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GENERALIZED ε-EFFICIENCY CONDITIONS AND SECOND ORDER B-$(b, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-INVEXITIES IN MINIMAX FRACTIONAL PROGRAMMING PROBLEMS

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Abstract. This communication deals with investigating the results on $\varepsilon-$efficiency conditions based on generalized second order B-$(b, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-invexities-a major generalization to Antczak type first order B-$(b, \tilde{p}, \tilde{r})$-invexities, which encompass most of the existing generalized invexity concepts in the literature. Then using these developing notions, a wide range of parametric sufficient efficiency conditions for multiobjective fractional programming are established. To the best of our knowledge, the obtained results seem to be most advanced on generalized invexities at the present time, while offer more challenging applications to other fields.

1. Introduction

The contribution of Antczak [1-3] on first order B-(p, r)-invexities is enormous to the context of nonlinear mathematical programming problems, which have been applied to a class of global parametric sufficient optimality conditions based on first order B-(p, r)-invexities for semiinfinite discrete minimax fractional programming problems. This was followed by Zalmai [41] who modified B-(p, r)-invexities introduced by Antczak [1-3], and applied to a class of global parametric sufficient optimality criteria using various assumptions for semiinfinite discrete minimax fractional programming problems. Recently,

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Verma [31] introduced a major second order generalization as B-$(b, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-invexities to Antczak type first order B-$(b, \eta, \tilde{p}, \tilde{r})$-invexities, and applied to exploring parametric sufficient efficiency conditions to semiinfinite minimax fractional programming problems, while Verma [30] introduced and investigated second order $(\Phi, \Psi, \rho, \eta, \theta)-$invexities to the context of parametric sufficient optimality conditions in semiinfinite discrete minimax fractional programming problems. Zalmai and Zhang [42] have established a set of necessary efficiency conditions and a fairly large number of global nonparametric sufficient efficiency results under various frameworks for generalized $(\eta, \rho)-$invexity for semi-infinite discrete minimax fractional programming problems. There exists an enormous amount of literature on generalized first order as well as second order generalized invexities with applications.

Verma [25] also developed a general framework for a class of $(\rho, \eta, \theta)-$invex functions to examine some parametric sufficient efficiency conditions for multiobjective fractional programming problems for weakly $\varepsilon-$efficient solutions, while Kim et al. [8] have established some $\varepsilon-$optimality conditions for multiobjective fractional optimization problems. Motivated by the recent advances on the second order B-$(b, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-invexities - a major generalization to Antczak type first order B-$(\tilde{p}, \tilde{r})$-invexities - well-explored in the literature, we intend to establish some advanced results on the $\varepsilon-$efficiency conditions based on the generalized B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities in different settings. The obtained results encompass most of the results in existing literature mainly because of the enormous generality power of the second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities, which may not be limited to applications to just $\varepsilon-$efficiency conditions and further applications.

We consider under the generalized framework of the second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities of functions, the following multiobjective fractional programming problem:

(P)

$$Minimize \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to $x \in Q = \{x \in X : H_j(x) \leq 0, j \in \{1, 2, \cdots, m\}\}$, where X is an open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), f_i and g_i for $i \in \{1, \cdots, p\}$ and H_j for $j \in \{1, \cdots, m\}$ are real-valued functions defined on X such that $f_i(x) \geq 0$, $g_i(x) > 0$ for $i \in \{1, \cdots, p\}$ and for all $x \in Q$. Here Q denotes the feasible set of (P).

Next, we observe that problem (P) is equivalent to the nonfractional programming problem:

(Pλ)

$$Minimize\Big(f_1(x)-\lambda_1 g_1(x),\cdots,f_p(x)-\lambda_p g_p(x)\Big)$$

subject to $x\in Q$ with

$$\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_p)=\left(\frac{f_1(x^*)}{g_1(x^*)},\frac{f_2(x^*)}{g_2(x^*)},\cdots,\frac{f_p(x^*)}{g_p(x^*)}\right),$$

where x^* is an efficient solution to (P).

The general theory of nonlinear programming problems serve a great purpose, not only in terms of theory, but also in terms of applications to decision and management sciences, game theory, statistical analysis, engineering design (including design of control systems, design of earthquakes-resistant structures, digital filters, and electronic circuits), random graphs, boundary value problems, wavelet analysis, environmental protection planning, optimal control problems, continuum mechanics, robotics, and data envelopment analysis. For more details, we refer the reader [1-45].

2. Preliminaries

The general invexity has been investigated in several directions. Recently, Verma [31] generalized the notion of the first order Antczak type B-$(b,\tilde{p},\tilde{r})$-invexiies to the case of the second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities. These notions of the second order invexity encompass most of the existing notions in the literature. Let f be a twice continuously differentiable real-valued function defined on X. Furthermore, let $\rho: X\times X\to\mathbb{R}$ and $\theta: X\times X\to\mathbb{R}^n$ be functions on $X\times X$.

Definition 2.1. The function f is said to be second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invex at $x^*\in X$ if there exist a function $\eta: X\times X\to\mathbb{R}^n$, a function $b: X\times X\to(0,\infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x\in X$ and $z\in\mathbb{R}^n$,

$$\begin{aligned}&b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\\&\geq\frac{1}{\tilde{p}}\Big(\big\langle\nabla f(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2}\langle e^{\tilde{p}z}-1,\nabla^2 f(x^*)z\rangle\Big)\\&\quad+\rho(x,x^*)\|\theta(x,x^*)\|^2\ \text{ for }\tilde{p}\neq0\text{ and }\tilde{r}\neq0.\end{aligned}$$

Definition 2.2. The function f is said to be second order strictly B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invex at $x^*\in X$ if there exist a function $\eta: X\times X\to\mathbb{R}^n$, a function

$b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} &b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \\ &> \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big) \\ &\quad +\rho(x, x^*)\|\theta(x, x^*)\|^2 \text{ for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.3. The function f is said to be second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \\ &\geq 0 \\ \Rightarrow \quad &b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.4. The function f is said to be second order strictly B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\begin{aligned} &\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \\ &\geq 0 \\ \Rightarrow \quad &b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) > 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0, \end{aligned}$$

equivalently,

$$\begin{aligned} &b(x, x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0 \\ \Rightarrow \quad &\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big) \\ &+\rho(x, x^*)\|\theta(x, x^*)\|^2 < 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0. \end{aligned}$$

Definition 2.5. The function f is said to be second order prestrictly B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$

and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 > 0$$
$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \geq 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0.$$

Definition 2.6. The function f is said to be second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex with respect to Ψ, η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0$$
$$\Rightarrow \quad \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big)$$
$$+\rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0.$$

Definition 2.7. The function f is said to be second order strictly B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)- quasiinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) \leq 0$$
$$\Rightarrow \quad \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big)$$
$$+\rho(x,x^*)\|\theta(x,x^*)\|^2 < 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0.$$

Definition 2.8. The function f is said to be second order prestrictly B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$,

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]} - 1\big)\Big) < 0$$
$$\Rightarrow \quad \frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 f(x^*)z\rangle\Big)$$
$$+\rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0 \quad \text{for } \tilde{p} \neq 0 \text{ and } \tilde{r} \neq 0,$$

equivalently,

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla f(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2}\langle e^{\tilde{p}z}-1,\nabla^2 f(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2$$
$$>0$$
$$\Rightarrow \quad b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[f(x)-f(x^*)]}-1\big)\Big)\geq 0 \quad \text{for } \tilde{p}\neq 0 \text{ and } \tilde{r}\neq 0.$$

Now we consider the ε-efficiency solvability conditions for (P) and (Pλ) problems motivated by the publications (see Verma [25]) and (Kim et al. [8]), where they have investigated the ε-efficiency as well as the weak ε-efficiency conditions for multiobjective fractional programming problems under constraint qualifications. Based on these developments in the literature, we plan to establish some parametric sufficient efficiency conditions for multiobjective fractional programming problem (P) under this framework of B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities. We need to recall some auxiliary results crucial to the problem on hand.

