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CONTROL FUNCTION BASED COUPLED AND COMMON COUPLED FIXED POINT THEOREMS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we aim to prove coupled and common coupled fixed point theorems for contractive type conditions in the context of partial metric spaces by means of a control function, and to provide some corollaries of the established results. This paper presents a number of results that generalize and extend previous work in the field. In order to better illustrate the process, we provide examples.

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1. Introduction and preliminaries

The fixed point theorem is fundamental to many well-known mathematical theories and is used in many contexts. After Banach's [5] fundamental result, commonly referred to as it Banach's contraction mapping principle, "the field of fixed point theory saw a significant uptick in progress". The following is a correct statement of this result.

Theorem 1.1. ([5]) *Let (X, d) be a complete metric space and $Q\colon X \to X$ be a map satisfying*

$$d(Q(\mu), Q(\nu)) \leq \beta\, d(\mu, \nu), \text{ for all } \mu, \nu \in X, \tag{1.1}$$

where $0 < \beta < 1$ is a constant. Then
(1) *Q has a unique fixed point p in X;*
(2) *The Picard iteration $\{\mu_n\}_{n=0}^{\infty}$ defined by*

$$\mu_{n+1} = Q\mu_n, \quad n = 0, 1, 2, \dots \tag{1.2}$$

converges to p, for any $\mu_0 \in X$.

Remark 1.2. (i) A self-map satisfying (1) and (2) is said to be a Picard operator (see, [25, 26]).
(ii) Inequality (1.1) also implies the continuity of Q.

The existence and uniqueness of a solution to an operator equation $Q\mu = \mu$ can be proven by applying Banach's contraction mapping principle. In light of these advancements, Bhashkar and Lakshmikantham [8] introduced the concept of coupled fixed point of a mapping $F\colon X \times X \to X$ and explored some coupled fixed point theorems in partially ordered complete metric spaces. They also presented the first proof of the mixed monotone property and a classical proof of a fixed point for a mapping satisfying that property. An associated first order differential equation with a periodic boundary value problem was investigated for its existence and uniqueness.

Some coupled fixed point theorems in metric spaces were later proven by Ciric and Lakshmikantham [7], Sabetghadam *et al.* [27] and Olaleru *et al.* [23]. When dealing with weakly compatible mappings in the context of cone metric spaces, Abbas *et al.* [1] established common coupled fixed point results. Common coupled fixed point results for generalized nonlinear contraction mappings with mixed monotone property in partially ordered metric spaces were proven by Kim and Chandok [12] (also see [9], [15], [16], [22], [20], [21], [28], [29], [30]), among other related works.

Aydi [3] proved some coupled fixed point theorems in partial metric spaces in 2011. Recently, in the context of partial metric spaces, Kim *et al.* [13] proved some general fixed point theorems for weak compatible mappings.

Matthews [18, 19] introduced the concept of partial metric space (PMS) to the field of denotational semantics of data flow networks. Model building in the theory of computation relies heavily on partial metric spaces (for examples, see [10], [23], and others). When it Matthews introduced the concept of partial metric space, he proved a version of the Banach fixed point theorem [5] that applied to this new setting. In the PMS, the distance between any two points, denoted by $d(\mu,\mu)$, need not be zero. For more information, (see [2], [17], [24]).

We will need some fundamental theorems, lemmas, and supplementary results in partial metric spaces for the continuation.

Definition 1.3. ([19]) Let X be a nonempty set and $p\colon X\times X\to\mathbb{R}^+$ be a self-mapping of X such that for all $\mu,\nu,\xi\in X$ the followings are satisfied:

(P1) $\mu=\nu \;\Leftrightarrow\; p(\mu,\mu)=p(\mu,\nu)=p(\nu,\nu)$,
(P2) $p(\mu,\mu)\leq p(\mu,\nu)$,
(P3) $p(\mu,\nu)=p(\nu,\mu)$,
(P4) $p(\mu,\nu)\leq p(\mu,\xi)+p(\xi,\nu)-p(\xi,\xi)$.

Then p is called partial metric on X and the pair (X,p) is called partial metric space (in short, PMS).

Remark 1.4. It is clear that if $p(\mu,\nu)=0$, then from $(P1)$, $(P2)$, and $(P3)$, $\mu=\nu$. But if $\mu=\nu$, $p(\mu,\nu)$ may not be 0.

If p is a partial metric on X, then the function $p^s\colon X\times X\to\mathbb{R}^+$ given by

$$p^s(\mu,\nu)=2p(\mu,\nu)-p(\mu,\mu)-p(\nu,\nu) \tag{1.3}$$

is a metric on X.

Example 1.5. ([4]) Let $X=\mathbb{R}^+$, where $\mathbb{R}^+=[0,+\infty)$ and $p\colon X\times X\to\mathbb{R}^+$ be given by $p(\mu,\nu)=\max\{\mu,\nu\}$ for all $\mu,\nu\in\mathbb{R}^+$. Then $(\mathbb{R}^+,p)$ is a partial metric space.

Example 1.6. ([4]) Let I denote the set of all intervals $[a_1,a_2]$ for any real numbers $a_1\leq a_2$. Let $p\colon I\times I\to[0,\infty)$ be a function such that

$$p\big([a_1,a_2],[b_1,b_2]\big)=\max\{a_2,b_2\}-\min\{a_1,b_1\}.$$

Then (I,p) is a partial metric space.

Example 1.7. ([6]) Let $X=\mathbb{R}$ and $p\colon X\times X\to\mathbb{R}^+$ be given by $p(\mu,\nu)=e^{\max\{\mu,\nu\}}$ for all $\mu,\nu\in\mathbb{R}$. Then (X,p) is a partial metric space.

Numerous authors have extensively studied the various applications of this space (see [14], [31] for details). Note also that each partial metric p on

X generates a T_0 topology τ_p on X, whose base is a family of open p-balls $\{B_p(\mu,\varepsilon) : \mu \in X, \varepsilon > 0\}$, where

$$B_p(\mu,\varepsilon) = \{\nu \in X : p(\mu,\nu) < p(\mu,\mu) + \varepsilon\}$$

for all $\mu \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as

$$B_p[\mu,\varepsilon] = \{\nu \in X : p(\mu,\nu) \leq p(\mu,\mu) + \varepsilon\}$$

for all $\mu \in X$ and $\varepsilon > 0$.

Definition 1.8. ([18]) Let (X,p) be a partial metric space. Then

(i) a sequence $\{s_n\}$ in (X,p) is said to be convergent to a point $s \in X$ if and only if $p(s,s) = \lim_{n\to\infty} p(s_n,s)$;

(ii) a sequence $\{s_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(s_m,s_n)$ exists and is finite;

(iii) (X,p) is said to be complete if every Cauchy sequence $\{s_n\}$ in X converges to a point $s \in X$ with respect to τ_p. Furthermore,

$$\lim_{m,n\to\infty} p(s_m,s_n) = \lim_{n\to\infty} p(s_n,s) = p(s,s);$$

(iv) a mapping $F\colon X \to X$ is said to be continuous at $s_0 \in X$ if for every $\varepsilon > 0$, there exists $a > 0$ such that $F\big(B_p(s_0,a)\big) \subset B_p\big(F(s_0),\varepsilon\big)$.

