)$ such that

$$\begin{aligned}&\|\Re(\varsigma,y(\varsigma),(\Lambda y)(\varsigma),(\vartheta y)(\varsigma))-\Re(\varsigma,\Im(\varsigma),(\Lambda\Im)(\varsigma),(\vartheta\Im)(\varsigma))\| \\ &\qquad\le m_1(\varsigma)\|y-\Im\|+m_2(\varsigma)\|\Lambda y-\Lambda\Im\|+m_3(\varsigma)\|\vartheta y-\vartheta\Im\|,\end{aligned}$$

for each $\varsigma\in I$, and all $y,\Im\in\mathtt{C}$.

(E3) There exists a real-valued function $h(\varsigma)\in L^{\frac{1}{\sigma_2}}(I,\mathtt{C})$ such that

$$\|\Re(\varsigma,\Im,\Lambda\Im,\vartheta\Im)\|\le h(\varsigma),$$

for each $\varsigma\in I$, and all $\Im\in\mathtt{C}$ and constant $\sigma_2\in(0,\sigma-\Upsilon+1)$.
Let's keep this brief $M=\left\|m_1+\varphi_0 m_2+\varphi_0^1 m_3\right\|_{L^{\frac{1}{\sigma_1}}(I,\mathtt{C})}$ and $H=\|h\|_{L^{\frac{1}{\sigma_2}}(I,\mathtt{C})}$.

(E4) There exists $G>0$ and $\rho\in\left[0,1-\frac{1}{p}\right)$ for some $1<p<\frac{1}{\Upsilon-\sigma}$ such that

$$\|\Re(\varsigma,c,\Lambda c,\vartheta c)\|\le G\left(1+\varphi_0\|c\|^{\rho}+\varphi_0^1\|c\|^{\rho}\right),\ \ \forall\ \varsigma\in I \text{ and } \forall\ c\in\mathtt{C}.$$

(E5) For all $\varsigma \in I$, the set

$$K_1 = \left\{ (\varsigma - \wp)^{\sigma-1} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) : \Im \in C(I, \complement), \wp \in [0, \varsigma] \right\}$$

and

$$K_2 = \left\{ (\varsigma - \wp)^{\sigma-\Upsilon} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) : \Im \in C(I, X), \wp \in [0, \varsigma] \right\}$$

are relatively compact.

Theorem 3.1. *Assume* (E1)-(E3) *are true. If*

$$\Phi_{\sigma,\alpha,\Upsilon} = \frac{M}{\Gamma(\sigma)} \frac{\alpha^{\sigma-\sigma_1}}{\left(\frac{\sigma-\sigma_1}{1-\sigma_1}\right)^{1-\sigma_1}} + \frac{M}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)} \frac{\alpha^{\sigma-\sigma_1}}{\left(\frac{\sigma-\sigma_1-\Upsilon+1}{1-\sigma_1}\right)^{1-\sigma_1}} < 1, \tag{3.1}$$

then the FV-FIDE (1.1) *has a unique solution on I.*

Proof. For all $\varsigma \in I$, we get

$$\begin{aligned}
&\int_0^\varsigma \left\| (\varsigma - \wp)^{\sigma-1} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) \right\| d\wp \\
&\leq \left(\int_0^\varsigma (\varsigma - \wp)^{\frac{\sigma-1}{1-\sigma_2}} d\wp \right)^{1-\sigma_2} \left(\int_0^\varsigma (h(\wp))^{\frac{1}{\sigma_2}} d\wp \right)^{\sigma_2} \\
&\leq \left(\int_0^\varsigma (\varsigma - \wp)^{\frac{\sigma-1}{1-\sigma_2}} d\wp \right)^{1-\sigma_2} \left(\int_0^\alpha (h(\wp))^{\frac{1}{\sigma_2}} d\wp \right)^{\sigma_2} \\
&\leq \frac{\alpha^{\sigma-\sigma_2} H}{\left(\frac{\sigma-\sigma_2}{1-\sigma_2}\right)^{1-\sigma_2}}.
\end{aligned}$$

