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NEW BEST PROXIMITY POINT RESULTS FOR DIFFERENT TYPES OF NONSELF PROXIMAL CONTRACTIONS WITH AN APPLICATION

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Abstract. A new variety of non-self generalized proximal contraction, called Hardy-Rogers α^+F-proximal contraction, is shown in this work. Also, with an example, we prove that such contractions satisfying some conditions must have a unique best proximity point. For some particular values of the constants, that we have used to generalize the proximal contraction, we conclude different α^+F-proximal contraction results of the types Ćirić, Chatterjea, Reich, Kannan, and Banach with proof, that all such type of contractions must have unique best proximity point. We also apply our result to solve a functional equation.

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1. Introduction

A classical best approximation result was started in 1969 by Fan [10]. For nonself nonexpansive mappings, nonself Kannan-type mappings, and nonself Chatterjea-type mappings, Basha et al. [6] demonstrated the best proximity point theorems. All of which generalized the very famous Banach's [5] fixed point result. To generalize the theory, by Zhang et al. [25] the concepts of $p-$property and weak $p-$property were initiated in 2013. In order to generalize a number of proximity results, Hussain et al. [12] proposed Suzuki type $\alpha^+ - \psi-$proximal contraction utilizing the idea of $\alpha-$proximal admissible mapping defined in [13]. Following that, Ungchittrakool [23] developed several best proximity point theorems for generalized nonself Chatterjea-type mappings, Lipschitzian mappings, and non-self Kannan-type mappings in complete metric spaces.

In 2019, Al-Sulami at el. [4] covered certain variational inequality problems and dynamical programming challenges as implementations of their fundamental conclusion. Zhou et al. [26] in 2022 described an intriguing use of (ψ-ϕ)-weak contraction in computer science, namely in the field of words. Recently, Alam et al. [2] introduced Ćirić $\alpha^+(\theta, \psi)-$proximal contraction results for nonself mappings and solved an integral equation. For more results in this direction, authors can see [19, 20, 21, 22].

Subsequently, Wardowski [24] presented a fascinating yet unique extension of the Banach contraction theorem through the use of a novel contractive inequality known as $F-$contraction. By placing less stringent auxiliary constraints on the self-map of a whole metric space on the mapping Piri at el. [17] extended the research of Wardowski in 2014. Based on the concept of $F-$contraction, Ahmad et al. [1] defined two classes of functions in 2015 and demonstrated certain results. Later in 2023, Alam et al. [3] solved two-point boundary value problem by introducing and utilizing (α, β, F^*) and $(\alpha, \beta, F^{**})-$weak Geraghty contractions. To understand the theory in this direction we recommend [1, 3, 16]

Motivated by the above discussion, we utilize the definitions of α^+-proximality and F-contraction to introduce Hardy-Rogers α^+F-proximal contraction, a new class of nonself contraction. With an illustrative example, we state and prove that such contractions and their consequent corollaries must have a best proximity point. Our main theorem is a generalization, unification, improvement, and extension of many results in the literature. As an application, we utilize our outcome to resolve a functional equation.

2. Preliminaries

Consider a non-self-mapping $H : P \longrightarrow Q$ in any metric space (V, d) with non-void subsets P and Q. $H\vartheta = \vartheta$ is unlikely to have a solution because H is not a self-mapping. Therefore, it is crucial to look for an element ϑ that is in some way closer to $H\vartheta$. Numerous academics have examined this issue and tried to determine whether there is a point $\vartheta^* \in P$ with the least amount of error, where $d(\vartheta^*, H\vartheta^*)$ is the smallest. If ϑ corresponds to $d(\vartheta^*, H\vartheta^*) = d(P, Q)$, then $\vartheta^* \in P$ is a best proximity point [9] of $H : P \to Q$, where $d(P, Q) = \inf\{d(\vartheta_1, \vartheta_2) : \vartheta_1 \in P, \vartheta_2 \in Q\}$. The mapping H is a self-mapping if $P = Q$, which leads to the best proximity point theorem producing a fixed point result that can be solved using Banach's [5] fixed point result. As a result, one of the intriguing issues in fixed point theory is the proximity point theory.

To comprehend this article correctly, the following notation [9] should be fixed:

$$P_0 = \{\vartheta_1 \in P : d(\vartheta_1, \vartheta_2) = d(P, Q) \text{ for a } \vartheta_2 \in Q\},$$
$$Q_0 = \{\vartheta_2 \in Q : d(\vartheta_1, \vartheta_2) = d(P, Q) \text{ for a } \vartheta_1 \in P\}.$$

Definition 2.1. ([25]) Let (P, Q) represents a pair of nonempty subsets of the metric space (V, d). Now for any $u_1, u_2 \in P$ and $\vartheta_1, \vartheta_2 \in Q$,

$$\left.\begin{array}{lcl} d(u_1, \vartheta_1) & = & d(P, Q) \\ d(u_2, \vartheta_2) & = & d(P, Q) \end{array}\right\} \quad \Rightarrow \quad d(u_1, u_2) \leq d(\vartheta_1, \vartheta_2)$$

if and only if the pair (P, Q) said to satisfy weak p-property.