Lemma 1.9. ([3, 18, 19]) *Let (X,p) be a partial metric space. Then*

(a) *a sequence $\{s_n\}$ in (X,p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,p^s),*

(b) *a partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete, furthermore, $\lim_{n\to\infty} p^s(s_n,s) = 0$ if and only if*

$$p(s,s) = \lim_{n\to\infty} p(s_n,s) = \lim_{n,m\to\infty} p(s_n,s_m). \tag{1.4}$$

Lemma 1.10. ([11]) *Let (X,p) be a partial metric space.*

(1) *If for all $\mu,\nu \in X$, $p(\mu,\nu) = 0$, then $\mu = \nu$.*

(2) *If $\mu \neq \nu$, then $p(\mu,\nu) > 0$.*

Definition 1.11. ([3]) An element $(\mu,\nu) \in X \times X$ is said to be a coupled fixed point of the mapping $F\colon X \times X \to X$ if $F(\mu,\nu) = \mu$ and $F(\nu,\mu) = \nu$.

Example 1.12. Let $X = [0,+\infty)$ and $F\colon X \times X \to X$ defined by $F(\mu,\nu) = \frac{\mu+\nu}{6}$ for all $\mu,\nu \in X$. One can easily see that F has a unique coupled fixed point $(0,0)$.

Example 1.13. Let $X = [0,+\infty)$ and $F\colon X \times X \to X$ be defined by $F(\mu,\nu) = \frac{\mu+\nu}{2}$ for all $\mu,\nu \in X$. Then we see that F has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.

Definition 1.14. ([1],[12]) An element $(\mu,\nu) \in X \times X$ is called

(1) a coupled coincidence point of mappings $F\colon X \times X \to X$ and $A\colon X \to X$ if $A(\mu) = F(\mu,\nu)$ and $A(\nu) = F(\nu,\mu)$, and $(A\mu, A\nu)$ is called a coupled point of coincidence;

(2) a common coupled fixed point of mappings $F\colon X \times X \to X$ and $A\colon X \to X$ if $\mu = A(\mu) = F(\mu,\nu)$ and $\nu = A(\nu) = F(\nu,\mu)$.

Definition 1.15. ([1]) The mappings $F\colon X \times X \to X$ and $A\colon X \to X$ are called weakly compatible if $A(F(\mu,\nu)) = F(A\mu, A\nu)$ and $A(F(\nu,\mu)) = F(A\nu, A\mu)$ for all $\mu,\nu \in X$, whenever $A(\mu) = F(\mu,\nu)$ and $A(\nu) = F(\nu,\mu)$.

Example 1.16. Let $X = [0,3]$ endowed with $p(\mu,\nu) = \max\{\mu,\nu\}$ for all $\mu,\nu \in X$. Define $F\colon X \times X \to X$ and $A\colon X \to X$ by

$$F(\mu,\nu) = \begin{cases} \mu+\nu, & \text{if } \mu,\nu \in [0,1), \\ 3, & \text{otherwise}, \end{cases}$$

for all $\mu,\nu \in X$ and

$$A(\mu) = \begin{cases} \mu, & \text{if } \mu \in [0,1), \\ 3, & \text{if } \mu \in [1,3], \end{cases}$$

for all $\mu,\nu \in X$. Then for any $\mu,\nu \in [1,3]$,

$$F(A\mu, A\nu) = F(3,3) = 3 = A(F(\mu,\nu)) = A(3) = 3.$$

Similarly, we have

$$F(A\nu, A\mu) = F(3,3) = 3 = A(F(\nu,\mu)) = A(3) = 3.$$

Thus,

$$F(A\mu, A\nu) = A(F(\mu,\nu)) \text{ and } F(A\nu, A\mu) = A(F(\nu,\mu)).$$

This shows that the mappings F and A are weakly compatible on $[0,3]$.

Example 1.17. Let $X = \mathbb{R}$ endowed with the usual metric $p(\mu,\nu) = \max\{\mu,\nu\}$ for all $\mu,\nu \in X$. Define $F\colon X \times X \to X$ and $A\colon X \to X$ by $F(\mu,\nu) = \mu+\nu$ and $A(\mu) = \mu^2$ for all $\mu,\nu \in X$. Then F and A are not weakly compatible maps on $\mathbb{R}$, since

$$F(A\mu, A\nu) = F(\mu^2,\nu^2) = \mu^2+\nu^2, \text{ but } A(F(\mu,\nu)) = A(\mu+\nu) = (\mu+\nu)^2.$$

Therefore,

$$F(A\mu, A\nu) \neq A(F(\mu,\nu)).$$

Hence the mappings F and A are not weakly compatible on $\mathbb{R}$.

Sabetghadam *et al.* [27] obtained the following result in cone metric space.

Theorem 1.18. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies the following contractive condition for all $\mu, \nu, u, v \in X$*

$$d(F(\mu,\nu), F(u,v)) \leq k_1 d(\mu,u) + k_2 d(\nu,v), \tag{1.5}$$

where k_1, k_2 are nonnegative constants with $k_1 + k_2 < 1$. Then F has a unique coupled fixed point.

Recently, Aydi [3] obtained the following results in partial metric space.

Theorem 1.19. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfies one of the following contractive conditions (Γ_1), (Γ_2), (Γ_3):*

(Γ_1) *for all $\mu, \nu, u, v \in X$ and nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$,*

$$p(F(\mu,\nu), F(u,v)) \leq k_1 p(\mu,u) + k_2 p(\nu,v), \tag{1.6}$$

(Γ_2) *for all $\mu, \nu, u, v \in X$ and nonnegative constants k_1, k_2 with $k_1 + k_2 < 1$,*

$$p(F(\mu,\nu), F(u,v)) \leq k_1 p(F(\mu,\nu),\mu) + k_2 p(F(u,v),u), \tag{1.7}$$

(Γ_3) *for all $\mu, \nu, u, v \in X$ and nonnegative constants k_1, k_2 with $k_1 + 2k_2 < 1$,*

$$p(F(\mu,\nu), F(u,v)) \leq k_1 p(F(\mu,\nu),u) + k_2 p(F(u,v),\mu). \tag{1.8}$$

Then F has a unique coupled fixed point.

Motivated by [3, 13] and others, the purpose of this paper is to establish coupled fixed point and common coupled fixed point theorems for contractive type conditions in the context of partial metric spaces utilizing the control function. Our findings extend and generalize a number of previously published findings (see, for example, [3, 13, 27] and numerous others).

2. Main results

In this section, we will demonstrate unique coupled fixed point and common coupled fixed point theorems for partial metric spaces. We shall define the control function Ψ as follows:

Definition 2.1. Let Ψ be the set of functions $\psi\colon [0,\infty) \to [0,\infty)$ satisfying the following conditions:

($\mathbf{\Psi_1}$) ψ is continuous;
($\mathbf{\Psi_2}$) $\psi(t) < t$ for all $t > 0$.

Obviously, if $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(t) \leq t$ for all $t \geq 0$.

Theorem 2.2. *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $\mu, \nu, u, v \in X$*

$$p(F(\mu,\nu), F(u,v)) \le \psi\big(M(\mu,\nu,u,v)\big), \tag{2.1}$$

where

$$\begin{aligned} M(\mu,\nu,u,v) \;\; &= \;\; a_1\, p(\mu,u) + a_2\,[p(F(\mu,\nu),\mu) + p(F(u,v),u)] \\ &\quad + a_3\,[p(F(\mu,\nu),u) + p(F(u,v),\mu)] \\ &\quad + a_4\, \frac{[1 + p(F(u,v),u)]p(F(\mu,\nu),\mu)}{1 + p(F(\mu,\nu),u)}, \end{aligned} \tag{2.2}$$

a_1, a_2, a_3, a_4 are nonnegative constants with $a_1 + 2a_2 + 2a_3 + a_4 < 1$ and ψ is defined as in Definition 2.1. Then F has a unique coupled fixed point.