Thus $\left\| (\varsigma - \wp)^{\sigma-1} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) \right\|$ is Lebesgue integrable with respect to $\wp \in [0, \varsigma]$ for all $\varsigma \in I$ and $\Im \in C(I, \complement)$. Hence, we know that $(\varsigma - \wp)^{\sigma-1} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp))$ is Bochner integrable with respect to $\wp \in [0, \varsigma]$ for all $\varsigma \in I$ due to Lemma 2.4,

$$\begin{aligned}
&\int_0^\alpha \left\| (\varsigma - \wp)^{\sigma-1} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) \right\| d\wp \\
&\leq \left(\int_0^\alpha (\alpha - \wp)^{\frac{\sigma-\Upsilon}{1-\sigma_2}} d\wp \right)^{1-\sigma_2} \left(\int_0^\alpha (h(\wp))^{\frac{1}{\sigma_2}} d\wp \right)^{\sigma_2} \\
&\leq \frac{\alpha^{\sigma-\sigma_2-\Upsilon+1} H}{\left(\frac{\sigma-\sigma_2-\Upsilon+1}{1-\sigma_2}\right)^{1-\sigma_2}}.
\end{aligned}$$

Thus, $\left\| (\alpha - \wp)^{\sigma-\Upsilon} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) \right\|$ is Lebesgue integrable with respect to $\wp \in [0, \alpha]$ for all $\varsigma \in I$ and $\Im \in C(I, X)$. Hence we know that $(\alpha - \wp)^{\sigma-\Upsilon} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp))$ is Bochner integrable with respect to

$\wp \in [0,\alpha]$, for all $\varsigma \in I$ due to Lemma 2.4. Therefore, the following fractional integral equation is what the FV-FIDE (1.1) represents.

$$\begin{aligned}\Im(\varsigma) &= \frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\quad - \frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\quad + \Im_0 + y_0^1\varsigma + \frac{\Im_0^2}{2!}\varsigma^2 + \cdots + \frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2} + \frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1}, \quad \varsigma \in I.\end{aligned}$$

Let

$$\begin{aligned}r \geq{}& \frac{H\alpha^{\sigma-\sigma_2}}{\Gamma(\sigma)\left(\frac{\sigma-\sigma_2}{1-\sigma_2}\right)^{1-\sigma_2}} + \frac{H\alpha^{\sigma-\sigma_1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\left(\frac{\sigma-\sigma_2-\Upsilon+1}{1-\sigma_2}\right)^{1-\sigma_2} \\ &+ \|\Im_0\| + \left\|\Im_0^1\right\|\alpha + \frac{\left\|\Im_0^2\right\|}{2!}\alpha^2 + \cdots + \frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\alpha^{\Upsilon-2} + \frac{\|\Im_\alpha\|}{(\Upsilon-1)!}\alpha^{\Upsilon-1}.\end{aligned}$$

Now, the operator is defined on $B_r := \{\Im \in C(I,\complement) : \|\Im\| \leq r\}$ as follows

$$\begin{aligned}&(\eth(\Im))(\varsigma) \\ &\quad = \frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\qquad - \frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\qquad + \Im_0 + \Im_0^1\varsigma + \frac{\varsigma_0^2}{2!}t^2 + \cdots + \frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2} + \frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1}, \quad \varsigma \in I. \quad (3.2)\end{aligned}$$

An operator's fixed point on B_r is therefore implied by the presence of a FV-FIDE (1.1) solution. There being a fixed point will be demonstrated using the Banach contraction principle. The evidence consists of two steps.

Step 1. $\Im \in B_r,\ \forall\ y \in B_r$.