Example 2.2. Let $V = \mathbb{R}^2$ with the usual metric. Then the pair (A, B), where $A = \left\{\left(0, \frac{1}{n}\right), n \in \mathbb{N}\right\} \cup \{(0, 0)\}$ and $B = \left\{\left(1, \frac{1}{n}\right), n \in \mathbb{N}\right\} \cup \{(1, 0)\}$ satisfies weak p-property.

Definition 2.3. ([24]) A self-mapping $H : V \longrightarrow V$ in any metric space (V, d) will be called as F-contraction, if for some $\tau > 0$ there exists $F \in \mathcal{F}$ and for all $\vartheta_1, \vartheta_2 \in V, d(H\vartheta_1, H\vartheta_2) > 0$ implies

$$\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(d(\vartheta_1, \vartheta_2)).$$

Where $\mathcal{F}$ consists maps $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying:

(F1) $r < s \Rightarrow F(r) < F(s), \; for \; all \; r, s \in \mathbb{R}_+$,
(F2) $\lim_{n\to\infty} r_n = 0$ iff $\lim_{n\to\infty} F(r_n) = -\infty$, where $\{r_n\}_{n\in\mathbb{N}}$ is any sequence in $\mathbb{R}_+$,
(F3) for some $\delta \in (0, 1)$, $\lim_{r\to 0^+} r^\delta F(r) = 0$.

Example 2.4. The functions $-\frac{1}{\sqrt{r}}$, $\ln r$, $\ln(r+r^2)$ are members of $\mathcal{F}$.

Definition 2.5. ([12]) Let $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ be any map and $H : P \longrightarrow Q$ be such that

$$\left.\begin{array}{rcl} \alpha(u_1, u_2) & \geq & 0 \\ d(u_1, H\vartheta_1) & = & d(P,Q) \\ d(u_2, H\vartheta_2) & = & d(P,Q) \end{array}\right\} \quad \Rightarrow \quad \alpha(\vartheta_1, \vartheta_2) \geq 0,$$

for any $u_1, u_2, \vartheta_1, \vartheta_2 \in P$. Then H is called α^+-proximal admissible.

3. Main result

We will use the definition of α^+-proximality and F-contraction to define Hardy-Rogers α^+F-proximal contraction, a new class of nonself contractions. Also, we will state and prove a result for mappings that satisfy such contraction condition must have a best proximity point. Our main theorem is a generalization, unification, improvement, and extension of Khammahawong et al. [15]. Also, a generalization of Basha et al. [6], Ungchittrakool et al. [23] and Hardy et al. [11]. Consequently, we will conclude some corollaries which are generalizations of Ćirić [8], Chatterjea [7], Reich [18], Kannan [14], Banach [5], and Wardowski [24] type contractions in the α^+-proximal contraction sense.

Definition 3.1. A mapping $H : P \longrightarrow Q$ is known as a Hardy-Rogers α^+F-proximal contraction when it applies to nonempty subsets P, Q of any metric space (V, d) if for some $\tau > 0, F \in \mathcal{F}$ we have

$$d(\vartheta_1, \vartheta_2) > 0 \quad \Rightarrow \quad F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M(\vartheta_1, \vartheta_2)) \tag{3.1}$$

for all $\vartheta_1, \vartheta_2 \in P$. Where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$\begin{aligned} M(\vartheta_1, \vartheta_2) = \zeta_1 d(\vartheta_1, \vartheta_2) &+ \zeta_2\Big\{d(\vartheta_1, H\vartheta_1) - d(P,Q)\Big\} \\ &+ \zeta_3\Big\{d(\vartheta_2, H\vartheta_2) - d(P,Q)\Big\} + \zeta_4\Big\{d(\vartheta_1, H\vartheta_2) - d(P,Q)\Big\} \\ &+ \zeta_5\Big\{d(\vartheta_2, H\vartheta_1) - d(P,Q)\Big\} \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \geq 0$, $\zeta_1 + \zeta_2 + \zeta_3 + 2\zeta_4 < 1$, $\zeta_4 \neq \zeta_5, \zeta_3 + \zeta_4 \neq 1$.

Now we demonstrate our main theorem.

Theorem 3.2. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is a Hardy-Rogers α^+F-proximal contraction such that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P,Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then there exists a best proximity point ϑ^* of H in P such that $d(\vartheta^*, H\vartheta^*) = d(P,Q)$.*

Proof. In line with the theorem's assertion, there are some points $\vartheta_0, \vartheta_1 \in P_0$ such that $\alpha(\vartheta_0, \vartheta_1) \geq 0$ and $d(\vartheta_1, H\vartheta_0) = d(P, Q)$. Also, $H(P_0) \subseteq Q_0$ implies for $\vartheta_1 \in P_0$ there will be $\vartheta_2 \in P_0$ with $d(\vartheta_2, H\vartheta_1) = d(P, Q)$.