Proof. Choose $\mu_0, \nu_0 \in X$. Set $\mu_1 = F(\mu_0, \nu_0)$ and $\nu_1 = F(\nu_0, \mu_0)$. Repeating this process, we obtain two sequences $\{\mu_n\}$ and $\{\nu_n\}$ in X such that $\mu_{n+1} = F(\mu_n, \nu_n)$ and $\nu_{n+1} = F(\nu_n, \mu_n)$. Then, from equations (2.1), (2.2) and using $(P2)$, $(P3)$, $(P4)$, we have

$$\begin{aligned} p(\mu_n, \mu_{n+1}) &= p(F(\mu_{n-1}, \nu_{n-1}), F(\mu_n, \nu_n)) \\ &\le \psi\big(M(\mu_{n-1}, \nu_{n-1}, \mu_n, \nu_n)\big), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(\mu_{n-1}, \nu_{n-1}, \mu_n, \nu_n) &= a_1\, p(\mu_{n-1}, \mu_n) + a_2\,[p(F(\mu_{n-1}, \nu_{n-1}), \mu_{n-1}) \\ &\quad + p((F(\mu_n, \nu_n), \mu_n)] \\ &\quad + a_3\,[p(F(\mu_{n-1}, \nu_{n-1}), \mu_n) + p(F(\mu_n, \nu_n), \mu_{n-1})] \\ &\quad + a_4\, \frac{[1 + p(F(\mu_n, \nu_n), \mu_n)]p(F(\mu_{n-1}, \nu_{n-1}), \mu_{n-1})}{1 + p(F(\mu_{n-1}, \nu_{n-1}), \mu_n)} \\ &= a_1\, p(\mu_{n-1}, \mu_n) + a_2\,[p(\mu_n, \mu_{n-1}) + p(\mu_{n+1}, \mu_n)] \\ &\quad + a_3\,[p(\mu_n, \mu_n) + p(\mu_{n+1}, \mu_{n-1})] \\ &\quad + a_4\, \frac{[1 + p(\mu_{n+1}, \mu_n)]p(\mu_n, \mu_{n-1})}{1 + p(\mu_n, \mu_n)} \\ &\le a_1\, p(\mu_{n-1}, \mu_n) + a_2\,[p(\mu_n, \mu_{n-1}) + p(\mu_{n+1}, \mu_n)] \\ &\quad + a_3\,[p(\mu_n, \mu_n) + p(\mu_{n+1}, \mu_n) + p(\mu_n, \mu_{n-1}) - p(\mu_n, \mu_n)] \\ &\quad + a_4\, \frac{[1 + p(\mu_{n+1}, \mu_n)]p(\mu_n, \mu_{n-1})}{1 + p(\mu_n, \mu_{n+1})} \\ &= (a_1 + a_2 + a_3 + a_4)\, p(\mu_{n-1}, \mu_n) \\ &\quad + (a_2 + a_3)\, p(\mu_n, \mu_{n+1}). \end{aligned} \tag{2.4}$$

From equations (2.3) and (2.4), we obtain

$$p(\mu_n,\mu_{n+1}) \le \psi\Big((a_1+a_2+a_3+a_4)\,p(\mu_{n-1},\mu_n) + (a_2+a_3)\,p(\mu_n,\mu_{n+1})\Big). \tag{2.5}$$

Using the property of ψ in the above equation, we get

$$p(\mu_n,\mu_{n+1}) \le (a_1+a_2+a_3+a_4)\,p(\mu_{n-1},\mu_n) + (a_2+a_3)\,p(\mu_n,\mu_{n+1}). \tag{2.6}$$

Similarly, we have

$$p(\nu_n,\nu_{n+1}) \le (a_1+a_2+a_3+a_4)\,p(\nu_{n-1},\nu_n) + (a_2+a_3)\,p(\nu_n,\nu_{n+1}). \tag{2.7}$$

Set

$$\zeta_n = p(\mu_n,\mu_{n+1}), \quad \omega_n = p(\nu_n,\nu_{n+1}) \tag{2.8}$$

and

$$\lambda_n = \zeta_n + \omega_n = p(\mu_n,\mu_{n+1}) + p(\nu_n,\nu_{n+1}). \tag{2.9}$$

From equations (2.5) - (2.9), we obtain

$$\lambda_n \le (a_1+a_2+a_3+a_4)\lambda_{n-1} + (a_2+a_3)\lambda_n.$$

This implies that

$$\lambda_n \le \Big(\frac{a_1+a_2+a_3+a_4}{1-a_2-a_3}\Big)\lambda_{n-1} = \delta\,\lambda_{n-1}, \tag{2.10}$$

where $\delta = \Big(\frac{a_1+a_2+a_3+a_4}{1-a_2-a_3}\Big) < 1$, since $a_1+2a_2+2a_3+a_4 < 1$. Then for each $n \in \mathbb{N}$, we have

$$\lambda_n \le \delta\lambda_{n-1} \le \delta^2\lambda_{n-2} \le \cdots \le \delta^n\lambda_0. \tag{2.11}$$

If $\lambda_0 = 0$, then $p(\mu_0,\mu_1) + p(\nu_0,\nu_1) = 0$. Hence, from Remark 1.4, we get $\mu_0 = \mu_1 = F(\mu_0,\nu_0)$ and $\nu_0 = \nu_1 = F(\nu_0,\mu_0)$, means that (μ_0,ν_0) is a coupled fixed point of F.

Now, we assume that $\lambda_0 > 0$. For each $n \ge m$, where $n, m \in \mathbb{N}$, we have, by using condition $(P4)$

$$\begin{aligned} p(\mu_n,\mu_m) &\le p(\mu_n,\mu_{n-1}) + p(\mu_{n-1},\mu_{n-2}) + \cdots + p(\mu_{m+1},\mu_m) \\ &\quad - p(\mu_{n-1},\mu_{n-1}) - p(\mu_{n-2},\mu_{n-2}) - \cdots - p(\mu_{m+1},\mu_{m+1}) \\ &\le p(\mu_n,\mu_{n-1}) + p(\mu_{n-1},\mu_{n-2}) + \cdots + p(\mu_{m+1},\mu_m). \end{aligned} \tag{2.12}$$

Similarly, we have

$$\begin{aligned} p(\nu_n,\nu_m) &\leq p(\nu_n,\nu_{n-1})+p(\nu_{n-1},\nu_{n-2})+\cdots+p(\nu_{m+1},\nu_m) \\ &\quad -p(\nu_{n-1},\nu_{n-1})-p(\nu_{n-2},\nu_{n-2})-\cdots-p(\nu_{m+1},\nu_{m+1}) \\ &\leq p(\nu_n,\nu_{n-1})+p(\nu_{n-1},\nu_{n-2})+\cdots+p(\nu_{m+1},\nu_m). \end{aligned} \tag{2.13}$$

Thus,

$$\begin{aligned} p(\mu_n,\mu_m)+p(\nu_n,\nu_m) &\leq \lambda_{n-1}+\lambda_{n-2}+\cdots+\lambda_m \\ &\leq (\delta^{n-1}+\delta^{n-2}+\cdots+\delta^m)\lambda_0 \\ &\leq \Big(\frac{\delta^m}{1-\delta}\Big)\lambda_0. \end{aligned} \tag{2.14}$$