Definition 2.9. A point $x^*\in Q$ is an ε-efficient solution to (P) if there exists no $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}\leq\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i,\quad \forall\, i=1,\cdots,p,$$
$$\frac{f_j(x)}{g_j(x)}<\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j \quad \text{for some } j\in\{1,\cdots,p\},$$

where $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$ with $\varepsilon_i\geq 0$ for $i=1,\cdots,p$.

Next to this context, we have the following auxiliary problem:

(P$\bar{\lambda}$)

$$Minimize_{x\in Q}\,(f_1(x)-\bar{\lambda}_1 g_1(x),\cdots,f_p(x)-\bar{\lambda}_p g_p(x)),$$

subject to $x\in Q$, where $\bar{\lambda}_i$ for $i\in\{1,\cdots,p\}$ are parameters, and $\bar{\lambda}_i=\frac{f_(x^*)}{g_i(x^*)}-\varepsilon_i$.

Next, we introduce the ε-efficient solvability conditions for (P$\bar{\lambda}$) problem.

Definition 2.10. A point $x^*\in Q$ is an $\bar{\varepsilon}$-efficient solution to (P$\bar{\lambda}$) if there exists no $x\in Q$ such that

$$f_i(x)-\bar{\lambda}g_i(x)\leq f_i(x^*)-\bar{\lambda}g_i(x^*)-\bar{\varepsilon}_i,\quad \forall\, i=1,\cdots,p,$$
$$f_j(x)-\bar{\lambda}g_j(x)<f_j(x^*)-\bar{\lambda}g_j(x^*)-\bar{\varepsilon}_j \quad \text{for some } j\in\{1,\cdots,p\},$$

where $\bar{\lambda_i}=\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i$, $\bar{\varepsilon_i}=\varepsilon_i g((x^*))$ with $\varepsilon_i\geq 0$ for $i=1,\cdots,p$, and $\varepsilon=(\varepsilon_1,\cdots,\varepsilon_p)$ with $\varepsilon_i\geq 0$ for $i=1,\cdots,p$.

Lemma 2.11. *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \varepsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) *x^* is an $\varepsilon-$efficient solution to (P).*
(ii) *x^* is an $\bar{\varepsilon}-$efficient solution to $(P\bar{\lambda})$, where*

$$\bar{\lambda} = \left(\frac{f_1(x^*)}{g_1(x^*)} - \varepsilon_1, \cdots, \frac{f_p(x^*)}{g_p(x^*)} - \varepsilon_p \right),$$

where $\varepsilon^* = (\varepsilon_1 g_1(x^*), \cdots, \varepsilon_p g_p(x^*))$.

Lemma 2.12. *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \varepsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) *x^* is an ε-efficient solution to (P).*
(ii) *There exists $c = (c_1, \cdots, c_p) \in \Re^p_+ \setminus \{0\}$ such that*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{p} c_i \left[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) g_i(x) \right] \\ &= \sum_{i=1}^{p} c_i \left[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) g_i(x^*) \right] - \sum_{i=1}^{p} c_i \varepsilon_i g_i(x^*), \end{aligned}$$

for any $x \in Q$.

Lemma 2.13. *Let $x^* \in Q$. Suppose that $f_i(x^*) \geq \varepsilon_i g_i(x^*)$ for $i = 1, \cdots, p$. Then the following statements are equivalent:*

(i) *x^* is an ε^*-efficient solution to $(P\bar{\lambda})$.*
(ii) *There exists $c = (c_1, \cdots, c_p) \in \Re^p_+ \setminus \{0\}$ such that*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{p} c_i \left[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) g_i(x) \right] \geq 0 \\ &= \sum_{i=1}^{p} c_i \left[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) g_i(x^*) \right] - \sum_{i=1}^{p} c_i \varepsilon_i g_i(x^*), \end{aligned}$$

for any $x \in Q$.

Next, we first present a specialization to the second order B-$(b, c, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-invexities when $\tilde{p} = 0$ and $\tilde{r} = 0$ as the second order B-(b, ρ, η, θ)-invexities.

Definition 2.14. The function f is said to be second order B-$(b, c, \rho, \eta, \theta)$-invex at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions

$b, c : X \times X \to (0, \infty)$ such that for all $x \in X$,

$$\begin{aligned} & b(x, x^*)\Big([f(x) - f(x^*)]\Big) \\ & \geq c(x, x^*)\Big(\big\langle \nabla f(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2. \end{aligned}$$

Definition 2.15. The function f is said to be second order B-$(b, c, \rho, \eta, \theta)$-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$,

$$\begin{aligned} & c(x, x^*)\Big(\big\langle \nabla f(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \\ & \geq 0 \\ \Rightarrow \quad & b(x, x^*)\Big([f(x) - f(x^*)]\Big) \geq 0. \end{aligned}$$

Definition 2.16. The function f is said to be second order B-$(b, c, \rho, \eta, \theta)$-quasiinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$,

$$\begin{aligned} & b(x, x^*)\Big([f(x) - f(x^*)]\Big) \leq 0 \\ \Rightarrow \quad & c(x, x^*)\Big(\big\langle \nabla f(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \\ & \leq 0. \end{aligned}$$

Definition 2.17. The function f is said to be second order strictly B-(b, ρ, η, θ)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$,

$$\begin{aligned} & \langle \nabla f(x^*), \eta(x, x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^2 \geq 0 \\ \Rightarrow \quad & b(x, x^*)\Big([f(x) - f(x^*)]\Big) > 0. \end{aligned}$$

Definition 2.18. The function f is said to be second order strictly B-$(b, c, \rho, \eta, \theta)$-quasiinvex with respect to η and b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty$ such that for all $x \in X$,

$$\begin{aligned} & b(x, x^*)\Big([f(x) - f(x^*)]\Big) \leq 0 \\ \Rightarrow \quad & c(x, x^*)\Big(\big\langle \nabla f(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 \\ & < 0. \end{aligned}$$

Now, we need recall the following result (Verma [28]) that is crucial to developing the results for the next section based on second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities.

Theorem 2.19. ([28]) *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z,t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z,s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\Big\{\Big(\nabla G_j(x^*,t), \langle y, \nabla^2 G_j(x^*,t)y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\Big\}$$
$$+span\Big\{\Big(\nabla H_k(x^*,s), \langle y, \nabla^2 H_k(x^*,s)y\rangle\Big) : s \in S_k, k \in \underline{r}\Big\},$$
$$where\ \hat{T}_j(x^*) \equiv \{t \in T_j\ :\ G_j(x^*,t) = 0\},$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0,\ \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \leq \nu_0^* \leq \nu^* \leq n+1$, such that there exist ν_0^* indices j_m, with $1 \leq j_m \leq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*}\backslash\underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^p u_i^*[\nabla f_i(x^*) - \lambda^*\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*,t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^*\nabla H_k(x^*,s^m) = 0, \tag{2.1}$$

$$\Big\langle y, \Big[\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - \lambda^*\nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*\nabla^2 G_{j_m}(x^*,t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^*\nabla^2 H_k(x^*,s^m)\Big]y\Big\rangle \geq 0, \tag{2.2}$$

where $\hat{T}_{j_m}(x^*) = \{t \in T_{j_m} : G_{j_m}(x^*,t) = 0\}$, $U = \{u \in \mathbb{R}^p : u \geq 0, \sum_{i=1}^p u_i = 1\}$, and $\underline{\nu^*}\backslash\underline{\nu_0^*}$ is the complement of the set $\underline{\nu_0^*}$ relative to the set $\underline{\nu^*}$.

