**Nonlinear Functional Analysis and Applications**
**Vol. 18, No. 3 (2013), pp. 359-381**

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm


# EXISTENCE OF SOLUTIONS OF NONLINEAR NEUTRAL INTEGRODIFFERENTIAL EQUATIONS OF SOBOLEV TYPE IN BANACH SPACES

**K. Shri Akiladevi[1], K. Balachandran[2] and J. K. Kim[3]**

[1]Department of Mathematics
Bharathiar University, Coimbatore 641 046, India
e-mail: shriakiladevi@gmail.com

[2]Department of Mathematics
Bharathiar University, Coimbatore 641 046, India
e-mail: kb.maths.bu@gmail.com

[3]Department of Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 631 701, Korea
e-mail: jongkyuk@kyungnam.ac.kr

**Abstract.** In this paper we prove the existence of mild solutions of the nonlinear neutral integrodifferential equations of Sobolev type and the nonlinear neutral evolution integrodifferential equations in Banach spaces. The results are obtained by using the Schaefer fixed point theorem. Examples are provided to illustrate the theory.

## 1. Introduction

The theory of integrodifferential equations has emerged as an active area of research due to its applications in many areas of science and engineering such as the study of unsteady aerodynamics and aeroelastic phenomena, viscoelastic panel in supersonic gas flow, fluid dynamics, electrodynamics of complex media, many models of complex growth, neural network modeling, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, theory of population dynamics and nuclear reactors.

[0]Received March 2, 2013. Revised June 15, 2013.
[0]2000 Mathematics Subject Classification: 34G20, 34K30, 34K40, 47J35.
[0]Keywords: Existence of solutions, neutral integrodifferential equation, Sobolev type, neutral evolution integrodifferential equation, semigroup theory, evolution operator, Schaefer fixed point theorem.

Systems with memory have the property that the mathematical physical description of their state at a given point of time includes such states in which the systems have been at earlier points of time. This leads to the inclusion of an integral term in the basic differential equation yielding an integrodifferential equation. Such models which contain past information are called hereditary systems.

In recent years there has been a resurgence of interest in the study of neutral hereditary systems motivated largely by new applications. For instance, a simplified model for compartmental systems with pipes is represented by nonlinear neutral integrodifferential equation. Compartmental models are frequently used in theoretical epidemiology, physiology, population dynamics, analysis of ecosystems. Sobolev type equation considered here occurs in thermodynamics [14], in the flow of fluid through fissured rocks [12], in the shear of second order fluids [29] and in soil mechanics [28]. The neutral evolution integrodifferential equation arises in viscoelasticity, in heat conduction in materials with memory and in control systems with a feedback control governed by a proportional integrodifferential law [1, 20].

The existence of solutions of functional differential equations of Sobolev type has been studied by many authors. Showalter [27] and Brill [13] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. Lightbourne and Rankin [21] discussed the solution of partial functional differential equation of Sobolev type in a Banach space. Balachandran *et al.* [6, 8] studied the existence of solutions for nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in a Banach space. Balachandran *et al.* [9] established the existence of mild solutions for neutral functional integrodifferential equations in Banach spaces. For the controllability of nonlinear systems, one can refer the survey paper [7].

The problem of existence of solutions for partial functional differential equations and partial neutral functional differential equations with delay has been studied by many researchers. Using the method of semigroups, existence and uniqueness of mild, strong and classical solutions of semilinear evolution equations has been discussed in Pazy [25]. Ntouyas *et al.* [22] studied the existence of solutions of the initial value problems for neutral functional differential equations and also the global existence of solutions for functional integrodifferential equations via Leray-Schauder alternative in [23, 24]. Hernandez and Henriquez [17] obtained some existence results for neutral functional differential equations in Banach spaces and in [18], they established the existence of periodic solutions for the same kind of equations by describing them as abstract functional differential equations. For more details regarding the existence of solutions of neutral differential equations the reader can refer the articles [3, 4], the book [16] and the references therein. Balachandran and Sakthivel [5], Dauer and Balachandran [15] investigated the existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces. Hernandez and Balachandran [19] obtained some existence results for abstract degenerate neutral functional differential equations. Annapoorani and Balachandran [2] studied the existence of solutions of partial neutral integrodifferential equations in Banach spaces. Balachandran *et al.* [10] established the existence results for nonlinear abstract neutral integrodifferential equation using the Schaefer fixed point theorem and the existence of mild solutions of neutral evolution integrodifferential equations has been discussed in [11].

This article contains six sections. In Section 2, we introduce some preliminary results on the nonlinear neutral integrodifferential equation of Sobolev type. In Section 3, we discuss the existence of mild solutions by the Schaefer fixed point theorem. The neutral evolution integrodifferential equation is considered in Section 4. The existence of mild solutions is established in Section 5 using the same technique as in Section 3. Finally examples are presented in Section 6 to show the applications of the obtained results.