Now α^+-proximal admissibility condition of H gives $\alpha(\vartheta_1, \vartheta_2) \geq 0$. Again, since $H(P_0) \subseteq Q_0$, for $\vartheta_2 \in P_0$ there will be $\vartheta_3 \in P_0$ with $d(\vartheta_3, H\vartheta_2) = d(P, Q)$ and α^+-proximal admissibility condition of H will give $\alpha(\vartheta_2, \vartheta_3) \geq 0$. Proceeding similarly, we will have a sequence $\{\vartheta_n\}$ of points in P_0 with

$$d(\vartheta_{n+1}, H\vartheta_n) = d(P, Q) \text{ and } \alpha(\vartheta_n, \vartheta_{n+1}) \geq 0 \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Due to the fact that, pair (P, Q) meets weak p-property, we obtain

$$d(\vartheta_n, \vartheta_{n+1}) \leq d(H\vartheta_{n-1}, H\vartheta_n), \ \forall n \in \mathbb{N}.$$

If $\vartheta_n = \vartheta_{n+1}$ for any $n \in \mathbb{N}$, then $d(\vartheta_{n+1}, H\vartheta_n) = d(P, Q)$, it implies that $d(\vartheta_n, H\vartheta_n) = d(P, Q)$. So that H has a best proximity point ϑ_n in $P_0 \subseteq P$. Hence we can assume $\vartheta_n \neq \vartheta_{n+1}$ for all $n \in \mathbb{N}$, that is, $d(\vartheta_n, \vartheta_{n+1}) \geq 0$. Being H is $\alpha^+ F$-proximal contraction, we obtain

$$\begin{aligned}
F(d(\vartheta_n, \vartheta_{n+1})) &\leq F(d(\vartheta_n, \vartheta_{n+1})) + \tau \\
&\leq F(d(H\vartheta_{n-1}, H\vartheta_n)) + \tau \\
&\leq F(d(H\vartheta_{n-1}, H\vartheta_n)) + \tau + \alpha(\vartheta_{n-1}, \vartheta_n) \\
&\leq F(M(\vartheta_{n-1}, \vartheta_n)),
\end{aligned}$$

where

$$\begin{aligned}
M(\vartheta_{n-1}, \vartheta_n) &= \zeta_1 d(\vartheta_{n-1}, \vartheta_n) + \zeta_2 \Big\{ d(\vartheta_{n-1}, H\vartheta_{n-1}) - d(P, Q) \Big\} \\
&\quad + \zeta_3 \Big\{ d(\vartheta_n, H\vartheta_n) - d(P, Q) \Big\} + \zeta_4 \Big\{ d(\vartheta_{n-1}, H\vartheta_n) - d(P, Q) \Big\} \\
&\quad + \zeta_5 \Big\{ d(\vartheta_n, H\vartheta_{n-1}) - d(P, Q) \Big\} \\
&\leq \zeta_1 d(\vartheta_{n-1}, \vartheta_n) + \zeta_2 \Big\{ \Big(d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, H\vartheta_{n-1}) \Big) - d(P, Q) \Big\} \\
&\quad + \zeta_3 \Big\{ \Big(d(\vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, H\vartheta_n) \Big) - d(P, Q) \Big\} \\
&\quad + \zeta_4 \Big\{ \Big(d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, H\vartheta_n) \Big) - d(P, Q) \Big\} \\
&\quad + \zeta_5 \Big\{ d(\vartheta_n, H\vartheta_{n-1}) - d(P, Q) \Big\}
\end{aligned}$$

$$\begin{aligned}
&= \zeta_1 d(\vartheta_{n-1},\vartheta_n) + \zeta_2\Big\{d(\vartheta_{n-1},\vartheta_n) + \Big(d(\vartheta_n, H\vartheta_{n-1}) - d(P,Q)\Big)\Big\} \\
&\quad + \zeta_3\Big\{d(\vartheta_n,\vartheta_{n+1}) + \Big(d(\vartheta_{n+1}, H\vartheta_n) - d(P,Q)\Big)\Big\} \\
&\quad + \zeta_4\Big\{\Big(d(\vartheta_{n-1},\vartheta_n) + d(\vartheta_n,\vartheta_{n+1})\Big) + \Big(d(\vartheta_{n+1}, H\vartheta_n) - d(P,Q)\Big)\Big\} \\
&\quad + \zeta_5\Big\{d(\vartheta_n, T\vartheta_{n-1}) - d(P,Q)\Big\} \\
&= \Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n) + \Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1}).
\end{aligned}$$

Thus

$$F(d(\vartheta_n,\vartheta_{n+1})) \le F\Big(\Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n) + \Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1})\Big).$$

Since F is strictly increasing

$$d(\vartheta_n,\vartheta_{n+1}) \le \Big(\zeta_1+\zeta_2+\zeta_4\Big)d(\vartheta_{n-1},\vartheta_n) + \Big(\zeta_3+\zeta_4\Big)d(\vartheta_n,\vartheta_{n+1}),$$

it implies that

$$d(\vartheta_n,\vartheta_{n+1}) \le \frac{\Big(\zeta_1+\zeta_2+\zeta_4\Big)}{\Big(1-\zeta_3-\zeta_4\Big)}d(\vartheta_{n-1},\vartheta_n).$$