3. Second Order sufficient optimality conditions

This section deals with some parametric sufficient efficiency conditions for problem (P) under the generalized frameworks of second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-invexities for generalized invex functions. We start with real-valued functions $E_i(., x^*, u^*)$ and $B_j(., v)$ defined by

$$E_i(x, x^*, u^*) = u_i\left[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)g_i(x)\right],\ i \in \{1, \cdots, p\}$$

and

$$B_j(., v) = v_j H_j(x),\ j = 1, \cdots, m.$$

Theorem 3.1. *Let $x^* \in Q$. Let f_i, g_i for $i \in \{1, \cdots, p\}$ with $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, $g_i(x^*) > 0$ and H_j for $j \in \{1, \cdots, m\}$ be twice continuously differentiable at $x^* \in Q$, and let there exist $u^* \in U = \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1\}$ and $v^* \in \mathbb{R}^m_+$ such that*

$$\sum_{i=1}^{p} u_i^*\left[\triangledown f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right)\triangledown g_i(x^*)\right] + \sum_{j=1}^{m} v_j^* \triangledown H_j(x^*) = 0, \tag{3.1}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*\Big[\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)\Big] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0, \tag{3.2}$$

where $z \in \mathbb{R}^n$, and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.3}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) *$E_i(.; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+ = (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\rho(x, x^*) \geq 0$. $B_j(., v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order B-$(b, \rho, \eta, \theta, \tilde{p}, \tilde{r})$-quasiinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+ = (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$, $\rho(x, x^*) \geq 0$.*

(ii) *$E_i(.; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ are second order B-(b, ρ_1, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudoinvex with respect to η and b at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, a function $b : X \times X \to \mathbb{R}_+ = (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$. $B_j(., v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order B-$(b, \rho_2, \eta, \theta, \tilde{p}, \tilde{r})$-quasiinvex with respect to η and b at $x^* \in X$ if there* *exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to \mathbb{R}_+ = (0, \infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$ *such that for all* $x \in X$*,* $\rho_1(x, x^*), \rho_2(x, x^*) \geq 0$ *with* $\rho_2(x, x^*) \geq \rho_1(x, x^*)$*.*

(iii) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are second order prestrictly B-(*b*,* ρ*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-pseudoinvex with respect to* η *and* b *at* $x^* \in X$ *if there exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to \mathbb{R}_+ = (0, \infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$*.* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are second order strictly B-*$(\rho, \eta, \theta, \tilde{p}, \tilde{r})$*-quasiinvex with respect to* η *and* b *at* $x^* \in X$ *if there exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to \mathbb{R}_+ = (0, \infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$ *such that for all* $x \in X$*,* $\rho(x, x^*) \geq 0$*.*

(iv) $E_i(.\,; x^*, u^*)\ \forall\, i \in \{1, \cdots, p\}$ *are second order prestrictly B-(*b*,* ρ*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-quasiinvex with respect to* η *and* b *at* $x^* \in X$ *if there exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to \mathbb{R}_+ = (0, \infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$*.* $B_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *are second order strictly B-(*b*,* ρ*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-pseudoinvex with respect to* η *and* b *at* $x^* \in X$ *if there exist a function* $\eta : X \times X \to \mathbb{R}^n$*, a function* $b : X \times X \to \mathbb{R}_+ = (0, \infty)$*, and real numbers* $\tilde{r}$ *and* $\tilde{p}$ *such that for all* $x \in X$*,* $\rho(x, x^*) \geq 0$*.*

(v) *For each* $i \in \{1, \cdots, p\}$*,* f_i *is second order B-(*b*,* ρ_1*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-invex and* $-g_i$ *is second order B-(*b*,* ρ_2*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-invex at* x^**.* $H_j(.\,, v^*)\ \forall\, j \in \{1, \cdots, m\}$ *is B-(*b*,* ρ_3*,* η*,* θ*,* $\tilde{p}$*,* $\tilde{r}$*)-quasi-invex at* x^**, and* $\sum_{j=1}^m v_j^* \rho_3 + \rho^* \geq 0$ *for* $\rho^* = \sum_{i=1}^p u_i^*(\rho_1 + \phi(x^*)\rho_2)$ *and for* $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$*.*

Then x^* *is an* $\varepsilon-$*efficient solution to (P).*

Proof. If (i) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\frac{1}{\tilde{p}}\left\langle \sum_{i=1}^p u_i^* \left[\nabla f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) \nabla g_i(x^*)\right], e^{\tilde{p}\eta(x,x^*)} - 1\right\rangle$$
$$+\frac{1}{\tilde{p}}\left\langle \sum_{j=1}^m v_j^* \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\right\rangle = 0, \quad \forall\, x \in Q, \tag{3.4}$$

$$\frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z} - 1, \Big[\sum_{i=1}^p u_i^*\Big[\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)\Big]$$
$$+\sum_{j=1}^m v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0. \tag{3.5}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big) \leq 0,$$

since $\tilde{r} \neq 0$ and $b(x,x^*) > 0$ for all $x \in Q$. In light of the B-$(b,\rho,\eta,\theta,\tilde{p},\tilde{r})$-quasiinvexity of $B_j(.,v^*)$ at x^*, it follows that

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z}-1, \nabla^2 H_j(x^*)z\rangle\Big) + \rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0,$$

and hence,

$$\frac{1}{\tilde{p}}\Big(\sum_{j=1}^{m}\big\langle \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z}-1, \sum_{j=1}^{m}\nabla^2 H_j(x^*)z\rangle\Big)$$
$$+\rho(x,x^*)\|\theta(x,x^*)\|^2 \leq 0. \tag{3.6}$$

It follows from (3.4), (3.5) and (3.6) that

$$\frac{1}{\tilde{p}}\Big(\langle\sum_{i=1}^{p} u_i^*[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)}-1\rangle$$
$$+\frac{1}{2}\Big\langle e^{\tilde{p}z}-1, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big)$$
$$\geq \rho(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.7}$$

Since $\rho(x,x^*) \geq 0$, applying B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudo-invexity at x^* to (3.7), we have

$$\frac{1}{\tilde{r}}b(x,x^*)\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big) \geq 0. \tag{3.8}$$

Since $b(x,x^*) > 0$, (3.8) implies

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^*\Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)g_i(x)\Big]\\
\geq\ &\sum_{i=1}^{p} u_i^*\Big[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)g_i(x^*)\Big]\\
\geq\ &\sum_{i=1}^{p} u_i^*\Big[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)g_i(x^*)\Big] - \sum_{i=1}^{p} u_i^*\varepsilon_i g_i(x^*)\\
=\ &0.
\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{p} u_i^*\Big[f_i(x)-\Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)g_i(x)\Big]\geq 0. \tag{3.9}$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-\Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)\leq 0,\quad \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-\Big(\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j\Big)<0,\quad \text{for some } j\in\{1,\cdots,p\}.$$

Hence, x^* is an ε-efficient solution to (P).