## 2. Preliminaries

Consider the nonlinear neutral integrodifferential equation of the form

$$\begin{aligned} &\frac{d}{dt}\Big[Ex(t) - g\Big(t, x_t, \int_0^t e_1(t,s,x_s)ds\Big)\Big] \\ &\qquad = Ax(t) + h(t,x_t) + f\Big(t, x_t, \int_0^t e_2(t,s,x_s)ds\Big),\ t \in J = [0,b], \qquad (2.1) \\ &x_0 = \phi \quad \text{on} \quad [-r,0], \end{aligned}$$

where $E$ and $A$ are linear operators with domains contained in a Banach space $X$ and ranges contained in a Banach space $Y$. The functions $g, f : J \times C \times C \to Y$, $e_1, e_2 : J \times J \times C \to C$, $h : J \times C \to Y$ are continuous functions. Here $C = C([-r,0],Y)$ is the Banach space of all continuous functions $\phi : [-r,0] \to Y$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}$. Also, for $x \in C([-r,b],Y)$, we have $x_t \in C$ for $t \in [0,b]$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. The norm of $X$ is denoted by $|.|$ and $Y$ by $\|.\|$.

The operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ satisfy the hypotheses:

(C1) $A$ and $E$ are closed linear operators.
(C2) $D(E) \subset D(A)$ and $E$ is bijective.

(C3) $E^{-1}: Y \to D(E)$ is continuous.

(C4) The resolvent $R(\lambda, AE^{-1})$ is a compact operator for some $\lambda \in \rho(AE^{-1})$, the resolvent set of $(AE^{-1})$.

The hypotheses $(C1)$, $(C2)$ and the closed graph theorem imply the boundedness of the linear operator $AE^{-1}: Y \to Y$.

**Lemma 2.1.** (See [25]) *Let $S(t)$ be a strongly continuous semigroup and $A$ be its infinitesimal generator. If the resolvent $R(\lambda : A)$ of $A$ is compact for some $\lambda \in \rho(A)$ and $S(t)$ is continuous in the uniform operator topology, then $S(t)$ is compact.*

From the above fact, $AE^{-1}$ generates a compact semigroup $T(t),\ t > 0$, on $Y$.

**Definition 2.2.** A solution $x : (-r, b) \to Y,\ b > 0$, is called a mild solution of the Cauchy problem (2.1) if

(i) $x_0 = \phi$;

(ii) the restriction of $x(\cdot)$ to the interval $[0, b)$ is continuous;

(iii) for each $0 \le t < b$, the function $AE^{-1}T(t-s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)$, $s \in [0, t)$, is integrable and

(iv) the integral equation

$$\begin{aligned} x(t) = {} & E^{-1}T(t)\left[E\phi(0) - g(0,\phi,0)\right] + E^{-1}g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right) \\ & + \int_0^t E^{-1}AE^{-1}T(t-s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)ds \\ & + \int_0^t E^{-1}T(t-s)h(s,x_s)ds \\ & + \int_0^t E^{-1}T(t-s)f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)ds,\ t \in J \end{aligned}$$

is satisfied.

Assume that the following hold:

(C5) The strongly continuous semigroup of bounded linear operators $T(t)$ generated by $A$ is compact and there exists a constant $M_1 \ge 1$ such that $\|T(t)\| \le M_1$, for $t > 0$.

(C6) There exist constants $c_1^*, c_2^* > 0$ and $c_3^* \ge 0$ with $c_1 = \max\{c_1^*, c_2^*\}$ such that
$\left\|AE^{-1}T(t-s)g(t,\phi,y)\right\| \le c_1^*\|\phi\| + c_2^*|y| + c_3^*$, for all $t \in J,\ \phi \in C,\ y \in$

$C$ and for $c_3 > 0$, $\left\|AE^{-1}T(t_1-s)g(s,\phi,y)-AE^{-1}T(t_2-s)g(s,\phi,y)\right\| \leq c_3(|t_1-t_2|)$, for $t_1, t_2 \in J$.

(C7) For each $s \in J,\ x \in C$, the function $e_1(.,s,x) : J \to C$ is completely continuous, the function $e_1(.,.,x) : J \times J \to C$ is strongly measurable and $\{t \to e_1(t,s,x_s)\}$ is equicontinuous in $C([0,b],Y)$.

(C8) For each $(t,s) \in J \times J$, the function $e_2(t,s,.) : C \to C$ is continuous and, for each $x \in C,\ e_2(.,.,x) : J \times J \to C$ is strongly measurable.