By definition of metric p^s, we have $p^s(\mu,\nu) \leq 2p(\mu,\nu)$, therefore for any $n \geq m$

$$\begin{aligned} p^s(\mu_n,\mu_m)+p^s(\nu_n,\nu_m) &\leq 2p(\mu_n,\mu_m)+2p(\nu_n,\nu_m) \\ &\leq \Big(\frac{2\delta^m}{1-\delta}\Big)\lambda_0, \end{aligned} \tag{2.15}$$

which implies that $\{\mu_n\}$ and $\{\nu_n\}$ are Cauchy sequences in (X,p^s) because $0 \leq \delta < 1$, where $\delta = a_1+2a_2+2a_3+a_4 < 1$. Since the partial metric space (X,p) is complete, by Lemma 1.9, the metric space (X,p^s) is complete, so there exist $g,h \in X$ such that

$$\lim_{n\to\infty} p^s(\mu_n,g) = \lim_{n\to\infty} p^s(\nu_n,h) = 0. \tag{2.16}$$

From Lemma 1.9, we obtain

$$p(g,g) = \lim_{n\to\infty} p(\mu_n,g) = \lim_{n\to\infty} p(\mu_n,\mu_n) \tag{2.17}$$

and

$$p(h,h) = \lim_{n\to\infty} p(\nu_n,h) = \lim_{n\to\infty} p(\nu_n,\nu_n). \tag{2.18}$$

But, from condition $(P2)$ and equation (2.11), we have

$$p(\mu_n,\mu_n) \leq p(\mu_n,\mu_{n+1}) \leq \lambda_n \leq \delta^n\lambda_0 \tag{2.19}$$

and since $0 \leq \delta < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} p(\mu_n,\mu_n) = 0$. It follows that

$$p(g,g) = \lim_{n\to\infty} p(\mu_n,g) = \lim_{n\to\infty} p(\mu_n,\mu_n) = 0. \tag{2.20}$$

Similarly, we obtain

$$p(h,h) = \lim_{n\to\infty} p(\nu_n,h) = \lim_{n\to\infty} p(\nu_n,\nu_n) = 0. \tag{2.21}$$

Now, using equations (2.1), (2.2), the conditions ($P3$) and ($P4$), we have

$$\begin{aligned} p(F(g,h),g) &\leq p(F(g,h),\mu_{n+1}) + p(\mu_{n+1},g) - p(\mu_{n+1},\mu_{n+1}) \\ &\leq p(F(g,h),\mu_{n+1}) + p(\mu_{n+1},g) \\ &= p(F(g,h),F(\mu_n,\nu_n)) + p(\mu_{n+1},g) \\ &= p(F(\mu_n,\nu_n),F(g,h)) + p(\mu_{n+1},g) \\ &\leq \psi\big(M(\mu_n,\nu_n,g,h)\big) + p(\mu_{n+1},g), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} M(\mu_n,\nu_n,g,h) &= a_1\, p(\mu_n,g) + a_2\,[p(F(\mu_n,\nu_n),\mu_n) \\ &\quad +p((F(g,h),g)] \\ &\quad +a_3\,[p(F(\mu_n,\nu_n),g) + p(F(g,h),\mu_n)] \\ &\quad +a_4\,\frac{[1+p(F(g,h),g)]p(F(\mu_n,\nu_n),\mu_n)}{1+p(F(\mu_n,\nu_n),g)} \\ &= a_1\, p(\mu_n,g) + a_2\,[p(\mu_{n+1},\mu_n) \\ &\quad +p((F(g,h),g)] \\ &\quad +a_3\,[p(\mu_{n+1},g) + p(F(g,h),\mu_n)] \\ &\quad +a_4\,\frac{[1+p(F(g,h),g)]p(\mu_{n+1},\mu_n)}{1+p(\mu_{n+1},g)}. \end{aligned} \tag{2.23}$$

From equations (2.22)-(2.23) and using the property of ψ, we obtain

$$\begin{aligned} p(F(g,h),g) &\leq \psi\Big(a_1\, p(\mu_n,g) + a_2\,[p(\mu_{n+1},\mu_n) + p((F(g,h),g)] \\ &\quad +a_3\,[p(\mu_{n+1},g) + p(F(g,h),\mu_n)] \\ &\quad +a_4\,\frac{[1+p(F(g,h),g)]p(\mu_{n+1},\mu_n)}{1+p(\mu_{n+1},g)}\Big) \\ &\quad +p(\mu_{n+1},g) \\ &\leq a_1\, p(\mu_n,g) + a_2\,[p(\mu_{n+1},\mu_n) \\ &\quad +p(F(g,h),g)] + a_3\,[p(\mu_{n+1},g) + p(F(g,h),\mu_n)] \\ &\quad +a_4\,\frac{[1+p(F(g,h),g)]p(\mu_{n+1},\mu_n)}{1+p(\mu_{n+1},g)} \\ &\quad +p(\mu_{n+1},g). \end{aligned} \tag{2.24}$$

Passing to the limit as $n \to \infty$ in equation (2.24) and using equation (2.20), we obtain

$$\begin{aligned} p(F(g,h),g) &\leq (a_2+a_3)\, p(F(g,h),g) \\ &\leq (a_1+2a_2+2a_3+a_4)\, p(F(g,h),g) \\ &< p(F(g,h),g), \end{aligned}$$

which is a contradiction, since $a_1 + 2a_2 + 2a_3 + a_4 < 1$. Hence, we have $p(F(g,h),g) = 0$, that is, $F(g,h) = g$. Similarly, we can show that $F(h,g) = h$. This shows that (g,h) is a coupled fixed point of F.

Now, we show the uniqueness. Suppose that (g',h') is another coupled fixed point of F such that $(g,h) \neq (g',h')$, then from equation (2.1), (2.2) and using (2.20), (2.21) and $(P3)$, we have

$$\begin{aligned} p(g,g') &= p(F(g,h),F(g',h')) \\ &\leq \psi\big(M(g,h,g',h')\big), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} M(g,h,g',h') &= a_1\, p(g,g') + a_2\,[p(F(g,h),g) + p(F(g',h'),g')] \\ &\quad + a_3\,[p(F(g,h),g') + p(F(g',h'),g)] \\ &\quad + a_4\,\frac{[1 + p(F(g',h'),g')]p(F(g,h),g)}{1 + p(F(g,h),g')} \\ &= a_1\, p(g,g') + a_2\,[p(g,g) + p(g',g')] \\ &\quad + a_3\,[p(g,g') + p(g',g)] \\ &\quad + a_4\,\frac{[1 + p(g',g')]p(g,g)}{1 + p(g,g')} \\ &= (a_1 + 2a_3)\, p(g,g'). \end{aligned} \tag{2.26}$$

From equations (2.25), (2.26) and using the property of ψ, we get

$$\begin{aligned} p(g,g') &\leq \psi\Big((a_1 + 2a_3)\, p(g,g')\Big) \\ &< (a_1 + 2a_3)\, p(g,g') \\ &\leq (a_1 + 2a_2 + 2a_3 + a_4)\, p(g,g') \\ &< p(g,g'), \end{aligned}$$

which is a contradiction, since $a_1 + 2a_2 + 2a_3 + a_4 < 1$. Hence, we have $p(g,g') = 0$, that is, $g = g'$. By similar fashion, we can show that $h = h'$. This shows that the coupled fixed point of F is unique. This completes the proof. □

If we set $\psi(t) = k\,t$ where $0 < k < 1$, $a_1 = a_3 = a_4 = 0$ and $ka_2 \to l$ where $l \in (0, \frac{1}{2})$ in Theorem 1.19, then we have the following result.