The proof for (ii) is similar to that of (i), but we include for the sake of the completeness. If (ii) holds, and if $x\in Q$, then it follows from (3.1) and (3.2) that

$$\frac{1}{\tilde{p}}\Big\langle\sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*)-\Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)\bigtriangledown g_i(x^*)\Big],e^{\tilde{p}\eta(x,x^*)}-1\Big\rangle$$
$$+\frac{1}{\tilde{p}}\Big\langle\sum_{j=1}^{m} v_j^*\bigtriangledown H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\Big\rangle=0,\quad \forall\, x\in Q, \tag{3.10}$$

$$\frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z}-1,\Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^{m} v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle\geq 0. \tag{3.11}$$

Since $v^*\geq 0$, $x\in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x)\leq 0=\sum_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big)\leq 0,$$

since $\tilde{r}\neq 0$ and $b(x,x^*)>0$ for all $x\in Q$. In light of the B-(b, ρ_2, η, θ, $\tilde{p}$, $\tilde{r}$)-quasiinvexity of $B_j(.,v^*)$ at x^*, it follows that

$$\frac{1}{\tilde{p}}\Big(\big\langle\nabla H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2}\langle e^{\tilde{p}z}-1,\nabla^2 H_j(x^*)z\rangle\Big)+\rho_2(x,x^*)\|\theta(x,x^*)\|^2\leq 0,$$

and hence,

$$\frac{1}{\tilde{p}}\Big(\sum_{j=1}^{m}\big\langle\nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2}\langle e^{\tilde{p}z}-1, \sum_{j=1}^{M}\nabla^2 H_j(x^*)z\rangle\Big)$$
$$+\rho_2(x,x^*)\|\theta(x,x^*)\|^2\leq 0. \tag{3.12}$$

It follows from (3.10), (3.11) and (3.12) that

$$\frac{1}{\tilde{p}}\Big(\langle\sum_{i=1}^{p}u_i^*[\triangledown f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\triangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)}-1\rangle$$
$$+\frac{1}{2}\Big\langle e^{\tilde{p}z}-1, \sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big)$$
$$\geq\rho_2(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.13}$$

Since $\rho_1(x,x^*),\rho_2(x,x^*)\geq 0$ with $\rho_2(x,x^*)\geq\rho_1(x,x^*)$, applying B-(b, ρ_1, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudo-invexity at x^* to (3.13), we have

$$\frac{1}{\tilde{r}}b(x,x^*)\big(e^{\tilde{r}[E_i(x,x^*,u^*)-E_i(x^*,x^*,u^*)]}-1\big)\geq 0. \tag{3.14}$$

Since $b(x,x^*)>0$, (3.14) implies

$$\sum_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]$$
$$\geq\sum_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)])$$
$$\geq 0.$$

Thus, we have

$$\sum_{i=1}^{p}u_i^*\left[f_i(x)-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)g_i(x)\right]\geq 0. \tag{3.15}$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)\leq 0,\quad\forall\, i=1,\cdots,p,$$
$$\frac{f_j(x)}{g_j(x)}-\left(\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j\right)<0,\quad\text{for some } j\in\{1,\cdots,p\}.$$

Hence, x^* is an ε-efficient solution to (P).

Next, we prove (iii) as follows: if (iii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\frac{1}{\tilde{p}}\left\langle \sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\bigtriangledown g_i(x^*)\Big], e^{\tilde{p}\eta(x,x^*)} - 1\right\rangle$$
$$+\frac{1}{\tilde{p}}\left\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\right\rangle = 0, \quad \forall\, x \in Q, \tag{3.16}$$

and

$$\frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z} - 1, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^{m} v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle \geq 0. \tag{3.17}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]} - 1\big)\Big) \leq 0.$$

Then, in light of the strict B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)- quasi-invexity of $B_j(., v^*)$ at x^*, we have

$$\frac{1}{\tilde{p}}\Big(\big\langle \nabla H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1\big\rangle + \frac{1}{2}\langle e^{\tilde{p}z} - 1, \nabla^2 H_j(x^*)z\rangle\Big)$$
$$+\rho(x,x^*)\|\theta(x,x^*)\|^2 < 0. \tag{3.18}$$

It follows from (3.3), (3.16), (3.17) and (3.18) that

$$\frac{1}{\tilde{p}}\Big(\langle\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\bigtriangledown g_i(x^*)], e^{\tilde{p}\eta(x,x^*)} - 1\rangle$$
$$+\frac{1}{2}\Big\langle e^{\tilde{p}z} - 1, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*)z - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big)$$
$$> \rho(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.19}$$

As a result, since $\rho(x,x^*)\geq 0$, applying the prestrict B-(b, $\rho,\eta,\theta,\tilde{p}$, $\tilde{r}$)-pseudo-invexity at x^* to (3.19), we have

$$\Big(\sum_{i=1}^{p} u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)}-\varepsilon_i)g_i(x)]-\sum_{i=1}^{p} u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}&\sum_{i=1}^{p} u_i^*\left[f_i(x)-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)g_i(x)\right]\\&\geq \sum_{i=1}^{p} u_i^*\left[f_i(x^*)-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)g_i(x^*)\right]\\&\geq 0.\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{p} u_i^*\left[f_i(x)-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)g_i(x)\right]\geq 0. \tag{3.20}$$

Since $u_i^*>0$ for each $i\in\{1,\cdots,p\}$, we conclude that there does not exist an $x\in Q$ such that

$$\frac{f_i(x)}{g_i(x)}-\left(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\right)\leq 0,\quad \forall\, i=1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)}-\left(\frac{f_j(x^*)}{g_j(x^*)}-\varepsilon_j\right)< 0,\quad \text{for some } j\in\{1,\cdots,p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

The proof applying (iv) is similar to that of (iii), but still we include it as follows: if $x\in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned}&\frac{1}{\tilde{p}}\left\langle\sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*)-\Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)\bigtriangledown g_i(x^*)\Big], e^{\tilde{p}\eta(x,x^*)}-1\right\rangle\\&+\frac{1}{\tilde{p}}\left\langle\sum_{j=1}^{m} v_j^*\bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)}-1\right\rangle=0,\quad \forall\, x\in Q,\end{aligned} \tag{3.21}$$

and

$$\frac{1}{2\tilde{p}}\Big\langle e^{\tilde{p}z}-1,\Big[\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)]$$
$$+\sum_{j=1}^{m}v_j^*\nabla^2 H_j(x^*)\Big]z\Big\rangle\geq 0. \tag{3.22}$$

Since $v^*\geq 0$, $x\in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m}v_j^*H_j(x)\leq 0=\sum_{j=1}^{m}v_j^*H_j(x^*),$$

which implies

$$b(x,x^*)\Big(\frac{1}{\tilde{r}}\big(e^{\tilde{r}[H_j(x)-H_j(x^*)]}-1\big)\Big)\leq 0.$$

Then, in light of the equivalent form for the strict B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-pseudo-invexity of $B_j(.,v^*)$ at x^*, we have

$$\frac{1}{\tilde{p}}\Big(\big\langle\nabla H_j(x^*),e^{\tilde{p}\eta(x,x^*)}-1\big\rangle+\frac{1}{2}\langle e^{\tilde{p}z}-1,\nabla^2 H_j(x^*)z\rangle\Big)+\rho(x,x^*)\|\theta(x,x^*)\|^2<0.$$

It follows from (3.3), (3.21) and (3.22) that

$$\frac{1}{\tilde{p}}\Big(\langle\sum_{i=1}^{p}u_i^*[\nabla f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\bigtriangledown g_i(x^*)],e^{\tilde{p}\eta(x,x^*)}-1\rangle$$
$$+\frac{1}{2}\Big\langle e^{\tilde{p}z}-1,\sum_{i=1}^{p}u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z]\Big\rangle\Big)$$
$$>\rho(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.23}$$

As a result, since $\rho(x,x^*)\geq 0$, applying the equivalent form for the prestrict B-(b, ρ, η, θ, $\tilde{p}$, $\tilde{r}$)-quasi-invexity of $E_i(.;x^*,u^*)$ at x^* to (3.23), we have

$$\Big(\sum_{i=1}^{p}u_i^*[f_i(x)-(\frac{f_i(x^*)}{g_i(g^*)}-\varepsilon_i)g_i(x)]-\sum_{i=1}^{p}u_i^*[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]\Big)\geq 0,$$

which implies

$$\begin{aligned}&\sum_{i=1}^{p} u_i^* \left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) g_i(x) \right] \\ &\geq \sum_{i=1}^{p} u_i^* \left[f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) g_i(x^*) \right] \\ &\geq 0.\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{p} u_i^* \left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) g_i(x) \right] \geq 0. \tag{3.24}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) \leq 0, \quad \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - \left(\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j \right) < 0, \quad \text{for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