(C9) For each $t \in J$, the function $h(t,.) : C \to Y$ is continuous and, for each $x \in C$, the function $h(.,x) : J \to Y$ is strongly measurable.

(C10) For each $t \in J$, the function $f(t,.,.) : C \times C \to Y$ is continuous and, for each $(x,y) \in C \times C$, the function $f(.,x,y) : J \to Y$ is strongly measurable.

(C11) There exist integrable functions $p_i : J \to [0,\infty),\ i = 0,1,2$, such that

$$\begin{aligned}\|f(t,x,y)\| &\leq p_1(t)\Omega_1(\|x\|) + p_2(t)\Omega_2(|y|),\ t \in J,\ x,y \in C,\\ \|h(t,x_t)\| &\leq p_0(t)\Omega_0(\|x_t\|),\ 0 \leq t \leq b,\ x_t \in C,\end{aligned}$$

where $\Omega_i : [0,\infty) \to (0,\infty),\ i = 0,1,2$, are continuously differentiable nondecreasing functions, such that $\lim_{s\to\infty} \Omega_0(s) = \infty$, $\Omega_i',\ i = 0,1,2$, (the first derivative of $\Omega_i$) are also nondecreasing and $\Omega_0'(\|E\phi(0)\| \left|E^{-1}\right| M_1) > 0$.

(C12) The function $g : J \times C \times C \to Y$ is completely continuous and, for any bounded set $D$ in $C([-r,b],X)$, the set

$$\left\{t \to g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right) : x \in D\right\}$$

is equicontinuous in $C([0,b],Y)$. There exist $\overline{c}_1, \overline{c}_2 > 0$ and $\overline{c}_3 \geq 0$ with $c_2 = \max\{\overline{c}_1, \overline{c}_2\}$ and $c_2 \in (0, \frac{1}{|E^{-1}|})$ such that $\|g(t,\phi,y)\| \leq \overline{c}_1 \|\phi\| + \overline{c}_2 |y| + \overline{c}_3$, for all $t \in J,\ \phi \in C,\ y \in C$. There exist $m_i : J \times J \to [0,\infty),\ i = 1,2$, differentiable a.e., with respect to the first variable, such that $\int_0^t m_i(t,s)ds,\ \int_0^t \frac{\partial m_i}{\partial t}(t,s)ds$ are bounded on $J$ and $\dfrac{\partial m_1}{\partial t}(t,s) \geq 0$, for a.e., $0 \leq s < t \leq b$. Moreover

$$\begin{aligned}|e_1(t,s,x)| &\leq m_1(t,s)\psi_1(\|x\|),\ 0 \leq s < t \leq b,\ x \in C,\\ |e_2(t,s,x)| &\leq m_2(t,s)\psi_2(\|x\|),\ 0 \leq s < t \leq b,\ x \in C,\end{aligned}$$

where $\psi_i : [0,\infty) \to (0,\infty),\ i = 1,2$ are continuous nondecreasing functions.

(C13) Let $p(t) = \max\left\{a(t), b(t), \frac{M_1|E^{-1}|}{1-|E^{-1}|c_2}q(t), \frac{|E^{-1}|c_1}{1-|E^{-1}|c_2}\right\}$ be such that

$$\int_0^b p(s)ds < \int_a^\infty \{[s + \psi_1(s) + \Omega_0(s) + \Omega_1(s) + \Omega_2(L_0\psi_2(s))] \times \left[1 + \frac{\Omega_1'(s)}{\Omega_0'(s)}\right] + \frac{\psi_2(s)}{\Omega_0'(s)}\Omega_2'(L_0\psi_2(s))\Big\}^{-1} ds,$$

where

$$a(t) = \frac{|E^{-1}|}{1 - |E^{-1}|\, c_2}\left\{c_2 m_1(t,t) + \int_0^t \left(c_2 \frac{\partial m_1}{\partial t}(t,s) + c_1 m_1(t,s)\right) ds\right\},$$

$$b(t) = m_2(t,t) + \int_0^t \left|\frac{\partial m_2}{\partial t}(t,s)\right| ds,$$

$$q(t) = \max\{p_0(t), p_1(t), p_2(t)\}.$$

$L_0$ is a finite bound for $\int_0^t m_2(t,s)ds$ and $a = \Omega_0^{-1}(\Omega_0(\alpha_0) + \Omega_1(\alpha_0) + \Omega_2(0))$ with $\alpha_0 = \frac{1}{(1 - |E^{-1}|\, c_2)}[(\|E\phi(0)\| + \overline{c}_1 \|\phi\|)\, |E^{-1}|\, M_1 + |E^{-1}|\, \overline{c}_3(1 + M_1) + |E^{-1}|\, c_3^* b]$.