Now, since $\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5 \ge 0, \zeta_1+\zeta_2+\zeta_3+2\zeta_4 < 1, \zeta_3+\zeta_4 \ne 1$, we have

$$\frac{\Big(\zeta_1+\zeta_2+\zeta_4\Big)}{\Big(1-\zeta_3-\zeta_4\Big)} < 1.$$

Therefore, we know that

$$d(\vartheta_n,\vartheta_{n+1}) \le d(\vartheta_{n-1},\vartheta_n).$$

Thus, we have

$$F(M(\vartheta_{n-1},\vartheta_n))) \le F(d(\vartheta_{n-1},\vartheta_n)).$$

Hence

$$F(d(\vartheta_n,\vartheta_{n+1})) \le F(d(\vartheta_n,\vartheta_{n+1})) + \tau \le F(M(\vartheta_{n-1},\vartheta_n)) \le F(d(\vartheta_{n-1},\vartheta_n)),$$

this implies that

$$\begin{aligned}
F(d(\vartheta_n,\vartheta_{n+1})) &\le F(d(\vartheta_{n-1},\vartheta_n)) - \tau \\
&\le F(d(\vartheta_{n-2},\vartheta_{n-1})) - 2\tau \\
&\vdots \\
&\le F(d(\vartheta_0,\vartheta_1)) - n\tau.
\end{aligned}$$

Limiting as $n \to +\infty$, we have

$$\lim_{n\to+\infty} F(d(\vartheta_n, \vartheta_{n+1})) = -\infty.$$

By (F2) of the Definition 2.3,

$$\lim_{n\to+\infty} d(\vartheta_n, \vartheta_{n+1}) = 0.$$

Now, set $\beta = d(\vartheta_n, \vartheta_{n+1})$, then $\beta \to 0$ if $n \to +\infty$. Hence by (F3) of the Definition 2.3, we get an $\delta \in (0,1)$ such that

$$\lim_{n\to+\infty} d(\vartheta_n, \vartheta_{n+1})^{\delta} F(d(\vartheta_n, \vartheta_{n+1})) = \lim_{\beta\to 0} \beta^{\delta} F(\beta) = 0. \tag{3.3}$$

Again,

$$F(d(\vartheta_n, \vartheta_{n+1})) \leq F(d(\vartheta_0, \vartheta_1)) - n\tau,$$

then we have

$$d(\vartheta_n, \vartheta_{n+1})^{\delta} F(d(\vartheta_n, \vartheta_{n+1})) \leq d(\vartheta_n, \vartheta_{n+1})^{\delta} F(d(\vartheta_0, \vartheta_1)) - n\tau d(\vartheta_n, \vartheta_{n+1})^{\delta} \leq 0.$$

Using (3.3) and limiting $n \to +\infty$, we have

$$\lim_{n\to+\infty} n\tau d(\vartheta_n, \vartheta_{n+1})^{\delta} = 0.$$

So we have

$$\lim_{n\to+\infty} n d(\vartheta_n, \vartheta_{n+1})^{\delta} = 0.$$

Hence, there exists $N \in \mathbb{N}$, for all $n \geq N$

$$d(\vartheta_n, \vartheta_{n+1}) \leq \frac{1}{n^{\frac{1}{\delta}}}.$$

Let $m > n > N$, where $m, n \in \mathbb{N}$. Then

$$\begin{aligned} d(\vartheta_n, \vartheta_m) &\leq d(\vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots + d(\vartheta_{n+m-1}, \vartheta_m) \\ &\leq \frac{1}{n^{\frac{1}{\delta}}} + \frac{1}{(n+1)^{\frac{1}{\delta}}} + \frac{1}{(n+2)^{\frac{1}{\delta}}} + \cdots + \frac{1}{(n+m-1)^{\frac{1}{\delta}}} \\ &= \sum_{j=n}^{n+m-1} \frac{1}{j^{\frac{1}{\delta}}} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j^{\frac{1}{\delta}}}. \end{aligned}$$

Since, $\delta \in (0,1)$, $\frac{1}{\delta} > 1$, thus the series $\sum_{j=0}^{\infty} \frac{1}{j^{\frac{1}{\delta}}}$ is convergent. So $d(\vartheta_n, \vartheta_m) \to 0$ as $m, n \to +\infty$, concludes the sequence $\{\vartheta_n\}$ is Cauchy in $P_0 \subseteq V$. Being

(V, d) is complete there must be $\vartheta^* \in V$, in fact, being P is closed in V and $\{\vartheta_n\}$ is Cauchy in $P_0 \subseteq P$, ϑ^* must be in P, with

$$\lim_{n\to+\infty} d(\vartheta_n, \vartheta^*) = 0.$$

Since H is continuous,

$$\lim_{n\to+\infty} d(H\vartheta_n, H\vartheta^*) = 0.$$

Again,

$$d(\vartheta_{n+1}, H\vartheta_n) = d(P, Q),$$

this implies,

$$\lim_{n\to+\infty} d(\vartheta_{n+1}, H\vartheta_n) = \lim_{n\to+\infty} d(P, Q).$$

Hence we have

$$d(\vartheta^*, H\vartheta^*) = d(P, Q),$$

which shows that ϑ^* is best proximity point of H in P. □

The additional requirement of the uniqueness of the best proximity point of Theorem 3.2 will follow from the following corollary.