Corollary 2.3. ([3]) *Let (X,p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $\mu, \nu, u, v \in X$*

$$p(F(\mu,\nu),F(u,v)) \leq l\,[p(F(\mu,\nu),\mu) + p(F(u,v),u)], \tag{2.27}$$

where $l \in (0, \frac{1}{2})$ is a constant. Then F has a unique coupled fixed point.

If we set $\psi(t) = k\,t$ where $0 < k < 1$, $a_1 = a_2 = a_4 = 0$ and $ka_3 \to m$ where $m \in (0, \frac{1}{2})$ in Theorem 1.19, then we have the following result.

Corollary 2.4. ([3]) *Let (X, p) be a complete partial metric space. Suppose that the mapping $F\colon X \times X \to X$ satisfying the following contractive condition for all $\mu, \nu, u, v \in X$,*

$$p(F(\mu,\nu), F(u,v)) \leq m\,[p(F(\mu,\nu), u) + p(F(u,v), \mu)], \tag{2.28}$$

where $m \in (0, \frac{1}{2})$ is a constant. Then F has a unique coupled fixed point.

Remark 2.5. (1) Theorem 2.2 extends the results of t Aydi [3].

(2) Theorem 2.2 also extends the results of Sabetghadam *et al.* [27] from cone metric space to the setting of partial metric space.

Example 2.6. Let $X = [0, +\infty)$ endowed with the usual partial metric p defined by $p\colon X \times X \to [0, +\infty)$ with $p(\mu, \nu) = \max\{\mu, \nu\}$. The partial metric space (X, p) is complete because (X, p^s) is complete. Indeed, for any $\mu, \nu \in X$,

$$\begin{aligned} p^s(\mu,\nu) &= 2p(\mu,\nu) - p(\mu,\mu) - p(\nu,\nu) \\ &= 2\max\{\mu,\nu\} - (\mu+\nu) = |\mu - \nu|. \end{aligned}$$

Thus, (X, p^s) is the Euclidean metric space which is complete. Consider the mapping $F\colon X \times X \to X$ defined by $F(\mu,\nu) = \frac{\mu+\nu}{6}$. Now, for any $\mu, \nu, u, v \in X$, we have

$$\begin{aligned} p(F(\mu,\nu), F(u,v)) &= \frac{1}{6}\max\{\mu+\nu, u+v\} \\ &\leq \frac{1}{6}[\max\{\mu+\nu, u\} + \max\{u+v, \mu\}] \\ &= \frac{1}{6}[p(F(\mu,\nu), u) + p(F(u,v), \mu)], \end{aligned}$$

which is the contractive condition of Corollary 2.4 for $m = 1/6 < 1/2$. Therefore, by Corollary 2.4, F has a unique coupled fixed point, which is $(0, 0)$.

Note that if the mapping $F\colon X \times X \to X$ is given by $F(\mu,\nu) = \frac{\mu+\nu}{2}$, then F satisfies contractive condition of Corollary 2.4 for $m = 1/2$, that is,

$$\begin{aligned} p(F(\mu,\nu), F(u,v)) &= \frac{1}{2}\max\{\mu+\nu, u+v\} \\ &\leq \frac{1}{2}[\max\{\mu+\nu, u\} + \max\{u+v, \mu\}] \\ &= \frac{1}{2}[p(F(\mu,\nu), u) + p(F(u,v), \mu)]. \end{aligned}$$

In this case $(0, 0)$ and $(1, 1)$ are both coupled fixed points of F, and hence, the coupled fixed point of F is not unique. This shows that the condition

$m < 1/2$ in Corollary 2.4, and hence $k_1 + 2k_2 < 1$ in Theorem 1.18 (Γ_3) can not be omitted in the statement of the aforesaid results.

Theorem 2.7. *Let (X, p) be a complete partial metric space. Suppose that the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ satisfy the following contractive condition for all $\mu, \nu, u, v \in X$,*

$$\begin{aligned} p(F(\mu,\nu),F(u,v)) \leq q_1\,\psi\Big(\frac{[1+p(F(\mu,\nu),A\mu)]p(F(u,v),Au)}{1+p(A\mu,Au)}\Big) \\ + q_2\,\psi\Big(\max\Big\{[p(A\mu,Au)+p(A\nu,Av)], \\ [p(F(\mu,\nu),A\mu)+p(F(u,v),Au)], \\ [p(F(\mu,\nu),Au)+p(F(u,v),A\mu)]\Big\}\Big), \end{aligned} \tag{2.29}$$

where q_1, q_2 are nonnegative constants with $q_1 + 2q_2 < 1$ and ψ is defined as in Definition 2.1. If $F(X \times X) \subseteq A(X)$ and $A(X)$ is a complete subset of X, then F and A have a coupled coincidence point in X. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point in X.

Proof. Since $F(X\times X) \subseteq A(X)$, for $\mu_0, \nu_0 \in X$, we can define $A\mu_1 = F(\mu_0, \nu_0)$ and $A\nu_1 = F(\nu_0, \mu_0)$. Repeating this process, we obtain two sequences $\{\mu_n\}$ and $\{\nu_n\}$ in X such that $A\mu_{n+1} = F(\mu_n, \nu_n)$ and $A\nu_{n+1} = F(\nu_n, \mu_n)$. Then, from equations (2.29) and using $(P3)$, $(P4)$, we have

$$\begin{aligned} p(A\mu_n, A\mu_{n+1}) &= p(F(\mu_{n-1},\nu_{n-1}),F(\mu_n,\nu_n)) \\ &\leq q_1\,\psi\Big(\frac{[1+p(F(\mu_{n-1},\nu_{n-1}),A\mu_{n-1})]p(F(\mu_n,\nu_n),A\mu_n)}{1+p(A\mu_{n-1},A\mu_n)}\Big) \\ &\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_{n-1},A\mu_n)+p(A\nu_{n-1},A\nu_n)], \\ &\qquad [p(F(\mu_{n-1},\nu_{n-1}),A\mu_{n-1})+p(F(\mu_n,\nu_n),A\mu_n)], \\ &\qquad [p(F(\mu_{n-1},\nu_{n-1}),A\mu_n)+p(F(\mu_n,\nu_n),A\mu_{n-1})]\Big\}\Big) \\ &= q_1\,\psi\Big(\frac{[1+p(A\mu_n,A\mu_{n-1})]p(A\mu_{n+1},A\mu_n)}{1+p(A\mu_{n-1},A\mu_n)}\Big) \\ &\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_{n-1},A\mu_n)+p(A\nu_{n-1},A\nu_n)], \\ &\qquad [p(A\mu_n,A\mu_{n-1})+p(A\mu_{n+1},A\mu_n)], \\ &\qquad [p(A\mu_n,A\mu_n)+p(A\mu_{n+1},A\mu_{n-1})]\Big\}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le q_1\,\psi\Big(p(A\mu_n, A\mu_{n+1})\Big) \\
&\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_{n-1}, A\mu_n)+p(A\nu_{n-1}, A\nu_n)], \\
&\qquad [p(A\mu_{n-1}, A\mu_n)+p(A\mu_n, A\mu_{n+1})], \\
&\qquad [p(A\mu_n, A\mu_n)+p(A\mu_{n-1}, A\mu_n) \\
&\qquad +p(A\mu_n, A\mu_{n+1})-p(A\mu_n, A\mu_n)]\Big\}\Big) \\
&= q_1\,\psi\Big(p(A\mu_n, A\mu_{n+1})\Big) \\
&\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_{n-1}, A\mu_n)+p(A\nu_{n-1}, A\nu_n)], \\
&\qquad [p(A\mu_{n-1}, A\mu_n)+p(A\mu_n, A\mu_{n+1})], \\
&\qquad [p(A\mu_{n-1}, A\mu_n)+p(A\mu_n, A\mu_{n+1})]\Big\}\Big). \qquad (2.30)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
p(A\nu_n, A\nu_{n+1}) &= p(F(\nu_{n-1}, \mu_{n-1}), F(\nu_n, \mu_n)) \\
&\le q_1\,\psi\Big(p(A\nu_n, A\nu_{n+1})\Big) \\
&\quad +q_2\,\psi\Big(\max\Big\{[p(A\nu_{n-1}, A\nu_n)+p(A\mu_{n-1}, A\mu_n)], \\
&\qquad [p(A\nu_{n-1}, A\nu_n)+p(A\nu_n, A\nu_{n+1})], \\
&\qquad [p(A\nu_{n-1}, A\nu_n)+p(A\nu_n, A\nu_{n+1})]\Big\}\Big). \qquad (2.31)
\end{aligned}
$$