**Schaefer Theorem:** (See [26]) Let $E$ be a convex subset of a normed linear space $V$ containing 0. If $H : E \to E$ is a completely continuous operator, then either $H$ has a fixed point or the subset

$$\zeta(H) = \{x \in E : x = \lambda Hx, \text{ for some } \lambda \in (0,1)\}$$

is unbounded.

## 3. Main Results

**Theorem 3.1.** *Assume that the hypotheses* $(C1) - (C13)$ *hold. Then, the problem* (2.1) *admits a mild solution on* $[-r, b]$.

*Proof.* Consider the space $C_b = C([-r, b] : Y)$ endowed with the norm,

$$\|x\|_1 = \sup\{|x(t)| : -r \le t \le b\}.$$

To prove the existence of mild solution of (2.1), let us consider the nonlinear operator equation,

$$x(t) = \lambda F x(t),\ 0 < \lambda < 1 \tag{3.1}$$

where $F : C_b \to C_b$ is given by

$$\begin{aligned}
Fx(t) \;=\;& E^{-1}T(t)[E\phi(0) - g(0,\phi,0)] + E^{-1}g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right) \\
&+ \int_0^t E^{-1}AE^{-1}T(t-s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right) ds \\
&+ \int_0^t E^{-1}T(t-s)h(s,x_s)ds \\
&+ \int_0^t E^{-1}T(t-s)f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right) ds, \quad t \in J.
\end{aligned}$$

Now,

$$\begin{aligned}
|x(t)| \;=\;& |\lambda Fx(t)| \\
\leq\;& \left|E^{-1}\right| M_1 \left[\|E\phi(0)\| + \overline{c}_1 \|\phi\| + \overline{c}_3\right] + \left|E^{-1}\right| c_2 \|x_t\| + \left|E^{-1}\right| \overline{c}_3 \\
&+ \left|E^{-1}\right| c_2 \int_0^t m_1(t,s)\psi_1(\|x_s\|)ds + \left|E^{-1}\right| c_1 \int_0^t \|x_s\| ds \\
&+ \left|E^{-1}\right| c_1 \int_0^t \left(\int_0^s m_1(s,\tau)\psi_1(\|x_\tau\|)d\tau\right) ds + \left|E^{-1}\right| c_3^* b \\
&+ \left|E^{-1}\right| M_1 \int_0^t p_0(s)\Omega_0(\|x_s\|)ds + \left|E^{-1}\right| M_1 \int_0^t p_1(s)\Omega_1(\|x_s\|)ds \\
&+ \left|E^{-1}\right| M_1 \int_0^t p_2(s)\Omega_2\left(\int_0^s m_2(s,\tau)\psi_2(\|x_\tau\|)d\tau\right) ds. \qquad (3.2)
\end{aligned}$$

Let us define the function $\mu$ by $\mu(t) = \sup\{|x(s)| : -r \leq s \leq t\}$, $t \in J$. If $\mu(t) = |x(t^*)|$, for some $t^* \in [0,b]$, then, from (3.2) and from our assumptions, we infer

$$\begin{aligned}
\mu(t) \;\leq\;& \left|E^{-1}\right| M_1 \left[\|E\phi(0)\| + \overline{c}_1 \|\phi\| + \overline{c}_3\right] + \left|E^{-1}\right| c_2 \mu(t) \\
&+ \left|E^{-1}\right| c_2 \int_0^{t^*} m_1(t^*,s)\psi_1(\mu(s))ds + \left|E^{-1}\right| \overline{c}_3 + \left|E^{-1}\right| c_1 \int_0^{t^*} \mu(s)ds \\
&+ \left|E^{-1}\right| c_1 \int_0^{t^*} \left(\int_0^s m_1(s,\tau)\psi_1(\mu(\tau))d\tau\right) ds + \left|E^{-1}\right| c_3^* b \\
&+ \left|E^{-1}\right| M_1 \int_0^{t^*} p_0(s)\Omega_0(\mu(s))ds + \left|E^{-1}\right| M_1 \int_0^{t^*} p_1(s)\Omega_1(\mu(s))ds \\
&+ \left|E^{-1}\right| M_1 \int_0^{t^*} p_2(s)\Omega_2\left(\int_0^s m_2(s,\tau)\psi_2(\mu(\tau))d\tau\right) ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(t) \leq\; & \left|E^{-1}\right| M_1 \left[\|E\phi(0)\| + \overline{c}_1 \|\phi\| + \overline{c}_3\right] + \left|E^{-1}\right| c_2 \mu(t) \\ & + \left|E^{-1}\right| c_2 \int_0^t m_1(t,s)\psi_1(\mu(s))ds + \left|E^{-1}\right| \overline{c}_3 + \left|E^{-1}\right| c_1 \int_0^t \mu(s)ds \\ & + \left|E^{-1}\right| c_1 \int_0^t \left(\int_0^s m_1(s,\tau)\psi_1(\mu(\tau))d\tau\right) ds + \left|E^{-1}\right| c_3^* b \\ & + \left|E^{-1}\right| M_1 \int_0^t p_0(s)\Omega_0(\mu(s))ds + \left|E^{-1}\right| M_1 \int_0^t p_1(s)\Omega_1(\mu(s))ds \\ & + \left|E^{-1}\right| M_1 \int_0^t p_2(s)\Omega_2 \left(\int_0^s m_2(s,\tau)\psi_2(\mu(\tau))d\tau\right) ds.\end{aligned}$$