Finally, we prove (v) as follows: since $x \in Q$, it follows that

$H_j(x) \leq H_j(x^*)$, which implies $\Big(H_j(x) - H_j(x^*)\Big) \leq 0$. Then applying the B-(b, ρ_3, η, $\tilde{p}$, $\tilde{r}$)- quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$\begin{aligned}&\frac{1}{\tilde{p}}\Big(\Big\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), e^{\tilde{p}\eta(x,x^*)} - 1 \Big\rangle \\ &\quad + \frac{1}{2} \Big\langle e^{\tilde{p}z} - 1, \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z \Big\rangle \Big) \\ &\leq - \sum_{j=1}^{m} v_j^* \rho_3 \|\theta(x, x^*)\|^2.\end{aligned}$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, it follows from B-(b, ρ_3, η, $\tilde{p}$, $\tilde{r}$)-invexity assumptions that

$$
\begin{aligned}
&b(x,x^*)\frac{1}{\tilde r}\Big(e^{\tilde r\sum_{i=1}^p u_i^*\big([f_i(x)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x)]-[f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)g_i(x^*)]\big)}-1\Big)\\
&=b(x,x^*)\frac{1}{\tilde r}\Big(e^{\tilde r\sum_{i=1}^p u_i^*\{[f_i(x)-f_i(x^*)]-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)[g_i(x)-g_i(x^*)]\}}-1\Big)\\
&\geq \frac{1}{\tilde p}\Big(\sum_{i=1}^p u_i^*\{\langle\nabla f_i(x^*)-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla g_i(x^*),e^{\tilde p\eta(x,x^*)}-1\rangle\}\\
&\quad+\frac{1}{2}\langle e^{\tilde p z}-1,\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*)z-(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i)\nabla^2 g_i(x^*)z\rangle]\Big)\\
&\quad+\sum_{i=1}^p u_i^*[\rho_1+\phi(x^*)\rho_2]\|\theta(x,x^*)\|^2\\
&\geq -\frac{1}{\tilde p}\Big[\Big\langle\sum_{j=1}^m v_j^*\nabla H_j(x^*),e^{\tilde p\eta(x,x^*)}-1\Big\rangle\\
&\quad+\frac{1}{2}\Big\langle e^{\tilde p z}-1,\sum_{j=1}^m v_j^*\nabla^2 H_j(x^*)z\Big\rangle\Big]\Big)+\sum_{i=1}^p u_i^*[\rho_1+\phi(x^*)\rho_2]\|\theta(x,x^*)\|^2\\
&\geq\left(\sum_{j=1}^m v_j^*\rho_3+\sum_{i=1}^p u_i^*[\rho_1+\phi(x^*)\rho_2]\right)\|\theta(x,x^*)\|^2\\
&=\left(\sum_{j=1}^m v_j^*\rho_3+\rho^*\right)\|\theta(x,x^*)\|^2\\
&\geq 0,
\end{aligned}
$$

where $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i$ and $\rho^*=\sum_{i=1}^p u_i^*(\rho_1+\phi(x^*)\rho_2)$. □

Next, we first present a specialization to Theorem 3.1 relating to the second order B-$(b,c,\rho,\eta,\theta,\tilde p,\tilde r)$-invexities when $\tilde p=0$ and $\tilde r=0$.

Theorem 3.2. *Let $x^*\in Q$. Let f_i,g_i for $i\in\{1,\cdots,p\}$ with $\phi(x^*)=\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\geq 0$, $g_i(x^*)>0$ and H_j for $j\in\{1,\cdots,m\}$ be twice continuously differentiable at $x^*\in Q$, and let there exist $u^*\in U=\{u\in\mathbb{R}^p: u>0,\sum_{i=1}^p u_i=1\}$ and $v^*\in\mathbb{R}^m_+$ such that*

$$
\sum_{i=1}^p u_i^*\Big[\nabla f_i(x^*)-\Big(\frac{f_i(x^*)}{g_i(x^*)}-\varepsilon_i\Big)\nabla g_i(x^*)\Big]+\sum_{j=1}^m v_j^*\nabla H_j(x^*)=0, \quad (3.25)
$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)\nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle \geq 0, \tag{3.26}$$

and

$$v_j^* H_j(x^*) = 0,\ j \in \{1, \cdots, m\}. \tag{3.27}$$

Suppose, in addition, that any one of the following assumptions holds:

(i) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are second order B-(b, c, ρ, η, θ)-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$ and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order B-(b,c, ρ, η, θ)- quasiinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$, $z \in \mathbb{R}^n$ and $\rho(x, x^*) \geq 0$.*

(ii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are second order B-(b, c, ρ_1, η, θ)-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order B-(b, c, ρ_2, η, θ)-quasiinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, and functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$, $z \in \mathbb{R}^n$, and $\rho_1(x, x^*), \rho_2(x, x^*) \geq 0$ with $\rho_2(x, x^*) \geq \rho_1(x, x^*)$.*

(iii) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are second order prestrictly B-(b, c, ρ, η, θ)-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions $b, c : X \times X \to (0, \infty)$, and real numbers $\tilde{r}$ and $\tilde{p}$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order strictly B-(b, c, ρ, η, θ)-quasiinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions $b, c : X \times X \to (0, \infty)$, such that for all $x \in X$, $z \in \mathbb{R}^n$, $\rho(x, x^*) \geq 0$.*

(iv) $E_i(.\,; x^*, u^*)$ $\forall\, i \in \{1, \cdots, p\}$ *are second order prestrictly B-(b, c, ρ, η,θ)-quasi-invex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions $b, c : X \times X \to (0, \infty)$ such that for all $x \in X$ and $z \in \mathbb{R}^n$, and $B_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ are second order strictly B-(b, c, ρ, η, θ)-pseudoinvex with respect to η, b and c at $x^* \in X$ if there exist a function $\eta : X \times X \to \mathbb{R}^n$, functions*

$b, c : X \times X \to (0,\infty)$ such that for all $x \in X$, $z \in \mathbb{R}^n$, $\rho(x, x^*) \geq 0$.

(v) *For each $i \in \{1, \cdots, p\}$, f_i is second order B-(b, c, ρ_1, η, θ)-invex and $-g_i$ is second order B-(b, c, ρ_2, η, θ)-invex at x^*. $H_j(.\,, v^*)$ $\forall\, j \in \{1, \cdots, m\}$ is B-(b, c, ρ_3, η, θ)-quasi-invex at x^*, and $\sum_{j=1}^{m} v_j^* \rho_3 + \rho^* \geq 0$ for $\rho^* = \sum_{i=1}^{p} u_i^*(\rho_1 + \phi(x^*)\rho_2)$ and for $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$.*

Then x^* is an ε-efficient solution to (P).