Let

$$
c_n = p(A\mu_n, A\mu_{n+1}), \quad d_n = p(A\nu_n, A\nu_{n+1}), \qquad (2.32)
$$

$$
\begin{aligned}
\mathcal{A}_1 &= \max\Big\{[p(A\mu_{n-1}, A\mu_n)+p(A\nu_{n-1}, A\nu_n)], \\
&\qquad [p(A\mu_{n-1}, A\mu_n)+p(A\mu_n, A\mu_{n+1})], \\
&\qquad [p(A\mu_{n-1}, A\mu_n)+p(A\mu_n, A\mu_{n+1})]\Big\} \\
&= \max\{c_{n-1}+d_{n-1}, c_{n-1}+c_n, c_{n-1}+c_n\} \qquad (2.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_2 &= \max\Big\{[p(A\nu_{n-1}, A\nu_n)+p(A\mu_{n-1}, A\mu_n)], \\
&\qquad [p(A\nu_{n-1}, A\nu_n)+p(A\nu_n, A\nu_{n+1})], \\
&\qquad [p(A\nu_{n-1}, A\nu_n)+p(A\nu_n, A\nu_{n+1})]\Big\} \\
&= \max\{d_{n-1}+c_{n-1}, d_{n-1}+d_n, d_{n-1}+d_n\}. \qquad (2.34)
\end{aligned}
$$

Again, we assume that

$$e_n = c_n + d_n = p(A\mu_n, A\mu_{n+1}) + p(A\nu_n, A\nu_{n+1}). \tag{2.35}$$

Then from equations (2.30) - (2.35), we have

$$e_n \leq q_1\,[\psi(c_n) + \psi(d_n)] + q_2\,[\psi(\mathcal{A}_1) + \psi(\mathcal{A}_2)]. \tag{2.36}$$

Consider the following possible cases:

Case I. If $\mathcal{A}_1 = c_{n-1} + d_{n-1}$ and $\mathcal{A}_2 = d_{n-1} + c_{n-1}$, then from (2.36) and using the property of ψ, we have

$$\begin{aligned} e_n &\leq q_1\,[\psi(c_n) + \psi(d_n)] + 2q_2\,\psi(c_{n-1} + d_{n-1}) \\ &\leq q_1\,[c_n + d_n] + 2q_2\,[c_{n-1} + d_{n-1}] \\ &= q_1\,e_n + 2q_2\,e_{n-1}, \end{aligned}$$

that is,

$$e_n \leq \Big(\frac{2q_2}{1-q_1}\Big)e_{n-1}. \tag{2.37}$$

Case II. If $\mathcal{A}_1 = c_{n-1} + c_n$ and $\mathcal{A}_2 = d_{n-1} + d_n$, then from (2.36) and using the property of ψ, we have

$$\begin{aligned} e_n &\leq q_1\,[\psi(c_n) + \psi(d_n)] + q_2\,[\psi(c_{n-1} + c_n) \\ &\quad +\psi(d_{n-1} + d_n)] \\ &\leq q_1\,[c_n + d_n] + q_2\,[c_{n-1} + c_n + d_n + d_{n-1}] \\ &= q_1\,e_n + q_2\,[e_{n-1} + e_n] \\ &= (q_1 + q_2)\,e_n + q_2\,e_{n-1}, \end{aligned}$$

that is,

$$e_n \leq \Big(\frac{q_2}{1-q_1-q_2}\Big)e_{n-1}. \tag{2.38}$$

Let

$$h = \max\Big\{\frac{2q_2}{1-q_1}, \frac{q_2}{1-q_1-q_2}\Big\}. \tag{2.39}$$

Then $h < 1$, since $q_1 + 2q_2 < 1$. Thus from equations (2.37) - (2.39), we obtain

$$e_n \leq h\,e_{n-1}. \tag{2.40}$$

Then for each $n \in \mathbb{N}$, we have

$$e_n \leq he_{n-1} \leq h^2 e_{n-2} \leq \cdots \leq h^n e_0. \tag{2.41}$$

If $e_0 = 0$, then $p(A\mu_0, A\mu_1) + p(A\nu_0, A\nu_1) = 0$. Hence, from Remark 1.4, we get $A\mu_0 = A\mu_1 = F(\mu_0, \nu_0)$ and $A\nu_0 = A\nu_1 = F(\nu_0, \mu_0)$, meaning that $(A\mu_0, A\nu_0)$ is a coupled fixed point of F and A.

Now, we assume that $e_0 > 0$. For each $n \geq m$, where $n, m \in \mathbb{N}$, we have, by using condition $(P4)$

$$\begin{aligned} p(A\mu_n, A\mu_m) &\leq p(A\mu_n, A\mu_{n-1}) + p(A\mu_{n-1}, A\mu_{n-2}) + \dots \\ &\quad + p(A\mu_{m+1}, A\mu_m) - p(A\mu_{n-1}, A\mu_{n-1}) - p(A\mu_{n-2}, A\mu_{n-2}) \\ &\quad - \dots - p(A\mu_{m+1}, A\mu_{m+1}) \\ &\leq p(A\mu_n, A\mu_{n-1}) + p(A\mu_{n-1}, A\mu_{n-2}) + \dots \\ &\quad + p(A\mu_{m+1}, A\mu_m). \end{aligned} \tag{2.42}$$

Similarly, we have

$$\begin{aligned} p(A\nu_n, A\nu_m) &\leq p(A\nu_n, A\nu_{n-1}) + p(A\nu_{n-1}, A\nu_{n-2}) + \dots \\ &\quad + p(A\nu_{m+1}, A\nu_m) - p(A\nu_{n-1}, A\nu_{n-1}) - p(A\nu_{n-2}, A\nu_{n-2}) \\ &\quad - \dots - p(A\nu_{m+1}, A\nu_{m+1}) \\ &\leq p(A\nu_n, A\nu_{n-1}) + p(A\nu_{n-1}, A\nu_{n-2}) + \dots \\ &\quad + p(A\nu_{m+1}, A\nu_m). \end{aligned} \tag{2.43}$$