The above estimate is still valid, if $t^* \in [-r, 0]$, since $\mu(t) = \|\phi\|$ and $M_1 \geq 1$. Hence,

$$\begin{aligned}\mu(t) \leq\; & \frac{1}{1 - |E^{-1}| c_2} \Big\{ \left|E^{-1}\right| M_1 \left[\|E\phi(0)\| + \overline{c}_1 \|\phi\| + \overline{c}_3\right] + \left|E^{-1}\right| \overline{c}_3 \\ & + \left|E^{-1}\right| c_2 \int_0^t m_1(t,s)\psi_1(\mu(s))ds + \left|E^{-1}\right| c_1 \int_0^t \mu(s)ds \\ & + \left|E^{-1}\right| c_1 \int_0^t \left(\int_0^s m_1(s,\tau)\psi_1(\mu(\tau))d\tau\right) ds + \left|E^{-1}\right| c_3^* b \\ & + \left|E^{-1}\right| M_1 \int_0^t p_0(s)\Omega_0(\mu(s))ds + \left|E^{-1}\right| M_1 \int_0^t p_1(s)\Omega_1(\mu(s))ds \\ & + \left|E^{-1}\right| M_1 \int_0^t p_2(s)\Omega_2 \left(\int_0^s m_2(s,\tau)\psi_2(\mu(\tau))d\tau\right) ds \Big\}. \qquad (3.3)\end{aligned}$$

Let us denote the right-hand side of (3.3) as $v(t)$. Then, clearly,

$$\begin{aligned}v(0) =\; & \frac{1}{(1 - |E^{-1}| c_2)} \Big\{ (\|E\phi(0)\| + \overline{c}_1 \|\phi\|) \left|E^{-1}\right| M_1 + \left|E^{-1}\right| \overline{c}_3 (1 + M_1) \\ & + \left|E^{-1}\right| c_3^* b \Big\} \equiv \alpha_0\end{aligned}$$

and $\mu(t) \leq v(t),\ t \in J$. In addition, we have

$$\begin{aligned}v'(t) =\; & \frac{1}{1 - |E^{-1}| c_2} \Big\{ \left|E^{-1}\right| c_2 m_1(t,t)\psi_1(\mu) + \left|E^{-1}\right| c_1 \mu(t) \\ & + \left|E^{-1}\right| c_2 \int_0^t \frac{\partial m_1}{\partial t}(t,s)\psi_1(\mu(s))ds + \left|E^{-1}\right| c_1 \int_0^t m_1(t,s)\psi_1(\mu)ds \\ & + \left|E^{-1}\right| M_1 p_0(t)\Omega_0(\mu(t)) + \left|E^{-1}\right| M_1 p_1(t)\Omega_1(\mu(t)) \\ & + \left|E^{-1}\right| M_1 p_2(t)\Omega_2 \left(\int_0^t m_2(t,s)\psi_2(\mu(s))ds\right) \Big\} \geq 0, \quad t \in J.\end{aligned}$$

Next, let $\omega(t)$ be, such that

$$\Omega_0(\omega) = \Omega_0(v) + \Omega_1(v) + \Omega_2\left(\int_0^t m_2(t,s)\psi_2(v)ds\right).$$

We have $\omega \geq v$, and by differentiation, taking into account $(C12)$, we get,