By using (E3) and Holder's inequality, we get $\Im \in B_r$ and $\theta > 0$.

$$\begin{aligned}&\|(\eth(\Im))(\varsigma+\theta) - (\eth(\Im))(\varsigma)\| \\ &\quad \leq \|\frac{1}{\Gamma(\sigma)}\int_0^{v+\theta}(\varsigma+\theta-\wp)^{\sigma-1}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp \\ &\qquad - \frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp\| \\ &\qquad + \|\frac{(\varsigma+\theta)^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\varsigma+\theta)^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))d\wp\| \\
&+\|\Im_0^1(\varsigma+\theta-t)+\frac{\Im_0^2}{2!}\left[(t+\theta)^2-\varsigma^2\right]+\dots \\
&+\frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\left[(\varsigma+\theta)^{\Upsilon-2}-\varsigma^{\Upsilon-2}\right]+\frac{\Im_\alpha}{(\Upsilon-1)!}\left[(\varsigma+\theta)^{\Upsilon-1}-\varsigma^{\Upsilon-1}\right]\| \\
\le{}&\frac{H}{\Gamma(\sigma)}\left(\frac{(\varsigma+\theta)^{\frac{\sigma-\sigma_2}{1-\sigma_2}}}{\frac{\sigma-\sigma_2}{1-\sigma_2}}-\frac{\theta^{\frac{\sigma-\sigma_2}{1-\sigma_2}}}{\frac{\sigma-\sigma_2}{1-\sigma_2}}-\frac{\varsigma^{\frac{\sigma-\sigma_2}{1-\sigma_2}}}{\frac{\sigma-\sigma_2}{1-\sigma_2}}\right)^{1-\sigma_2} \\
&+\frac{H}{\Gamma(\sigma)}\left(\frac{\theta^{\frac{\sigma-\sigma_2}{1-\sigma_2}}}{\frac{\sigma-\sigma_2}{1-\sigma_2}}\right)^{1-\sigma_2}+\frac{\left[(\varsigma+\theta)^{\Upsilon-1}-\varsigma^{\Upsilon-1}\right]}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\frac{\alpha^{\sigma-\sigma_2-\Upsilon+1}H}{\left(\frac{\sigma-\sigma_2-\Upsilon+1}{1-\sigma_2}\right)^{1-\sigma_2}} \\
&+\left\|\Im_0^1\right\|(\varsigma+\theta-\varsigma)+\frac{\left\|\Im_0^2\right\|}{2!}\left[(\varsigma+\theta)^2-\varsigma^2\right]+\dots \\
&+\frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\left[(\varsigma+\theta)^{\Upsilon-2}-\varsigma^{\Upsilon-2}\right]+\frac{\|\Im\alpha\|}{(\Upsilon-1)!}\left[(\varsigma+\theta)^{\Upsilon-1}-\varsigma^{\Upsilon-1}\right] \\
\longrightarrow{}& 0 \text{ as } \theta\to 0.
\end{aligned}
$$

Thus, $\eth$ is continuous on I, that is, $\eth(\Im)\in C(I,X)$. Also, for $\Im\in B_r$ and for all $t\in I$, we get

$$
\begin{aligned}
&\|(\eth(\Im))(\varsigma)\| \\
\le{}&\frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1}\|\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))\|d\wp \\
&+\frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon}\|\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))\|d\wp \\
&+\|\Im_0\|+\left\|\Im_0^1\right\|\alpha+\frac{\left\|\Im_0^2\right\|}{2!}\alpha^2+\dots+\frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\alpha^{\Upsilon-2}+\frac{\|\Im_\alpha\|}{(\Upsilon-1)!}\alpha^{\Upsilon-1} \\
\le{}& r,
\end{aligned}
$$

it suggests that $\|\eth(\Im)\|_\infty\le r$, Consequently, we may say that for all $\Im\in B_r,\eth(\Im)\in B_r$, that is, $\boldsymbol{N}:B_r\to B_r$.

Step 2. F is a contraction operator on B_r.