Proof. If (i) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\left\langle \sum_{i=1}^{p} u_i^* \left[\triangledown f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) \triangledown\, g_i(x^*) \right], \eta(x, x^*) \right\rangle + \left\langle \sum_{j=1}^{m} v_j^* \triangledown\, H_j(x^*), \eta(x, x^*) \right\rangle = 0, \quad \forall\, x \in Q, \tag{3.28}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* [\nabla^2 f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \nabla^2 g_i(x^*)] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z \Big\rangle \geq 0. \tag{3.29}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x, x^*) \Big(([H_j(x) - H_j(x^*)]) \Big) \leq 0,$$

since $b(x, x^*) > 0$ for all $x \in Q$. In light of the B-(b, c, ρ, η, θ)-quasiinvexity of $B_j(., v^*)$ at x^*, and $c(x, x^*) > 0$, it follows that

$$c(x, x^*) \Big(\big\langle \nabla H_j(x^*), \eta(x, x^*) \big\rangle + \frac{1}{2} \langle z, \nabla^2 H_j(x^*) z \rangle \Big) + \rho(x, x^*) \|\theta(x, x^*)\|^2 \leq 0,$$

and hence,

$$c(x, x^*) \Big(\big\langle \sum_{j=1}^{m} \nabla H_j(x^*), \eta(x, x^*) \big\rangle + \frac{1}{2} \langle z, \sum_{j=1}^{m} \nabla^2 H_j(x^*) z \rangle \Big) + \rho(x, x^*) \|\theta(x, x^*)\|^2 \leq 0. \tag{3.30}$$

It follows from (3.28), (3.29) and (3.30) that

$$c(x,x^*)\Big(\Big\langle \sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\bigtriangledown g_i(x^*)\Big], \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*\Big[\nabla^2 f_i(x^*)z - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)z\Big]\Big\rangle\Big) \geq \rho(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.31}$$

Since $\rho(x,x^*) \geq 0$, applying B-(b, c, ρ, η, θ)-pseudo-invexity at x^* to (3.31), we have

$$b(x,x^*)\big([E_i(x,x^*,u^*) - E_i(x^*,x^*,u^*)]\big) \geq 0. \tag{3.32}$$

Since $b(x,x^*) > 0$, (3.32) implies

$$\sum_{i=1}^{p} u_i^*\left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x)\right] \geq \sum_{i=1}^{p} u_i^*\left[f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x^*)\right] \geq 0.$$

Thus, we have

$$\sum_{i=1}^{p} u_i^*\left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x)\right] \geq 0. \tag{3.33}$$

Since $u_i^* > 0$ for each $i \in \{1,\cdots,p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \leq 0, \quad \forall\, i = 1,\cdots,p,$$

$$\frac{f_j(x)}{g_j(x)} - \Big(\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j\Big) < 0, \quad \text{for some } j \in \{1,\cdots,p\}.$$

Hence, x^* is an ε-efficient solution to (P).

The proof for (ii) is similar to that of (i), but we include for the sake of the completeness. If (ii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\Big\langle \sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*) - \big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\bigtriangledown g_i(x^*)\Big], \eta(x,x^*)\Big\rangle + \Big\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x,x^*)\Big\rangle = 0, \quad \forall\, x \in Q, \tag{3.34}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* \Big[\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)\Big] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle$$
$$\geq 0. \tag{3.35}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

and so

$$b(x, x^*)\Big(\big([H_j(x) - H_j(x^*)]\big)\Big) \leq 0,$$

since $b(x, x^*) > 0$ for all $x \in Q$. In light of the B-(b, c, ρ_2, η, θ)-quasiinvexity of $B_j(., v^*)$ at x^*, it follows that

$$c(x, x^*)\Big(\big\langle \nabla H_j(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle\Big) + \rho_2(x, x^*)\|\theta(x, x^*)\|^2 \leq 0,$$

and hence,

$$c(x, x^*)\Big(\sum_{j=1}^{m} \big\langle \nabla H_j(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \sum_{j=1}^{m} \nabla^2 H_j(x^*) z\rangle\Big)$$
$$+\rho_2(x, x^*)\|\theta(x, x^*)\|^2 \leq 0. \tag{3.36}$$

It follows from (3.34), (3.35) and (3.36) that

$$c(x, x^*)\Big(\Big\langle \sum_{i=1}^{p} u_i^* \Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \bigtriangledown g_i(x^*)\Big], \eta(x, x^*)\Big\rangle$$
$$+\frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^* \Big[\nabla^2 f_i(x^*) z - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*) z\Big]\Big\rangle\Big)$$
$$\geq \rho_2(x, x^*)\|\theta(x, x^*)\|^2. \tag{3.37}$$

Since $\rho_1(x, x^*), \rho_2(x, x^*) \geq 0$ with $\rho_2(x, x^*) \geq \rho_1(x, x^*)$, and $c(x, x^*) > 0$, applying B-(b, c, ρ_1, η, θ)-pseudo-invexity at x^* to (3.37), we have

$$b(x, x^*)\Big(\big([E_i(x, x^*, u^*) - E_i(x^*, x^*, u^*)]\big)\Big) \geq 0. \tag{3.38}$$

Since $b(x, x^*) > 0$, (3.38) implies

$$\sum_{i=1}^{p} u_i^* \left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x)\right]$$
$$\geq \sum_{i=1}^{p} u_i^* \left[f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x^*)\right]$$
$$\geq 0.$$

Thus, we have

$$\sum_{i=1}^{p} u_i^* \left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) g_i(x) \right] \geq 0. \tag{3.39}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \right) \leq 0, \quad \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - \left(\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j \right) < 0, \quad \text{for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an ε-efficient solution to (P).

Next, we start off the proof for (iii) as follows: if (iii) holds, and if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\begin{aligned} &\left\langle \sum_{i=1}^{p} u_i^* \Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) \bigtriangledown g_i(x^*) \Big], \eta(x, x^*) \right\rangle \\ &+ \left\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*) \right\rangle = 0, \quad \forall\, x \in Q, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* \Big[\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \Big) \nabla^2 g_i(x^*) \Big] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) \Big] z \Big\rangle \\ &\geq 0. \end{aligned} \tag{3.41}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x, x^*) \Big(\big([H_j(x) - H_j(x^*)] \big) \Big) \leq 0.$$

Then, in light of the strict B-(b, c, ρ, η, θ)-quasi-invexity of $B_j(., v^*)$ at x^*, we have

$$\begin{aligned} &c(x, x^*) \Big(\big\langle \nabla H_j(x^*), \eta(x, x^*) \big\rangle + \frac{1}{2} \langle z, \nabla^2 H_j(x^*) z \rangle \Big) \\ &+ \rho(x, x^*) \|\theta(x, x^*)\|^2 < 0. \end{aligned} \tag{3.42}$$

It follows from (3.3), (3.40), (3.41) and (3.42) that

$$\Big(\Big\langle \sum_{i=1}^{p} u_i^*\Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \bigtriangledown g_i(x^*)\Big], \eta(x,x^*)\Big\rangle$$

$$+\frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*\Big[\nabla^2 f_i(x^*)z - \Big(\frac{f_i(x^*)}{g_i(x)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)z\Big]\Big\rangle\Big)$$

$$> \rho(x,x^*)\|\theta(x,x^*)\|^2. \tag{3.43}$$

As a result, since $\rho(x,x^*) \geq 0$, applying the prestrict B-(b, c, ρ, η, θ)-pseudo-invexity at x^* to (3.43), we have

$$\Big(\sum_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \varepsilon_i)g_i(x)] - \sum_{i=1}^{p} u_i^*[f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x^*)]\Big) \geq 0,$$

which implies

$$\sum_{i=1}^{p} u_i^* \left[f_i(x) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x)\right]$$

$$\geq \sum_{i=1}^{p} u_i^* \left[f_i(x^*) - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) g_i(x^*)\right]$$

$$\geq 0.$$

Thus, we have

$$\sum_{i=1}^{p} u_i^*\Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)g_i(x)\Big] \geq 0. \tag{3.44}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \leq 0, \quad \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - \Big(\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j\Big) < 0, \quad \text{for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an $\varepsilon-$efficient solution to (P).