$$\begin{aligned}
\Omega_0'(\omega)\omega'(t) &= (\Omega_0'(v) + \Omega_1'(v))v' + \Omega_2'\left(\int_0^t m_2(t,s)\psi_2(v)ds\right) \\
&\quad \times\left\{m_2(t,t)\psi_2(v) + \int_0^t \frac{\partial m_2}{\partial t}(t,s)\psi_2(v)ds\right\} \\
&\leq \frac{\Omega_0'(\omega)+\Omega_1'(\omega)}{1-|E^{-1}|\,c_2}\Bigg\{\left|E^{-1}\right| c_2 m_1(t,t)\psi_1(\omega) \\
&\quad +\left|E^{-1}\right| c_1\omega + \left|E^{-1}\right| c_2\int_0^t \frac{\partial m_1}{\partial t}(t,s)\psi_1(\omega)ds \\
&\quad +\left|E^{-1}\right| c_1 \int_0^t m_1(t,s)\psi_1(\omega)ds + \left|E^{-1}\right| M_1 q(t)\,(\Omega_0(\omega) \\
&\quad +\Omega_1(\omega) + \Omega_2\left(\int_0^t m_2(t,s)\psi_2(\omega)ds\right)\Big)\Bigg\} \\
&\quad +\left\{m_2(t,t)+\int_0^t\left|\frac{\partial m_2}{\partial t}(t,s)\right|ds\right\}\psi_2(\omega) \\
&\quad \times\Omega_2'\left(\psi_2(\omega)\int_0^t m_2(t,s)ds\right). \qquad (3.4)
\end{aligned}$$

Moreover, by our assumptions on $\Omega_0'$, we have

$$\Omega_0'(\omega) \geq \Omega_0'(v) \geq \Omega_0'(\alpha_0) \geq \Omega_0'(\|E\phi(0)\|\left|E^{-1}\right| M_1) > 0.$$

Therefore, inequality (3.4) implies that

$$\begin{aligned}
\omega'(t) &\leq \Bigg[\frac{\left|E^{-1}\right|\psi_1(\omega)}{1-|E^{-1}|\,c_2}\left\{c_2 m_1(t,t) + \int_0^t\left(c_2\frac{\partial m_1}{\partial t}(t,s) + c_1 m_1(t,s)\right)ds\right\} \\
&\quad +\frac{\left|E^{-1}\right| M_1 q(t)}{1-|E^{-1}|\,c_2}\left(\Omega_0(\omega)+\Omega_1(\omega)+\Omega_2\left(\int_0^t m_2(t,s)\psi_2(\omega)ds\right)\right) \\
&\quad +\frac{\left|E^{-1}\right| c_1\omega}{1-|E^{-1}|\,c_2}\Bigg] \times \left(1+\frac{\Omega_1'(\omega)}{\Omega_0'(\omega)}\right) \\
&\quad +\frac{\psi_2(\omega)}{\Omega_0'(\omega)}\left(m_2(t,t)+\int_0^t\left|\frac{\partial m_2}{\partial t}(t,s)\right|ds\right)\Omega_2'\left(\psi_2(\omega)\int_0^t m_2(t,s)ds\right)
\end{aligned}$$

or, using the notation in (C13), one finds that

$$\begin{aligned}
\omega'(t) \;\leq\; & \left\{ \frac{M_1 \left|E^{-1}\right| q(t)}{1-\left|E^{-1}\right| c_2} \left( \Omega_0(\omega) + \Omega_1(\omega) + \Omega_2 \left( \int_0^t m_2(t,s)\psi_2(\omega) ds \right) \right) \right. \\
& \left. + a(t)\psi_1(\omega) + \frac{\left|E^{-1}\right| c_1 \omega}{1-\left|E^{-1}\right| c_2} \right\} \left( 1 + \frac{\Omega_1'(\omega)}{\Omega_0'(\omega)} \right) \\
& + \frac{\psi_2(\omega)}{\Omega_0'(\omega)} b(t) \Omega_2' \left( L_0 \psi_2(\omega) \right) \\
\leq\; & p(t) \left\{ \left[ \omega + \psi_1(\omega) + \Omega_0(\omega) + \Omega_1(\omega) + \Omega_2(L_0\psi_2(\omega)) \right] \left[ 1 + \frac{\Omega_1'(\omega)}{\Omega_0'(\omega)} \right] \right. \\
& \left. + \frac{\psi_2(\omega)}{\Omega_0'(\omega)} \Omega_2'(L_0\psi_2(\omega)) \right\}.
\end{aligned}$$

Thus, by (C13), for $0 \leq t \leq b$,

$$\begin{aligned}
& \int_{\omega(0)}^{\omega(t)} \left\{ [s+\psi_1(s)+\Omega_0(s)+\Omega_1(s)+\Omega_2(L_0\psi_2(s))] \left[ 1+\frac{\Omega_1'(s)}{\Omega_0'(s)} \right] + \frac{\psi_2(s)}{\Omega_0'(s)} \Omega_2'(L_0\psi_2(s)) \right\}^{-1} ds \\
& \leq \int_0^b p(s) ds \\
& < \int_a^\infty \left\{ [s+\psi_1(s)+\Omega_0(s)+\Omega_1(s)+\Omega_2(L_0\psi_2(s))] \left[ 1+\frac{\Omega_1'(s)}{\Omega_0'(s)} \right] + \frac{\psi_2(s)}{\Omega_0'(s)} \Omega_2'(L_0\psi_2(s)) \right\}^{-1} ds.
\end{aligned}$$