Corollary 3.3. *In addition to the statement of the Theorem* 3.2 *if* $\zeta_1+\zeta_4+\zeta_5 < 1$, *then the best proximity point will become unique in* P.

Proof. Suppose $\vartheta^*, \vartheta^{**}$ are two distinct best proximity point of H in $P_0 \neq \emptyset$. Then $d(\vartheta^*, \vartheta^{**}) \geq 0$ and

$$d(\vartheta^*, H\vartheta^*) = d(P, Q),$$

$$d(\vartheta^{**}, H\vartheta^{**}) = d(P, Q).$$

Since the pair (P, Q) meets weak p-property we have

$$d(\vartheta^*, \vartheta^{**}) \leq d(T\vartheta^*, T\vartheta^{**}).$$

Therefore,

$$\begin{aligned} F(d(\vartheta^*, \vartheta^{**})) &\leq F(d(H\vartheta^*, H\vartheta^{**})) \\ &\leq F(d(H\vartheta^*, H\vartheta^{**}) + \tau + \alpha(\vartheta^*, \vartheta^{**})) \\ &\leq F(M(\vartheta^*, \vartheta^{**})), \end{aligned}$$

where

$$\begin{aligned} M(\vartheta^*, \vartheta^{**}) &= \zeta_1 d(\vartheta^*, \vartheta^{**}) + \zeta_2 \Big\{ d(\vartheta^*, H\vartheta^*) - d(P,Q) \Big\} \\ &\quad + \zeta_3 \Big\{ d(\vartheta^{**}, H\vartheta^{**}) - d(P,Q) \Big\} + \zeta_4 \Big\{ d(\vartheta^*, H\vartheta^{**}) - d(P,Q) \Big\} \\ &\quad + \zeta_5 \Big\{ d(\vartheta^{**}, H\vartheta^*) - d(P,Q) \Big\} \\ &\le \zeta_1 d(\vartheta^*, \vartheta^{**}) + \zeta_4 \Big\{ \Big(d(\vartheta^*, \vartheta^{**}) + d(\vartheta^{**}, H\vartheta^{**}) \Big) - d(P,Q) \Big\} \\ &\quad + \zeta_5 \Big\{ \Big(d(\vartheta^{**}, \vartheta^*) + d(\vartheta^*, H\vartheta^*) \Big) - d(P,Q) \Big\} \\ &= \zeta_1 d(\vartheta^*, \vartheta^{**}) + \zeta_4 \Big\{ d(\vartheta^*, \vartheta^{**}) + \Big(d(\vartheta^{**}, H\vartheta^{**}) - d(P,Q) \Big) \Big\} \\ &\quad + \zeta_5 \Big\{ d(\vartheta^{**}, \vartheta^*) + \Big(d(\vartheta^*, H\vartheta^*) - d(P,Q) \Big) \Big\} \\ &= \Big(\zeta_1 + \zeta_4 + \zeta_5 \Big) d(\vartheta^*, \vartheta^{**}). \end{aligned}$$

Hence, by $(F1)$

$$F(M(\vartheta^*, \vartheta^{**})) \le F\Big(\Big(\zeta_1 + \zeta_4 + \zeta_5 \Big) d(\vartheta^*, \vartheta^{**}) \Big),$$

which gives

$$F(d(\vartheta^*, \vartheta^{**})) \le F(M(\vartheta^*, \vartheta^{**})) \le F\Big(\Big(\zeta_1 + \zeta_4 + \zeta_5 \Big) d(\vartheta^*, \vartheta^{**}) \Big).$$

Then,

$$d(\vartheta^*, \vartheta^{**}) \le \Big(\zeta_1 + \zeta_4 + \zeta_5 \Big) d(\vartheta^*, \vartheta^{**}),$$

it implies that

$$\Big\{ 1 - \Big(\zeta_1 + \zeta_4 + \zeta_5 \Big) \Big\} d(\vartheta^*, \vartheta^{**}) \le 0.$$

Since $\zeta_1 + \zeta_4 + \zeta_5 < 1$, we have $d(\vartheta^*, \vartheta^{**}) \le 0$, so $\vartheta^* = \vartheta^{**}$. This completes the proof. □

Remark 3.4. Theorem 3.2 and Corollary 3.3 together is a generalization, unification, improvement, and extension of Khammahawong et al. [15]. Also, the generalization of Basha et al. [6], Ungchittrakool et al. [23], Hardy et al. [11], and many others which are demonstrated in the consequent results and corollaries.

Now we present an example that is concluded using Theorem 3.2 and because the example satisfies Corollary 3.3 the uniqueness of the proximity follows.