Thus,

$$\begin{aligned} p(A\mu_n, A\mu_m) + p(A\nu_n, A\nu_m) &\leq e_{n-1} + e_{n-2} + \dots + e_m \\ &\leq (h^{n-1} + h^{n-2} + \dots + h^m) e_0 \\ &\leq \Big(\frac{h^m}{1-h}\Big) e_0. \end{aligned} \tag{2.44}$$

By definition of metric p^s, we have $p^s(A\mu, A\nu) \leq 2p(A\mu, A\nu)$, therefore for any $n \geq m$,

$$\begin{aligned} p^s(A\mu_n, A\mu_m) + p^s(A\nu_n, A\nu_m) &\leq 2p(A\mu_n, A\mu_m) + 2p(A\nu_n, A\nu_m) \\ &\leq \Big(\frac{2h^m}{1-h}\Big) e_0, \end{aligned} \tag{2.45}$$

which implies that $\{A\mu_n\}$ and $\{A\nu_n\}$ are Cauchy sequences in (X, p^s) because $0 \leq h < 1$, where $h = q_1 + 2q_2 < 1$. Since the partial metric space (X, p) is complete, by Lemma 1.9, the metric space (X, p^s) is complete, so there exist $p_1, p_2 \in X$ such that

$$\lim_{n\to\infty} p^s(A\mu_n, Ap_1) = \lim_{n\to\infty} p^s(A\nu_n, Ap_2) = 0. \tag{2.46}$$

From Lemma 1.9, we obtain

$$p(Ap_1, Ap_1) = \lim_{n\to\infty} p(A\mu_n, Ap_1) = \lim_{n\to\infty} p(A\mu_n, A\mu_n) \tag{2.47}$$

and

$$p(Ap_2, Ap_2) = \lim_{n\to\infty} p(A\nu_n, Ap_2) = \lim_{n\to\infty} p(A\nu_n, A\nu_n). \tag{2.48}$$

But, from condition $(P2)$ and equation (2.41), we have

$$p(A\mu_n, A\mu_n) \leq p(A\mu_n, A\mu_{n+1}) \leq e_n \leq h^n e_0, \tag{2.49}$$

and since $0 \leq h < 1$, hence letting $n \to \infty$, we get $\lim_{n\to\infty} p(A\mu_n, A\mu_n) = 0$. It follows that

$$p(Ap_1, Ap_1) = \lim_{n\to\infty} p(A\mu_n, Ap_1) = \lim_{n\to\infty} p(A\mu_n, A\mu_n) = 0. \tag{2.50}$$

Similarly, we obtain

$$p(Ap_2, Ap_2) = \lim_{n\to\infty} p(A\nu_n, Ap_2) = \lim_{n\to\infty} p(A\nu_n, A\nu_n) = 0. \tag{2.51}$$

Therefore, using equation (2.29), the conditions $(P3)$ and $(P4)$, we have

$$\begin{aligned} p(F(p_1,p_2), Ap_1) &\leq p(F(p_1,p_2), A\mu_{n+1}) + p(A\mu_{n+1}, Ap_1) \\ &\quad -p(A\mu_{n+1}, A\mu_{n+1}) \\ &\leq p(F(p_1,p_2), A\mu_{n+1}) + p(A\mu_{n+1}, Ap_1) \\ &= p(F(p_1,p_2), F(\mu_n,\nu_n)) + p(A\mu_{n+1}, Ap_1) \\ &= p(F(\mu_n,\nu_n), F(p_1,p_2)) + p(A\mu_{n+1}, Ap_1) \\ &\leq q_1\,\psi\Big(\frac{[1+p(F(\mu_n,\nu_n), A\mu_n)]p(F(p_1,p_2), Ap_1)}{1+p(A\mu_n, Ap_1)}\Big) \\ &\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_n, Ap_1)+p(A\nu_n, Ap_2)], \\ &\qquad [p(F(\mu_n,\nu_n), A\mu_n)+p(F(p_1,p_2), Ap_1)], \\ &\qquad [p(F(\mu_n,\nu_n), Ap_1)+p(F(p_1,p_2), A\mu_n)]\Big\}\Big) \\ &\qquad +p(A\mu_{n+1}, Ap_1) \\ &= q_1\,\psi\Big(\frac{[1+p(A\mu_{n+1}, A\mu_n)]p(Ap_1, Ap_1)}{1+p(A\mu_n, Ap_1)}\Big) \\ &\quad +q_2\,\psi\Big(\max\Big\{[p(A\mu_n, Ap_1)+p(A\nu_n, Ap_2)], \\ &\qquad [p(A\mu_{n+1}, A\mu_n)+p(Ap_1, Ap_1)], \\ &\qquad [p(A\mu_{n+1}, Ap_1)+p(Ap_1, A\mu_n)]\Big\}\Big) \\ &\qquad +p(A\mu_{n+1}, Ap_1). \end{aligned} \tag{2.52}$$

Letting $n \to \infty$ in equation (2.52) and using equations (2.46), (2.50) and the property of ψ, we obtain

$$p(F(p_1,p_2), Ap_1) \leq 0.$$

Hence, we have $p(F(p_1,p_2), Ap_1) = 0$, that is, $F(p_1,p_2) = Ap_1$. Since the pair (F, A) is weakly compatible, so by weak compatibility of F and A, we have

$$A(F(p_1,p_2)) = F(Ap_1, Ap_2) \text{ and } A(F(p_2,p_1)) = F(Ap_2, Ap_1).$$

Hence (Ap_1, Ap_2) is a common coupled fixed point of F and A.

Now, we show the uniqueness of the common coupled fixed point of F and A. Assume that (At_1, At_2) is another common coupled fixed point of F and A with $Ap_1 \neq At_1$ and $Ap_2 \neq At_2$, that is, $(Ap_1, Ap_2) \neq (At_1, At_2)$. Then by using equations (2.29) and using equations (2.49), (2.50), $(P3)$ and the property of ψ, we have

$$\begin{aligned} p(Ap_1, At_1) &= p(F(p_1, p_2), F(t_1, t_2)) \\ &\leq q_1\,\psi\Big(\frac{[1 + p(F(p_1, p_2), Ap_1)]p(F(t_1, t_2), At_1)}{1 + p(Ap_1, At_1)}\Big) \\ &\quad + q_2\,\psi\Big(\max\Big\{[p(Ap_1, At_1) + p(Ap_2, At_2)], \\ &\qquad [p(F(p_1, p_2), Ap_1) + p(F(t_1, t_2), At_1)], \\ &\qquad [p(F(p_1, p_2), At_1) + p(F(t_1, t_2), Ap_1)]\Big\}\Big) \\ &= q_1\,\psi\Big(\frac{[1 + p(Ap_1, Ap_1)]p(At_1, At_1)}{1 + p(Ap_1, At_1)}\Big) \\ &\quad + q_2\,\psi\Big(\max\Big\{[p(Ap_1, At_1) + p(Ap_2, At_2)], \\ &\qquad [p(Ap_1, Ap_1) + p(At_1, At_1)], \\ &\qquad [p(Ap_1, At_1) + p(At_1, Ap_1)]\Big\}\Big) \\ &= q_2\,\psi\Big(\max\Big\{[p(Ap_1, At_1) + p(Ap_2, At_2)], 0, \\ &\qquad 2p(Ap_1, At_1)\Big\}\Big). \end{aligned} \tag{2.53}$$