This implies that $\omega(t)$ must be bounded by some positive constant $L$ on $[0, b]$. Consequently, $\|x\|_1 \leq L$.
We shall now prove that the operator $F : C_b \to C_b$ defined by

$$\begin{aligned}
(Fx)(t) &= \phi(t),\ t \in [-r, 0], \\
(Fx)(t) &= E^{-1}T(t)[E\phi(0) - g(0,\phi,0)] + E^{-1} g\Big(t, x_t, \int_0^t e_1(t,s,x_s) ds\Big) \\
&\quad + \int_0^t E^{-1}AE^{-1}T(t-s) g\Big(s, x_s, \int_0^s e_1(s,\tau,x_\tau) d\tau\Big) ds \\
&\quad + \int_0^t E^{-1}T(t-s) h(s,x_s) ds \\
&\quad + \int_0^t E^{-1}T(t-s) f\Big(s, x_s, \int_0^s e_2(s,\tau,x_\tau) d\tau\Big) ds,
\end{aligned}$$

is a completely continuous operator for all $t \in J$. Let $B_q = \{x \in C_b : \|x\|_1 \leq q\}$ for some $q \geq 1$. We first show that $F$ maps $B_q$ into an equicontinuous family. Let $x \in B_q$ and $t_1, t_2 \in [0, b]$. Then, if $0 \leq t_1 < t_2 \leq b$, (the other cases $t_1 < t_2 < 0$ and $t_1 < 0 < t_2$ may be treated similarly), we have

$$
\begin{aligned}
&\|(Fx)(t_1)-(Fx)(t_2)\| \\
&\leq \left|E^{-1}\right| \|[T(t_1)-T(t_2)][E\phi(0)-g(0,\phi,0)]\| \\
&\quad +\left|E^{-1}\right| \left\| g\left(t_1, x_{t_1}, \int_0^{t_1} e_1(t_1,s,x_s)ds\right) - g\left(t_2, x_{t_2}, \int_0^{t_2} e_1\left(t_2,s,x_s\right)ds\right)\right\| \\
&\quad + \int_0^{t_1} \left|E^{-1}\right| \left\| AE^{-1}T(t_1-s) g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)\right. \\
&\quad \left. -AE^{-1}T(t_2-s) g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)\right\| ds \\
&\quad + \int_{t_1}^{t_2} \left|E^{-1}\right| \left\| AE^{-1}T(t_2-s) g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)\right\| ds \\
&\quad + \int_0^{t_1} \left|E^{-1}\right| \|T(t_1-s)-T(t_2-s)\| \, \|h(s,x_s)\| \, ds \\
&\quad + \int_{t_1}^{t_2} \left|E^{-1}\right| \|T(t_2-s)\| \, \|h(s,x_s)\| \, ds \\
&\quad + \int_0^{t_1} \left|E^{-1}\right| \|[T(t_1-s)-T(t_2-s)]\| \left\| f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)\right\| ds \\
&\quad + \int_{t_1}^{t_2} \left|E^{-1}\right| \|T(t_2-s)\| \left\| f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)\right\| ds \\
&\leq \left|E^{-1}\right| \|[T(t_1)-T(t_2)][E\phi(0)-g(0,\phi,0)]\| \\
&\quad +\left|E^{-1}\right| \left\| g\left(t_1, x_{t_1}, \int_0^{t_1} e_1\left(t_1,s,x_s\right)ds\right) - g\left(t_2, x_{t_2}, \int_0^{t_2} e_1\left(t_2,s,x_s\right)ds\right)\right\| \\
&\quad +\left|E^{-1}\right| c_3 \int_0^{t_1} |t_1-t_2| \, ds + \left|E^{-1}\right| \int_{t_1}^{t_2} \rho_1(s) ds \\
&\quad +\left|E^{-1}\right| \int_0^{t_1} \|T(t_1-s)-T(t_2-s)\| \beta(s) ds + \left|E^{-1}\right| \int_{t_1}^{t_2} \|T(t_2-s)\| \, \beta(s) ds \\
&\quad +\left|E^{-1}\right| \int_0^{t_1} \|T(t_1-s)-T(t_2-s)\| \, \gamma(s) ds \\
&\quad +\left|E^{-1}\right| \int_{t_1}^{t_2} \|T(t_2-s)\| \, \gamma(s) ds,
\end{aligned}
$$

where
$$
\begin{aligned}
\rho_1(s) &= c_1^* \|x_s\| + c_2^* \int_0^s m_1(s,\tau)\psi_1(\|x_\tau\|)d\tau + c_3^*, \\
\beta(s) &= p_0(s)\Omega_0(\|x_s\|), \\
\gamma(s) &= p_1(s)\Omega_1(\|x_s\|) + p_2(s)\Omega_2\Big(\int_0^s m_2(s,\tau)\psi_2(\|x_\tau\|)d\tau\Big).
\end{aligned}
$$