Example 3.5. Consider two nonempty subsets

$$P = \{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 : 0 \leq \vartheta_1 \leq 1, \vartheta_2 = 1\},$$
$$Q = \{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 : 0 \leq \vartheta_1 \leq 1, \vartheta_2 = 0\}$$

of a complete metric space $(\mathbb{R}^2, d_u)$, where d_u is usual metric of $\mathbb{R}^2$. Then $P_0 = P \neq \emptyset$ and $Q_0 = Q$ also the pair (P, Q) meets weak p-property. Suppose $H : P \longrightarrow Q$ is given by

$$H(\vartheta_1, \vartheta_2) = \Big(\frac{\vartheta_1}{2}, 0\Big), \ \ \forall (\vartheta_1, \vartheta_2) \in P$$

and $\alpha^+ : \mathbb{R} \times \mathbb{R} \longrightarrow (-\infty, +\infty)$ is given by

$$\alpha(r_1, r_2) = \frac{1}{8}.$$

Then for $F(r) = -\frac{1}{\sqrt{r}} \in \mathcal{F}$ and $\tau = \frac{1}{8} > 0$, H satisfies all conditions of Theorem 3.2 for $\zeta_1 = \frac{3}{4}$, $\zeta_2 = \frac{1}{8}$, $\zeta_3 = \frac{1}{33}$, $\zeta_4 = \frac{1}{89}$, $\zeta_5 = \frac{1}{12}$. Clearly $(0, 1)$ is a best proximity point of H.

Remark 3.6. Note that, in Example 3.5, we used $\alpha(r_1, r_2) = \frac{1}{8}$ and $\tau = \frac{1}{8}$ so that we can say that H is not only an $F-$contraction for $\tau = \frac{1}{8}$ in the proximity sense, but there is still some positive amount to define another function α in addition to τ. Thus, our results generalize $F-$contraction results in the proximity sense.

Here are some subsequent outcomes concluded from Theorem 3.2.

Corollary 3.7. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that for some $\tau > 0$ and $F \in \mathcal{F}$, a continuous mapping $H : P \longrightarrow Q$ satisfies*

$$d(\vartheta_1, \vartheta_2) > 0 \ \ \Rightarrow \ \ F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M(\vartheta_1, \vartheta_2)) \quad (3.4)$$

for all $\vartheta_1, \vartheta_2 \in P$, where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$\begin{aligned} M(\vartheta_1, \vartheta_2) = \zeta_1 d(\vartheta_1, \vartheta_2) &+ \zeta_2 \Big\{ d(\vartheta_1, H\vartheta_1) - d(P, Q) \Big\} \\ &+ \zeta_3 \Big\{ d(\vartheta_2, H\vartheta_2) - d(P, Q) \Big\} + \zeta_4 \Big\{ d(\vartheta_1, H\vartheta_2) - d(P, Q) \Big\} \\ &+ \zeta_5 \Big\{ d(\vartheta_2, H\vartheta_1) - d(P, Q) \Big\} \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 + 2\zeta_4 < 1, \zeta_4 = \zeta_5, \zeta_3 + \zeta_4 \neq 1$ such that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P, Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then best proximity point ϑ^* of H exist in P such that $d(\vartheta^*, H\vartheta^*) = d(P, Q)$.

Proof. Letting $\zeta_4 = \zeta_5$ in Theorem 3.2, the result follows. ∎

Remark 3.8. As $\zeta_1 + 2\zeta_4 < 1$ following Corollary 3.3 proximity point of Theorem 3.2 is unique. Corollary 3.7 is a generalization of Ćirić [8] type contraction in the α^+F-proximal contraction sense.

Corollary 3.9. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that for some $\tau > 0$ and $F \in \mathcal{F}$, a continuous mapping $H : P \longrightarrow Q$ satisfies*

$$d(\vartheta_1, \vartheta_2) > 0 \quad \Rightarrow \quad F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M_1(\vartheta_1, \vartheta_2))$$

for all $\vartheta_1, \vartheta_2 \in P$, where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$M_1(\vartheta_1, \vartheta_2) = \zeta\Big\{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1) - 2d(P, Q)\Big\}$$

with $0 \leq \zeta < \frac{1}{2}$ such that $T(P_0) \in Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P, Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then best proximity point ϑ^* of H exist in P such that $d(\vartheta^*, H\vartheta^*) = d(P, Q)$.

Proof. Assuming $\zeta_1 = \zeta_2 = \zeta_3 = 0, \zeta_4 = \zeta_5 = \zeta$ in Corollary 3.7, Corollary 3.9 is immediate. □

Remark 3.10. Note that the constant ζ fulfills the inequality in Corollary 3.3 so that the proximity point in Corollary 3.9 is unique. In the sense of α^+F-proximal contraction Corollary 3.9 is a generalization of Chatterjea [7] type contraction.

Corollary 3.11. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that for some $\tau > 0$ and $F \in \mathcal{F}$ a continuous mapping $H : P \longrightarrow Q$ satisfies*

$$d(\vartheta_1, \vartheta_2) > 0 \quad \Rightarrow \quad F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M_2(\vartheta_1, \vartheta_2))$$

for all $\vartheta_1, \vartheta_2 \in P$, where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$\begin{aligned} M_2(\vartheta_1, \vartheta_2) = \zeta_1 d(\vartheta_1, \vartheta_2) + \zeta_2\Big\{d(\vartheta_1, H\vartheta_1) - d(P, Q)\Big\} \\ + \zeta_3\Big\{d(\vartheta_2, H\vartheta_2) - d(P, Q)\Big\} \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 < 1, \zeta_3 \neq 1$ such that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P, Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then best proximity point ϑ^* of H exist in P such that $d(\vartheta^*, H\vartheta^*) = d(P, Q)$.