For $y,\Im\in B_r$ and $\forall\ \varsigma\in I$, using **(E2)** and inequality of Holder, we get

$$\begin{aligned}
&\|(\eth(y))(\varsigma)-(\eth(\Im))(\varsigma)\| \\
&\quad\le \frac{1}{\Gamma(\sigma)}\int_0^{\varsigma}(\varsigma-\wp)^{\sigma-1} \\
&\qquad\times\|\Re(\wp,y(\wp),(\Lambda y)(\wp),(\vartheta y)(\wp))-\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))\|d\wp \\
&\qquad+\frac{\alpha^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^{\alpha}(\alpha-\wp)^{\sigma-\Upsilon} \\
&\qquad\times\|\Re(\wp,y(\wp),(\Lambda y)(\wp),(\vartheta y)(\wp))-\Re(\wp,\Im(\wp),(\Lambda\Im)(\wp),(\vartheta\Im)(\wp))\|d\wp \\
&\quad\le\frac{\|y-\Im\|_\infty}{\Gamma(\sigma)}\left(\int_0^{\varsigma}(\varsigma-\wp)^{\frac{\sigma-1}{1-\sigma_1}}d\wp\right)^{1-\sigma_1} \\
&\qquad\times\left(\int_0^{\varsigma}\left(m_1(\wp)+\varphi_0 m_2(\wp)+\varphi_0^1 m_3(\wp)\right)^{\frac{1}{\sigma_1}}d\wp\right)^{\sigma_1} \\
&\qquad+\frac{\left[(\varsigma+\theta)^{\Upsilon-1}-\varsigma^{\Upsilon-1}\right]}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\left(\int_0^{\alpha}(\alpha-\wp)^{\frac{\sigma-\Upsilon}{1-\sigma_1}}d\wp\right)^{1-\sigma_1} \\
&\qquad\times\left(\int_0^{\alpha}\left(m_1(\wp)+\varphi_0 m_2(\wp)+\varphi_0^1 m_3(\wp)\right)^{\frac{1}{\sigma_1}}d\wp\right)^{\sigma_1}.
\end{aligned}$$

So, we get

$$\|\eth(y)-\eth(\Im)\|_\infty\le\Phi_{\sigma,\alpha,\Upsilon}\|y-\Im\|_\infty.$$

As a result of the criterion (3.1), $\boldsymbol{W}$ is contraction. We may infer from the Banach contraction principle that possesses a singular fixed point, which is the singular solution of the FV-FIDE (1.1). □

Theorem 3.2. *Let's assume that* (E1), (E4) *and* (E5) *are true. Then there exists at least one solution to the FV-FIDE* (1.1) *on* I.

Proof. Let's turn the FV-FIDE (1.1) into a fixed point issue. Take into account the operator $\eth : C(I,\complement)\to C(I,\complement)$ defined as (.). It is evident that F is precisely defined thanks to (3.2), Holder's inequality and lemma 2.4.

For clarity, the proof has been broken down into many steps.

Step 1. $\boldsymbol{N}$ is a continuous operator.

Let $\{\Im_\Upsilon\}$ be a sequence such that $\Im_\Upsilon\to\Im$ in $C(I,\complement)$. Then for all $\varsigma\in I$, using the continuity of $\Re$, we get

$$\begin{aligned}
\|\eth(\Im)_\Upsilon-\eth(\Im)\|_\infty\le&\left(\frac{\alpha^\sigma}{\Gamma(\sigma+1)}+\frac{\alpha^\sigma}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+2)}\right)\\
&\times\|\Re(\cdot,\Im_\Upsilon(\cdot),(\Lambda\Im_\Upsilon)(\cdot),(\vartheta\Im_\Upsilon)(\cdot))\\
&-\Re(\cdot,\Im(\cdot),(\Lambda\Im)(\cdot),(\vartheta\Im)(\cdot))\|_\infty\\
\to&\ 0\text{ as }\Upsilon\to\infty.
\end{aligned}$$

Step 2. $\eth$ operators bounded sets into bounded sets in $C(I,\complement)$.