Similarly, we obtain

$$\begin{aligned} p(Ap_2, At_2) &\leq q_2\,\psi\Big(\max\Big\{[p(Ap_2, At_2) + p(Ap_1, At_1)], 0, \\ &\qquad 2p(Ap_2, At_2)\Big\}\Big). \end{aligned} \tag{2.54}$$

Let

$$N = p(Ap_1, At_1) + p(Ap_2, At_2), \tag{2.55}$$

$$N_1 = \max\Big\{[p(Ap_1, At_1) + p(Ap_2, At_2)], 0, 2p(Ap_1, At_1)\Big\} \tag{2.56}$$

and

$$N_2 = \max\Big\{[p(Ap_2, At_2) + p(Ap_1, At_1)], 0, 2p(Ap_2, At_2)\Big\}. \tag{2.57}$$

Then from equations (2.53)-(2.57), we have

$$N \leq q_2\,[\psi(N_1) + \psi(N_2)]. \tag{2.58}$$

Consider the following possible cases:

Case $\mathbf{I^0}$. If $N_1 = p(Ap_1, At_1) + p(Ap_2, At_2)$, $N_2 = p(Ap_2, At_2) + p(Ap_1, At_1)$, then from (2.58) and using the property of ψ, we have

$$\begin{aligned} N &\le 2q_2\,[\psi(p(Ap_1, At_1) + p(Ap_2, At_2))] \\ &= 2q_2\,\psi(N) \\ &\le 2q_2\,N < (q_1 + 2q_2)N \\ &< N, \end{aligned} \tag{2.59}$$

which is a contradiction, since $q_1 + 2q_2 < 1$. Hence, we have $N = 0$, that is, $p(Ap_1, At_1) + p(Ap_2, At_2) = 0$ and so $Ap_1 = At_1$ and $Ap_2 = At_2$.

Case $\mathbf{II^0}$. If $N_1 = 0$ and $N_2 = 0$, then from (2.58) and using the property of ψ, we have

$$\begin{aligned} N &\le q_2\,[\psi(0) + \psi(0)] \\ &= q_2.0 = 0, \end{aligned}$$

and hence $N = 0$, that is, $p(Ap_1, At_1) + p(Ap_2, At_2) = 0$ and so $Ap_1 = At_1$ and $Ap_2 = At_2$.

Case $\mathbf{III^0}$. If $N_1 = 2p(Ap_1, At_1)$ and $N_2 = 2p(Ap_2, At_2)$, then from (2.58) and using the property of ψ, we have

$$\begin{aligned} N &\le q_2\,[2\psi(p(Ap_1, At_1)) + 2\psi(p(Ap_2, At_2))] \\ &\le 2q_2[p(Ap_1, At_1) + p(Ap_2, At_2)] \\ &\le (q_1 + 2q_2)N < N, \end{aligned}$$

which is a contradiction, since $q_1 + 2q_2 < 1$. Hence, we have $N = 0$, that is, $p(Ap_1, At_1) + p(Ap_2, At_2) = 0$ and so $Ap_1 = At_1$ and $Ap_2 = At_2$. Thus in all the above cases we get $Ap_1 = At_1$ and $Ap_2 = At_2$. Consequently, F and A have a unique common coupled fixed point. This completes the proof. □

If we set $q_2 = 0$ in Theorem 2.7, then we have the following result.

Corollary 2.8. *Let (X, p) be a complete partial metric space. Suppose that the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ satisfy the following contractive condition for all $\mu, \nu, u, v \in X$*

$$p(F(\mu,\nu), F(u,v)) \le q_1\,\psi\Big(\frac{[1 + p(F(\mu,\nu), A\mu)]p(F(u,v), Au)}{1 + p(A\mu, Au)}\Big),$$

where $q_1 \in (0,1)$ is a constant and ψ is defined as in Definition 2.1*. If $F(X \times X) \subseteq A(X)$ and $A(X)$ is a complete subset of X, then F and A have a coupled coincidence point in X. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point in X.*

If we set $q_1 = 0$ in Theorem 2.7, then we have the following result.

Corollary 2.9. *Let (X, p) be a complete partial metric space. Suppose that the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ satisfy the following contractive condition for all $\mu, \nu, u, v \in X$*

$$\begin{aligned} p(F(\mu,\nu), F(u,v)) \;\; &\leq \;\; q_2\, \psi\Big(\max\Big\{[p(A\mu, Au) + p(A\nu, Av)], \\ &\qquad [p(F(\mu,\nu), A\mu) + p(F(u,v), Au)], \\ &\qquad [p(F(\mu,\nu), Au) + p(F(u,v), A\mu)]\Big\}\Big), \end{aligned}$$

where $q_2 \in (0, \frac{1}{2})$ is a constant and ψ is defined as in Definition 2.1. If $F(X \times X) \subseteq A(X)$ and $A(X)$ is a complete subset of X, then F and A have a coupled coincidence point in X. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point in X.

Remark 2.10. (1) Theorem 2.7 is an extension and a generalization of the results of *Aydi* [3].

(2) Theorem 2.7 also extends the results of *Sabetghadam et al.* [27] from cone metric space to the setting of partial metric space.

(3) Theorem 2.7 is also an extension of the results of *Kim et al.* [13].

Example 2.11. Let $X = [0, +\infty)$ endowed with the usual partial metric p defined by $p\colon X \times X \to [0, +\infty)$ with $p(\mu, \nu) = \max\{\mu, \nu\}$. The partial metric space (X, p) is complete because (X, p^s) is complete. Indeed, for any $\mu, \nu \in X$,

$$\begin{aligned} p^s(\mu,\nu) &= 2p(\mu,\nu) - p(\mu,\mu) - p(\nu,\nu) \\ &= 2\max\{\mu,\nu\} - (\mu+\nu) = |\mu - \nu|. \end{aligned} \tag{2.60}$$

Thus, (X, p^s) is the Euclidean metric space which is complete. Consider the mappings $F\colon X \times X \to X$ and $A\colon X \to X$ defined by

$$A(\mu) = \begin{cases} 6\mu, & \text{if } \; \mu \in [0,1], \\ 4\mu, & \text{if } \; \mu \in (1,\infty), \end{cases}$$

and

$$F(\mu,\nu) = \begin{cases} \frac{\mu}{6} + \frac{\nu}{6}, & \text{if } \; \mu \in [0,1] \text{ and } \nu \in \mathbb{R}, \\ \frac{\mu}{8} + \frac{\nu}{8}, & \text{if } \; \mu \in (1,\infty) \text{ and } \nu \in \mathbb{R}. \end{cases}$$

Clearly, F and A satisfies all the conditions of Theorem 2.7, by taking $q_1 = \frac{2}{5}$ and $q_2 = \frac{1}{6}$ with $q_1 + 2q_2 < 1$. Moreover $(0, 0)$ is common coupled coincidence point of F and A.

3. Conclusion

In the context of partial metric spaces, we prove a unique coupled fixed point theorem and a unique common coupled fixed point theorem, and we provide some corollaries of the established results. In addition, we provide an illustration to back up the result. Our findings generalize and expand upon a number of previously published findings in the literature (such as those found in [3, 13, 27], and many others).

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