Proof. If we put $\zeta_4 = \zeta_5 = 0$ in the Corollary 3.7, we will have our result. □

Remark 3.12. The corollary stated above in α^+F-proximal contraction form is none but a generalized sense of Reich [18] type contraction. Also, it is clear that the proximity point of Corollary 3.11 is unique.

Corollary 3.13. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that for some $\tau > 0$ and $F \in \mathcal{F}$, a continuous mapping $H : P \longrightarrow Q$ satisfies*

$$d(\vartheta_1, \vartheta_2) > 0 \;\Rightarrow\; F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M_3(\vartheta_1, \vartheta_2))$$

for all $\vartheta_1, \vartheta_2 \in P$, where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$M_3(\vartheta_1, \vartheta_2) = \zeta\Big\{d(\vartheta_1, H\vartheta_1) + d(\vartheta_2, H\vartheta_2) - 2d(P, Q)\Big\}$$

with $\zeta \in [0, \frac{1}{2})$ such that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P, Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then best proximity point ϑ^* of H exist in P such that $d(\vartheta^*, H\vartheta^*) = d(P, Q)$.

Proof. In the Corollary 3.7, if we put $\zeta_1 = \zeta_4 = \zeta_5 = 0, \zeta_2 = \zeta_3 = \zeta$, then the result is immediate. □

Remark 3.14. The above corollary concluding unique proximity point is a generalization of Kannan [14] type contraction in α^+F-proximal contraction sense.

Corollary 3.15. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that for some $\tau > 0$ and $F \in \mathcal{F}$, a continuous mapping $H : P \longrightarrow Q$ satisfies*

$$d(\vartheta_1, \vartheta_2) > 0 \;\Rightarrow\; F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M_4(\vartheta_1, \vartheta_2))$$

for all $\vartheta_1, \vartheta_2 \in P$, where $\alpha : P \times P \longrightarrow (-\infty, +\infty)$ and

$$M_4(\vartheta_1, \vartheta_2) = \zeta d(\vartheta_1, \vartheta_2)$$

with $0 \leq \zeta < 1$ such that $H(P_0) \subseteq Q_0$, $d(\vartheta_1, H\vartheta_0) = d(P, Q)$ and $\alpha(\vartheta_0, \vartheta_1) \geq 0$, for some $\vartheta_0, \vartheta_1 \in P_0$. Then best proximity point ϑ^* of H exist in P such that $d(\vartheta^*, H\vartheta^*) = d(P, Q)$.

Proof. Putting $\zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = 0, \zeta_1 = \zeta$, in the Corollary 3.7, we will have the proof. □

Remark 3.16. The obtained proximity point in Corollary 3.15 is unique as the constant ζ is the same as in Corollary 3.3. Also, Corollary 3.15 is generalizing the result of Banach [5] in the sense of α^+F-proximal contraction.

Corollary 3.17. *Consider two nonempty subsets P, Q satisfying weak p-property in any complete metric space (V, d) with $P_0 \neq \emptyset$. Assume that a continuous mapping $H : P \longrightarrow Q$ is such that, for some $\tau > 0, F \in \mathcal{F}$ we have*

$$d(\vartheta_1, \vartheta_2) > 0 \;\Rightarrow\; F(d(H\vartheta_1, H\vartheta_2)) + \tau \leq F(M(\vartheta_1, \vartheta_2))$$

for all $\vartheta_1, \vartheta_2 \in P$ *with* $H(P_0) \subseteq Q_0$, $d(\vartheta', H\vartheta'') = d(P,Q)$, *for some* $\vartheta', \vartheta'' \in P_0$, *where*

$$\begin{aligned} M(\vartheta_1, \vartheta_2) = \zeta_1 d(\vartheta_1, \vartheta_2) &+ \zeta_2 \Big\{ d(\vartheta_1, H\vartheta_1) - d(P,Q) \Big\} \\ &+ \zeta_3 \Big\{ d(\vartheta_2, H\vartheta_2) - d(P,Q) \Big\} + \zeta_4 \Big\{ d(\vartheta_1, H\vartheta_2) - d(P,Q) \Big\} \\ &+ \zeta_5 \Big\{ d(\vartheta_2, H\vartheta_1) - d(P,Q) \Big\} \end{aligned}$$

with $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \geq 0, \zeta_1 + \zeta_2 + \zeta_3 + 2\zeta_4 < 1, \zeta_4 \neq \zeta_5, \zeta_3 + \zeta_4 \neq 1$. *Then best proximity point* ϑ^* *of* H *exist in* P *such that* $d(\vartheta^*, H\vartheta^*) = d(P,Q)$. *In addition, if* $\zeta_1 + \zeta_4 + \zeta_5 < 1$, *then the best proximity point becomes unique in* P.

Proof. In the Theorem 3.2, if $\alpha = 0$ on P, then we get the result. □

Remark 3.18. Corollary 3.17 is a generalization of Wardowski [24] in the sense of $\alpha^+ F$-proximal contraction in the field of proximity point theory.

Remark 3.19. Following Corollary 3.17 if $\alpha = 0$ on P, we can get more corollaries from Corollaries 3.7-3.15.