In fact, it suffices to demonstrate that for all $\xi^*>0$, there exists $l>0$ such that for all $\Im\in B_{\xi^*}=\left\{\Im\in C(I,\complement):\|\Im\|_\infty\leq\xi^*\right\}$, we get $\|\Im\|_\infty\leq l$.

For all $\varsigma\in I$, by (E4), we have

$$\begin{aligned}\|(\eth(\Im))(\varsigma)\| \leq & \left(\frac{1}{\Gamma(\sigma+1)}+\frac{1}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+2)}\right)\alpha^\sigma\varphi_0 N\left(1+(\xi^*)^\rho\right)\\ & +\|\Im_0\|+\left\|\Im_0^1\right\|\alpha+\frac{\left\|\Im_0^2\right\|}{2!}\alpha^2+\cdots+\frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\alpha^{\Upsilon-2}+\frac{\|\Im_\alpha\|}{(\Upsilon-1)!}\alpha^{\Upsilon-1}\\ & := l,\end{aligned}$$

it implies that $\|\eth(\Im)\|_\infty\leq l$.

Step 3. $\eth$ operators bounded sets into equicontinuous sets of $C(I,\complement)$.

Let $0\leq\varsigma_1<\varsigma_2\leq\alpha,\Im\in B_{\xi^*}$. Using (E4) again, we get

$$\begin{aligned}&\|(\eth(\Im))(\varsigma_2)-(\eth(\Im))(\varsigma_1)\|\\ &\leq\frac{\varphi_0 N\left(1+(\xi^*)^\rho\right)}{\Gamma(\sigma+1)}\left(\varsigma_2^\sigma-\varsigma_1^\sigma\right)+\frac{\alpha^{\sigma-\Upsilon+1}\varphi_0 N\left(1+(\xi^*)^\rho\right)}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+2)}\left(\varsigma_2^2-\varsigma_1^2\right)\\ &\quad+\left\|\Im_0^1\right\|(\varsigma_2-\varsigma_1)+\frac{\left\|\Im_0^2\right\|}{2!}\left(\varsigma_2^2-\varsigma_1^2\right)+\cdots+\frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\left(\varsigma_2^{\Upsilon-2}-\varsigma_1^{\Upsilon-2}\right)\\ &\quad+\frac{\|\Im_\alpha\|}{(\Upsilon-1)!}\left(\varsigma_2^{\Upsilon-1}-\varsigma_1^{\Upsilon-1}\right)\\ &\longrightarrow 0\quad\text{as}\ \ \varsigma_2\to\varsigma_1.\end{aligned}$$

Therefore $\eth$ is equicontinuous.

Let's now $\{\Im_\Upsilon\},\Upsilon=1,2,\cdots$ be a sequence on B_{ξ^*}, and

$$(\eth(\Im)_\Upsilon)(\varsigma)=(\eth_1\Im_\Upsilon)(\varsigma)+(\eth_2\Im_\Upsilon)(\varsigma)+(\eth_3\Im)(\varsigma),\ \ \varsigma\in I,$$

where

$$\begin{aligned}(\eth_1\Im_\Upsilon)(\varsigma)&=\frac{1}{\Gamma(\sigma)}\int_0^\varsigma(\varsigma-\wp)^{\sigma-1}\Re\left(\wp,\Im_\Upsilon(\wp),(\Lambda\Im_\Upsilon)(\wp),(\vartheta\Im_\Upsilon)(\wp)\right)d\wp,\ \ \varsigma\in I,\\ (\eth_2\Im_\Upsilon)(\varsigma)&=-\frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\int_0^\alpha(\alpha-\wp)^{\sigma-\Upsilon}\\ &\qquad\times\Re\left(\wp,\Im_\Upsilon(\wp),(\Lambda\Im_\Upsilon)(\wp),(\vartheta\Im_\Upsilon)(\wp)\right)d\wp,\ \ \varsigma\in I,\\ (\eth_3\Im)(\varsigma)&=\Im_0+\Im_0^1\varsigma+\frac{\Im_0^2}{2!}\varsigma^2+\cdots+\frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2}+\frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1},\ \ \varsigma\in I.\end{aligned}$$