The proof applying (iv) is similar to that of (iii), but still we include it as follows: if $x \in Q$, then it follows from (3.1) and (3.2) that

$$\Big\langle \sum_{i=1}^{p} u_i^* \Big[\bigtriangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \bigtriangledown g_i(x^*)\Big], \eta(x, x^*)\rangle$$

$$+\Big\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\Big\rangle = 0, \quad \forall\, x \in Q, \tag{3.45}$$

$$\Big\langle z, \Big[\sum_{i=1}^{p} u_i^* \Big[\nabla^2 f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)\Big] + \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*)\Big] z\Big\rangle$$
$$\geq 0. \tag{3.46}$$

Since $v^* \geq 0$, $x \in Q$ and (3.3) holds, we have

$$\sum_{j=1}^{m} v_j^* H_j(x) \leq 0 = \sum_{j=1}^{m} v_j^* H_j(x^*),$$

which implies

$$b(x, x^*)\Big(([H_j(x) - H_j(x^*)])\Big) \leq 0.$$

Then, in light of the equivalent form for the strict B-(b, c, ρ, η, θ)-pseudo-invexity of $B_j(., v^*)$ at x^*, we have

$$c(x, x^*)\Big(\big\langle \nabla H_j(x^*), \eta(x, x^*)\big\rangle + \frac{1}{2}\langle z, \nabla^2 H_j(x^*) z\rangle\Big) + \rho(x, x^*)\|\theta(x, x^*)\|^2 < 0.$$

It follows from (3.3), (3.45) and (3.46) that

$$\Big(\Big\langle \sum_{i=1}^{p} u_i^* \Big[\bigtriangledown f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) \bigtriangledown g_i(x^*)\Big], \eta(x, x^*)\Big\rangle$$

$$+\frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^* \Big[\nabla^2 f_i(x^*) z - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*) z\Big]\Big\rangle\Big)$$

$$> \rho(x, x^*)\|\theta(x, x^*)\|^2. \tag{3.47}$$

As a result, since $\rho(x, x^*) \geq 0$, applying the equivalent form for the prestrict B-(b, c, ρ, η, θ)-quasi-invexity of $E_i(.; x^*, u^*)$ at x^* to (3.47), we have

$$\Big(\sum_{i=1}^{p} u_i^* [f_i(x) - (\frac{f_i(x^*)}{g_i(g^*)} - \varepsilon_i) g_i(x)] - \sum_{i=1}^{p} u_i^* [f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) g_i(x^*)]\Big) \geq 0,$$

which implies

$$\sum_{i=1}^{p} u_i^* \Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x)\Big]$$
$$\geq \sum_{i=1}^{p} u_i^* \Big[f_i(x^*) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x^*)\Big]$$
$$\geq 0.$$

Thus, we have

$$\sum_{i=1}^{p} u_i^* \Big[f_i(x) - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big) g_i(x)\Big] \geq 0. \tag{3.48}$$

Since $u_i^* > 0$ for each $i \in \{1, \cdots, p\}$, we conclude that there does not exist an $x \in Q$ such that

$$\frac{f_i(x)}{g_i(x)} - \left(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\right) \leq 0, \quad \forall\, i = 1, \cdots, p,$$

$$\frac{f_j(x)}{g_j(x)} - \left(\frac{f_j(x^*)}{g_j(x^*)} - \varepsilon_j\right) < 0, \quad \text{for some } j \in \{1, \cdots, p\}.$$

Hence, x^* is an ε-efficient solution to (P).

Finally, to prove (v), we start with: since $x \in Q$, it follows that $H_j(x) \leq H_j(x^*)$, i.e., $H_j(x) - H_j(x^*) \leq 0$, which implies

$$b(x, x^*)\Big(\big([H_j(x) - H_j(x^*)]\big)\Big) \leq 0.$$

Then applying the B-(b, c, ρ_3, η, θ)-quasi-invexity of H_j at x^* and $v^* \in R_+^m$, we have

$$c(x, x^*)\Big(\Big\langle \sum_{j=1}^{m} v_j^* \bigtriangledown H_j(x^*), \eta(x, x^*)\Big\rangle$$
$$+\frac{1}{2}\Big\langle z, \sum_{j=1}^{m} v_j^* \nabla^2 H_j(x^*) z\Big\rangle\Big)$$
$$\leq -\sum_{j=1}^{m} v_j^* \rho_3 \|\theta(x, x^*)\|^2.$$

Since $u^* \geq 0$ and $\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i \geq 0$, it follows from B-$(b, c, \rho_3, \eta, \theta)$-invexity assumptions that

$$\begin{aligned}
& b(x,x^*)\Big(\sum_{i=1}^{p} u_i^*[f_i(x) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)g_i(x)]\Big) \\
&= b(x,x^*)\Big(\sum_{i=1}^{p} u_i^*\{[f_i(x) - f_i(x^*)] - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i)[g_i(x) - g_i(x^*)]\}\Big) \\
&\geq c(x,x^*)\Big(\sum_{i=1}^{p} u_i^*\{\langle \triangledown f_i(x^*) - (\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i) \triangledown g_i(x^*), \eta(x,x^*)\rangle\} \\
&\quad + \frac{1}{2}\Big\langle z, \sum_{i=1}^{p} u_i^*\Big[\nabla^2 f_i(x^*)z - \Big(\frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i\Big)\nabla^2 g_i(x^*)z\Big\rangle\Big]\Big) \\
&\quad + \sum_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\
&\geq -c(x,x^*)\Big[\Big\langle \sum_{j=1}^{m} v_j^* \triangledown H_j(x^*), \eta(x,x^*)\Big\rangle + \frac{1}{2}\Big\langle z, \sum_{j=1}^{m} v_j^*\nabla^2 H_j(x^*)z\Big\rangle\Big] \\
&\quad + \sum_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\|\theta(x,x^*)\|^2 \\
&\geq \Big(\sum_{j=1}^{m} v_j^*\rho_3 + \sum_{i=1}^{p} u_i^*[\rho_1 + \phi(x^*)\rho_2]\Big)\|\theta(x,x^*)\|^2 \\
&= \Big(\sum_{j=1}^{m} v_j^*\rho_3 + \rho^*\Big)\|\theta(x,x^*)\|^2 \\
&\geq 0,
\end{aligned}$$

where $\phi(x^*) = \frac{f_i(x^*)}{g_i(x^*)} - \varepsilon_i$ and $\rho^* = \sum_{i=1}^{p} u_i^*(\rho_1 + \phi(x^*)\rho_2)$. □

Remark 3.3. The obtained results in this communication present challenging applications to the context of multiobjective fractional subset programming, while further generalizations to the case of rapidly developing notions of generalized invexities of higher orders seem to be feasible.

References

[1] T. Antczak, *A class of B-(p, r)-invex functions and mathematical programming,* J. Math. Anal. Appl., **268** (2003), 187–206.

[2] T. Antczak, *B-(p, r)-pre-invex functions,* Folia Math. Acta Univ. Lodziensis, **11** (2004), 3–15.

[3] T. Antczak, *Generalized B-(p, r)-invexity functions and nonlinear mathematical programming,* Numer. Funct. Anal. Optim., **30** (2009), 1–22.

[4] A. Ben-Israel and B. Mond *What is the invexity?* Journal of Australian Mathematical Society Ser. B, **28** (1986), 1–9.

[5] L. Caiping and Y. Xinmin, *Generalized $(\rho, \theta, \eta)-$invariant monotonicity and generalized $(\rho, \theta, \eta)-$invexity of non-differentiable functions,* Journal of Inequalities and Applications, **2009** (2009), Article ID 393940, 16 pages.

[6] M.A Hanson, *On sufficiency of the Kuhn-Tucker conditions,* Journal of Mathematical Analysis and Applications **80** (1985), 545–550.