4. Application

Functional equations over the space of bounded functions can have various other forms and conditions depending on the specific problem context, but they generally involve finding functions that satisfy certain algebraic relationships while being bounded on their domain. In fluid dynamics, certain equations governing the behavior of fluid flow can be formulated as functional equations over bounded functions. In economic modeling, it describes relationships between economic variables. In control theory, functional equations are used to describe dynamic systems and their responses to external inputs. The versatility of these equations makes them valuable tools for modeling, analyzing, and solving a wide range of real-world problems. Solving such equations often requires techniques from functional analysis and advanced mathematical methods.

In order to show the usefulness of our findings, we explore that there is a solution to the following functional equation over the space of all bounded real-valued functions.

$$\vartheta(v) = \sup_{u \in U} \{ f(v,u) + g(v,u,\vartheta(\gamma(v,u))) \}, \quad v \in V, \tag{4.1}$$

where V, U are complete normed linear spaces; $f : V \times U \longrightarrow \mathbb{R}$ and $g : V \times U \times \mathbb{R} \longrightarrow \mathbb{R}$ are bounded functions; $\gamma : V \times U \longrightarrow V$ is any function. These kinds of equations have applications in many fields including mathematical optimization, dynamic programming, and computer programming.

Let $(B(V), ||\cdot||)$ be the complete normed linear space of all bounded real-valued functions on V with respect to the usual "sup" norm and define $H : B(V) \longrightarrow B(V)$ by

$$H\vartheta(v) = \sup_{u\in U}\{f(v,u) + g(v,u,\vartheta(\gamma(v,u)))\}, \quad \vartheta(v) \in B(V).$$

Being f, g bounded, the function H is well defined. Then finding a solution of (4.1) is the same as to find a best proximity point of H, that is, a solution to the equation $d(H\vartheta(v), \vartheta(v)) = d(B(V), B(V))$, where metric d is given by $d(\vartheta_1, \vartheta_2) = ||\vartheta_1 - \vartheta_2||$.

Theorem 4.1. *The problem* (4.1) *will have a solution if for some* $\tau > 0$, $0 < \zeta < 1$, $g : V \times U \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies*

$$|g(v,u_1,\vartheta_1(\gamma(v,u_1))) - g(v,u_2,\vartheta_2(\gamma(v,u_2)))| + \tau \leq \zeta|\vartheta_1(v) - \vartheta_2(v)|e^{-\tau}$$

for all $\gamma : V \times U \longrightarrow V$, $\vartheta_1, \vartheta_2 \in B(V), v \in V$ *and* $u_1, u_2 \in U$.

Proof. From the definition of H, for $\vartheta_1(v), \vartheta_2(v) \in B(V)$ there will be $\delta > 0$ for some $u_1, u_2 \in U$ such that

$$H\vartheta_1(v) \leq f(v,u_1) + g(v,u_1,\vartheta_1(\gamma(v,u_1))) + \delta,$$
$$H\vartheta_2(v) \leq f(v,u_2) + g(v,u_2,\vartheta_2(\gamma(v,u_2))) + \delta.$$

Again, $H\vartheta_1(v) \geq f(v,u_1) + g(v,u_1,\vartheta_1(\gamma(v,u_1)))$ and $H\vartheta_2(v) \geq f(v,u_2) + g(v,u_2,\vartheta_2(\gamma(v,u_2)))$ implies

$$\begin{aligned}|H\vartheta_1(v) - H\vartheta_2(v)| &\leq |g(v,u_1,\vartheta_1(\gamma(v,u_1))) - g(v,u_2,\vartheta_2(\gamma(v,u_2)))| + \delta \\ &\leq \zeta|\vartheta_1(v) - \vartheta_2(v)|e^{-\delta}.\end{aligned}$$

Now, if $F(r) = \ln r$, taking supremum both sides we see for $\zeta = \zeta_1$, and any choice of $\zeta_2, \zeta_3, \zeta_4, \zeta_4 \geq 0$, H satisfies the condition

$$F(d(H\vartheta_1, H\vartheta_2)) + \alpha(\vartheta_1, \vartheta_2) + \tau \leq F(M(\vartheta_1, \vartheta_2))$$

of the Theorem 3.2 for $\alpha((\vartheta_1, \vartheta_2)) = 0$ and $\tau = \delta$. Thus there is best proximity point of H, that is, a solution of (4.1). □

5. Conclusions

We utilized the definitions of α^+-proximality and F-contraction and introduced Hardy-Rogers α^+F-proximal contraction, a new class of non-self contraction. With an illustrative example, we stated and proved that such contractions and their consequent corollaries must have a best proximity point. Our main theorem is a generalization, unification, improvement, and extension of Khammahawong et al. [15]. Also, the generalization of Basha et al. [6], Ungchittrakool et al. [23] and Hardy et al. [11]. Consequently, the consequent corollaries are generalizations of Ćirić [8], Chatterjea [7], Reich [18], Kannan [14], Banach [5], and Wardowski [24] type contractions in the α^+-proximal contraction sense. We also, explored that there is a solution to a functional equation, which provides tools for addressing boundary value issues arising in the physical sciences and engineering.

Open Question: Is it possible to remove the continuity condition of the nonself mapping in Theorem 3.2?

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