From (E5), we have that $\overline{con\varsigma}K_1$ is compact. For all $\varsigma^* \in I$, we have

$$\begin{aligned}(\eth_1\Im_\Upsilon)(\varsigma^*) &= \frac{1}{\Gamma(\sigma)}\lim_{k\to\infty}\sum_{j=1}^{k}\frac{\varsigma^*}{k}\left(\varsigma^* - \frac{it^*}{k}\right)^{\sigma-1} \\ &\quad \times \Re\left(\frac{it^*}{k}, \Im_\Upsilon\left(\frac{it^*}{k}\right), (\Lambda\Im_\Upsilon)\left(\frac{it^*}{k}\right), (\vartheta\Im_\Upsilon)\left(\frac{it^*}{k}\right)\right) \\ &= \frac{\Im^*}{\Gamma(\sigma)}\omega_{\Upsilon 1},\end{aligned}$$

where

$$\omega_{\Upsilon 1} = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(\varsigma^* - \frac{it^*}{k}\right)^{\sigma-1} f\left(\frac{it^*}{k}, \Im_\Upsilon\left(\frac{it^*}{k}\right), (\Lambda\Im_\Upsilon)\left(\frac{it^*}{k}\right), (\vartheta\Im_\Upsilon)\left(\frac{it^*}{k}\right)\right).$$

Since $\overline{con\varsigma}K_1$ is convex and compact, we know that $\omega_{\Upsilon_1} \in \overline{con\varsigma}K_1$. Hence, the set $\{(\eth_1\Im_\Upsilon)(\varsigma^*)\}$ is relatively compact, for any $\varsigma^* \in I$. From Ascoli-Arzela theorem, every $\{(\eth_1\Im_\Upsilon)(\varsigma)\}$ contains a uniformly convergent subsequence $\{(\eth_1\Im_{\Upsilon_k})(\varsigma)\}, k = 1, 2, \cdots$ on I. Thus, the set $\{\eth_1\Im : \Im \in B_{\xi^*}\}$ is relatively compact.

Set

$$\begin{aligned}\left(\overline{\eth_2}\Im_\Upsilon\right)(\varsigma) &= -\frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)} \\ &\quad \times \int_0^{\varsigma}(\varsigma-\wp)^{\sigma-\Upsilon}\Re\left(\wp, \Im_\Upsilon(\wp), (\Lambda\Im_\Upsilon)(\wp), (\vartheta\Im_\Upsilon)(\wp)\right)d\wp, \quad \varsigma \in I.\end{aligned}$$

Then, for any $\varsigma^* \in I$,

$$\begin{aligned}\left(\overline{\eth_2}\Im_\Upsilon\right)(\varsigma^*) &= -\frac{(\varsigma^*)^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\lim_{k\to\infty}\sum_{j=1}^{k}\frac{\varsigma^*}{k}\left(\varsigma^* - \frac{it^*}{k}\right)^{\sigma-\Upsilon} \\ &\quad \times \Re\left(\frac{it^*}{k}, \Im_\Upsilon\left(\frac{it^*}{k}\right), (\Lambda\Im_\Upsilon)\left(\frac{it^*}{k}\right), (\vartheta\Im_\Upsilon)\left(\frac{it^*}{k}\right)\right) \\ &= -\frac{(\varsigma^*)^{\Upsilon}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)}\omega_{\Upsilon 2},\end{aligned}$$

where

$$\begin{aligned}\omega_{\Upsilon_2} &= \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(\varsigma^* - \frac{it^*}{k}\right)^{\sigma-\Upsilon} \\ &\quad \times \Re\left(\frac{it^*}{k}, \Im_\Upsilon\left(\frac{it^*}{k}\right), (\Lambda\Im_\Upsilon)\left(\frac{it^*}{k}\right), (\vartheta\Im_\Upsilon)\left(\frac{it^*}{k}\right)\right).\end{aligned}$$