[7] V. Jeyakumar, *Strong and weak invexity in mathematical programming,* Methods Oper. Res., **55** (1985), 109–125.

[8] M.H. Kim, G.S. Kim and G.M. Lee, *On $\varepsilon-$optimality conditions for multiobjective fractional optimization problems,* Fixed Point Theory & Applications, 2011:6 doi:10.1186/1687-1812-2011-6.

[9] G.S. Kim and G.M. Lee, *On $\varepsilon-$optimality theorems for convex vector optimization problems,* Journal of Nonlinear and Convex Analysis, **12(3)** (2013), 473–482.

[10] J.C. Liu, *Second order duality for minimax programming,* Utilitas Math., **56** (1999), 53–63

[11] O.L. Mangasarian, *Second- and higher-order duality theorems in nonlinear programming,* J. Math. Anal. Appl., **51** (1975), 607–620.

[12] S.K. Mishra, *Second order generalized invexity and duality in mathematical programming,* Optimization, **42** (1997), 51–69.

[13] S.K. Mishra, *Second order symmetric duality in mathematical programming with F-convexity,* European J. Oper. Res., **127** (2000), 507–518.

[14] S.K. Mishra and N. G. Rueda, *Higher-order generalized invexity and duality in mathematical programming,* J. Math. Anal. Appl., **247** (2000), 173–182.

[15] S.K. Mishra and N. G. Rueda, *Second-order duality for nondifferential minimax programming involving generalized type I functions,* J. Optim. Theory Appl., **130** (2006), 477–486.

[16] S.K. Mishra, M. Jaiswal and Pankaj, *Optimality conditions for multiple objective fractional subset programming with invex and related non-convex functions,* Communications on Applied Nonlinear Analysis, **17(3)** (2010), 89–101.

[17] B. Mond and T. Weir, *Generalized convexity and higher-order duality,* J. Math. Sci., **16-18** (1981-1983), 74–94.

[18] B. Mond and J. Zhang, *Duality for multiobjective programming involving second-order V-invex functions,* in *Proceedings of the Optimization Miniconference II* (B. M. Glover and V. Jeyakumar, eds.), University of New South Wales, Sydney, Australia, 1997, pp. 89–100.

[19] B. Mond and J. Zhang, *Higher order invexity and duality in mathematical programming*, in Generalized Convexity, Generalized Monotonicity: Recent Results (J. P. Crouzeix, *et al.*, eds.), Kluwer Academic Publishers, printed in the Netherlands, 1998, pp. 357–372.

[20] R.B. Patel, *Second order duality in multiobjective fractional programming,* Indian J. Math., **38** (1997), 39–46.

[21] M.K. Srivastava and M. Bhatia, *Symmetric duality for multiobjective programming using second order (F,ρ)-convexity,* Opsearch, **43** (2006), 274–295.

[22] K.K. Srivastava and M. G. Govil, *Second order duality for multiobjective programming involving $(F,\rho,\sum)$-type I functions,* Opsearch, **37** (2000), 316–326.

[23] S.K. Suneja, C. S. Lalitha, and S. Khurana, *Second order symmetric duality in multiobjective programming,* European J. Oper. Res., **144** (2003), 492–500.

[24] M.N. Vartak and I. Gupta, *Duality theory for fractional programming problems under η-convexity,* Opsearch, **24** (1987), 163–174.

[25] R.U. Verma, *Weak ε-efficiency conditions for multiobjective fractional programming,* Applied Mathematics and Computation, **219** (2013), 6819–6927.

[26] R.U. Verma, *A generalization to Zalmai type second order univexities and applications to parametric duality models to discrete minimax fractional programming,* Advances in Nonlinear Variational Inequalities, **15(2)** (2012), 113–123.

[27] R.U. Verma, *Advanced higher order univexities with applications to parametric duality models in discrete minimax fractional programming,* Transactions on Mathematical Programming and Applications, **1(3)** (2013), 77–99.

[28] R.U. Verma, *Generalized $(\mathcal{G},b,\beta,\phi,h,\rho,\theta)$-univexities with applications to parametric duality models for discrete minimax fractional programming,* Transactions on Mathematical Programming and Applications, **1(1)** (2013), 1–14.

[29] R.U. Verma, *New $\varepsilon-$optimality conditions for multiobjective fractional subset programming problems,* Transactions on Mathematical Programming and Applications, **1(1)** (2013), 69–89.

[30] R.U. Verma, *Second order $(\Phi,\Psi,\rho,\eta,\theta)-$invexity frameworks and $\varepsilon-$ efficiency conditions for multiobjective fractional programming,* Theory and Applications of Mathematics & Computer Science, **2(1)** (2013), 31–47.

[31] R.U. Verma, *Role of second order $B-(\rho,\eta,\theta,\tilde{p},\tilde{r})-$invexities and parametric sufficient conditions in semiinfinite minimax fractional programming,* Transactions on Mathematical Programming and Applications, **1(2)** (2013), 13–45.

[32] X.M. Yang, *On second order symmetric duality in nondifferentiable multiobjective programming,* J. Ind. Manag. Optim., **5** (2009), 697–703.

[33] X.M. Yang and S. H. Hou, *Second-order symmetric duality in multiobjective programming,* Appl. Math. Lett., **14** (2001), 587–592.

[34] X.M. Yang, K.L. Teo and X.Q. Yang, *Higher-order generalized convexity and duality in nondifferentiable multiobjective mathematical programming,* J. Math. Anal. Appl., **297** (2004), 48–55.

[35] X.M. Yang, X.Q. Yang and K.L. Teo, *Nondifferentiable second order symmetric duality in mathematical programming with F-convexity,* European J. Oper. Res., **144** (2003), 554–559.

[36] X.M. Yang, X.Q. Yang and K.L. Teo, *Huard type second-order converse duality for nonlinear programming,* Appl. Math. Lett., **18** (2005), 205–208.

[37] X.M. Yang, X.Q. Yang and K.L. Teo, *Higher-order symmetric duality in multiobjective programming with invexity,* J. Ind. Manag. Optim., **4** (2008), 385–391.

[38] X.M. Yang, X.Q. Yang K.L. Teo and S.H. Hou, *Second order duality for nonlinear programming,* Indian J. Pure Appl. Math., **35** (2004), 699–708.

[39] K. Yokoyama, *ε-approximate solutions for multiobjective programming problems,* Journal of Mathematical Analysis and Applications, **203(1)** (1996), 142–149.

[40] G.J. Zalmai, *General parametric sufficient optimality conditions for discrete minmax fractional programming problems containing generalized (θ,η,ρ)-V-invex functions and arbitrary norms*, Journal of Applied Mathematics & Computing, **23(1-2)** (2007), 1–23.

[41] G.J. Zalmai, *Hanson-Antczak-type generalized invex functions in semiinfinte minmax fractional programming, Part I: Sufficient optimality conditions,* Communications on Applied Nonlinear Analysis, **19(4)** (2012), 1–36.

[42] G.J. Zalmai and Q. Zhang, *Global nonparametric sufficient optimality conditions for semiinfinite discrete minmax fractional programming problems involving generalized* $(\eta, \rho)-$*invex functions,* Numerical Functional Analysis and Optimization, **28** (2007), 173–209.

[43] J. Zhang and B. Mond, *Second order b-invexity and duality in mathematical programming,* Utilitas Math., **50** (1996), 19–31.

[44] J. Zhang and B. Mond, *Second order duality for multiobjective nonlinear programming involving generalized convexity,* in *Proceedings of the Optimization Miniconference III* (B. M. Glover, B. D. Craven, and D. Ralph, eds.), University of Ballarat, (1997), pp. 79–95.

[45] E. Zeidler, *Nonlinear Functional Analysis and its Applications* III, Springer-Verlag, New York, New York, 1985.