Since $\overline{con\varsigma}K_2$ is compact and convex, we have that $\omega_{\Upsilon 2} \in \overline{\text{conv}}\, K_2$. Hence, the set $\left\{\left(\bar{\eth}_2\Im_\Upsilon\right)(\varsigma^*)\right\}$ is relatively compact for all $\varsigma^* \in I$. From Ascoli-Arzela theorem, every $\left\{\left(\bar{\eth}_2\Im_\Upsilon\right)(\varsigma)\right\}$ contains a uniformly convergent subsequence $\left\{\left(\bar{\eth}_2\Im_{\Upsilon_k}\right)(\varsigma)\right\}, k = 1, 2, \cdots$ on I. Particularly, $\{(\eth_2\Im_\Upsilon)(\varsigma)\}$ contains a uniformly convergent subsequence $\{(\eth_2\Im_{\Upsilon_k})(\varsigma)\}, k = 1, 2, \cdots$ on I. Then, the set $\{\eth_2\Im : \Im \in B_{\xi^*}\}$ is relatively compact.

Clearly, the set $\{\eth_3\Im : \Im \in B_{\xi^*}\}$ is relatively compact. Consequently, the set $\{\eth(\Im) : \Im \in B_{\xi^*}\}$ is relatively compact. As a result of **Step 1-3**, we get that $\eth$ is continuous and then completely continuous.

Step 4. A priori bounds.

Now, demonstrating that the set

$$E(\eth) = \left\{\Im \in C(I, \complement) : \Im = \rho^*\eth(\Im), \quad \text{for some} \quad \rho^* \in [0,1]\right\}$$

is bounded. Let $\Im \in E(\eth)$. Then $\Im = \rho^*\eth(\Im)$ for some $\rho^* \in [0,1]$. Hence, for all $\varsigma \in I$, we get

$$\begin{aligned}\Im(\varsigma) = \rho^* \Bigg(&\frac{1}{\Gamma(\sigma)} \int_0^\varsigma (\varsigma - \wp)^{\sigma-1} \Re\left(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)\right) d\wp \\ &- \frac{\varsigma^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)} \int_0^\alpha (\alpha-\wp)^{\sigma-\Upsilon} \Re(\wp, \Im(\wp), (\Lambda\Im)(\wp), (\vartheta\Im)(\wp)) d\wp \\ &+ \Im_0 + \Im_0^1\varsigma + \frac{\Im_0^2}{2!}\varsigma^2 + \cdots + \frac{\Im_0^{\Upsilon-2}}{(\Upsilon-2)!}\varsigma^{\Upsilon-2} + \frac{\Im_\alpha}{(\Upsilon-1)!}\varsigma^{\Upsilon-1} \Bigg).\end{aligned}$$

For all $\varsigma \in I$, we get

$$\begin{aligned}\|\Im(\varsigma)\| \leq & \|(\eth(\Im))(\varsigma)\| \\ \leq & \frac{\varphi_0 N\alpha^\sigma}{\Gamma(\sigma+1)} + \frac{\varphi_0 N\alpha^\sigma}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)} + \|\Im_0\| + \left\|\Im_0^1\right\| \alpha \\ & + \frac{\left\|\Im_0^2\right\|}{2!}\alpha^2 + \cdots + \frac{\left\|\Im_0^{\Upsilon-2}\right\|}{(\Upsilon-2)!}\alpha^{\Upsilon-2} + \frac{\|\Im_\alpha\|}{(\Upsilon-1)!}\alpha^{\Upsilon-1} \\ & + \frac{\varphi_0 N}{\Gamma(\sigma)} \int_0^\varsigma (\varsigma-\wp)^{\sigma-1} \|\Im(\wp)\|^\rho d\wp \\ & + \frac{\varphi_0 N\alpha^{\Upsilon-1}}{(\Upsilon-1)!\Gamma(\sigma-\Upsilon+1)} \int_0^\alpha (\alpha-\wp)^{\sigma-\Upsilon} \|\Im(\wp)\|^\rho d\wp.\end{aligned}$$

From Lemma 2.9 in [10], there exists $M^* > 0$ such that $\|\Im(\varsigma)\| \leq M^*, \varsigma \in I$. Then for all $\varsigma \in I$, we get $\|\Im\|_\infty \leq M^*$.

By using Schaefer's fixed point theorem to demonstrate that $\eth$ has a fixed point that is a solution of the FV-FIDE (1.1). □

4. An example

In this section, we give one instances to illustrate how useful our key findings are.

Example 4.1. Let's think about the FBVPs listed below:

$$\begin{cases} {}^cD^{\sigma}\Im(\varsigma) = \dfrac{e^{-\varsigma}|\Im(\varsigma)|}{(1+ke^{\varsigma})(1+|\Im(\varsigma)|)} + \displaystyle\int_0^{\varsigma} \dfrac{e^{-\varsigma}}{16}\wp\dfrac{|\Im(\wp)|}{1+|\Im(\wp)|}d\wp \\ \qquad + \displaystyle\int_0^{\alpha} \dfrac{e^{-(\varsigma+\wp)}}{16}\dfrac{|\Im(\wp)|}{1+|\Im(\wp)|}d\wp, \\ \qquad \varsigma \in I = [0,\alpha], \sigma \in (3,4), k > 0, \\ \Im'(0) = 0, \quad \Im''(0) = 0, \quad \Im'''(\alpha) = 0, \\ \Im(0) = \Im(1) = 0 \quad \varsigma > 0, \Im > 0. \end{cases} \tag{4.1}$$

Set

$$\Re(\varsigma) = \frac{e^{-\varsigma}\varsigma}{(1+ke^{\varsigma})(1+\varsigma)}, \quad \varsigma \in I_1 \times [0,\infty), \quad k(\varsigma) = \frac{e^{-\varsigma}}{16}\wp.$$

Let $\Im_1, \Im_2 \in [0,\infty)$ and $\varsigma \in I_1$. Then, we get

$$|\Re(\varsigma, \Im_1, \Lambda\Im_1, \vartheta\Im_1) - \Re(\varsigma, \Im_2, \Lambda\Im_2, \vartheta\Im_2)| \leq \frac{9e^{-\varsigma}}{16}|\Im_1 - \Im_2|.$$

Naturally, for all $\Im \in [0,\infty)$ and each $\varsigma \in I_1$,

$$|\Re(\varsigma, \Im, \Lambda\Im, \vartheta\Im)| \leq \frac{9e^{-\varsigma}}{16}.$$

For $\varsigma \in I_1, \zeta \in (0, \sigma-3)$, let $m_1(\varsigma) = m_2(\varsigma) = m_3(\varsigma) = h(\varsigma) = \frac{e^{-\varsigma}}{32} \in L^{\frac{1}{\zeta}}(I_1, \mathbb{R})$, $M = \left\|\frac{9e^{-\varsigma}}{16}\right\|_{L^{\frac{1}{\zeta}}(I_1,\mathbb{R})}$. Selecting an appropriate $\alpha \geq 1$ and big enough $k > 0$ and an appropriate $\sigma \in (3,4)$ and $\zeta \in (0, \sigma-3)$. Using the following inequality as a starting point

$$\Omega_{\sigma,\alpha} = \frac{M}{\Gamma(\sigma)}\frac{\alpha^{\sigma-\zeta}}{\left(\frac{\sigma-\zeta}{1-\zeta}\right)^{1-\zeta}} + \frac{M}{3!\Gamma(\sigma-3)}\frac{\alpha^{\sigma-\zeta}}{\left(\frac{\sigma-\zeta-2}{1-\zeta}\right)^{1-\zeta}} < 1,$$

Clearly, Theorem 3.1's presumptions are all true. Our conclusions can be utilized to address the problem with (4.1).

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