Since $T(t),\ t>0$, is compact and continuous in the uniform operator topology and by the assumptions on $m_i,\psi_i,p_i,\Omega_i$ and the complete continuity of $g$ imply that the right-hand side of the above inequality goes to zero as $t_2-t_1\to 0$ independent of $x$ in $B_q$. Therefore, the family $\{FB_q\}$ is equicontinuous. Moreover, for $x$ in $B_q$,

$$\begin{aligned}\|(Fx)(t)\| \le\ & \left|E^{-1}\right|\|T(t)\|\|E\phi(0)-g(0,\phi,0)\|+\left|E^{-1}\right|\rho_2(t)+\int_0^t\left|E^{-1}\right|\rho_1(s)ds\\ &+\int_0^t\left|E^{-1}\right|\|T(t-s)\|\beta(s)ds+\int_0^t\left|E^{-1}\right|\|T(t-s)\|\gamma(s)ds\\ \le\ & \left|E^{-1}\right|M_1[\|E\phi(0)\|+\bar{c}_1\|\phi\|+\bar{c}_3]+\left|E^{-1}\right|\rho_2(t)\\ &+\left|E^{-1}\right|\int_0^t\rho_1(s)ds+\left|E^{-1}\right|M_1\int_0^t\beta(s)ds+\left|E^{-1}\right|M_1\int_0^t\gamma(s)ds\end{aligned}$$

where $\rho_2(t)=\bar{c}_1\|x_t\|+\bar{c}_2\int_0^t m_1(t,s)\psi_1(\|x_s\|)ds+\bar{c}_3$. Hence, $FB_q$ is uniformly bounded, and consequently, according to Arzela-Ascoli's Theorem, it suffices to show that $FB_q$ is precompact in $X$. Next, for a fixed $t\in(0,b]$ and $\epsilon$, such that $0<\epsilon<t$, we define, for $x\in B_q$,

$$\begin{aligned}(F_\epsilon x)(t) =\ & E^{-1}T(t)[E\phi(0)-g(0,\phi,0)]+E^{-1}g\Big(t,x_t,\int_0^t e_1(t,s,x_s)ds\Big)\\ &+\int_0^{t-\epsilon}E^{-1}AE^{-1}T(t-s)g\Big(s,x_s,\int_0^s e_1(s,\tau,x_\tau)d\tau\Big)ds\\ &+\int_0^{t-\epsilon}E^{-1}T(t-s)h(s,x_s)ds\\ &+\int_0^{t-\epsilon}E^{-1}T(t-s)f\Big(s,x_s,\int_0^s e_2(s,\tau,x_\tau)d\tau\Big)ds.\end{aligned}$$

As $T(t)$ is a compact operator, we see that $F_\epsilon B_q$ is precompact in $X$, for every $\epsilon$, such that $0<\epsilon<t$. On the other hand, we have

$$\begin{aligned}\|(Fx)(t)-(F_\epsilon x)(t)\| \le\ & \int_{t-\epsilon}^t\left|E^{-1}\right|\left\|AE^{-1}T(t-s)g\Big(s,x_s,\int_0^s e_1(s,\tau,x_\tau)d\tau\Big)\right\|ds\\ &+\int_{t-\epsilon}^t\left|E^{-1}\right|\|T(t-s)\|\|h(s,x_s)\|ds\\ &+\int_{t-\epsilon}^t\left|E^{-1}\right|\|T(t-s)\|\left\|f\Big(s,x_s,\int_0^s e_2(s,\tau,x_\tau)d\tau\Big)\right\|ds\\ \le\ & \left|E^{-1}\right|\int_{t-\epsilon}^t\rho_1(s)ds+\left|E^{-1}\right|M_1\int_{t-\epsilon}^t\beta(s)ds\\ &+\left|E^{-1}\right|M_1\int_{t-\epsilon}^t\gamma(s)ds.\end{aligned}$$

This shows that $FB_q$ may be arbitrarily approached by precompact sets and therefore, it is a precompact subset of X.

Finally, we want to prove that $F : C_b \to C_b$ is continuous. Let $\{x_n\}_{n=0}^{\infty} \subseteq C_b$ with $x_n \to x$ in $C_b$. Then, there is an integer $r$ such that $\|x_n(t)\| \le r$ for all $n$ and $t \in J$. So, $x_n \in B_r$ and $x \in B_r$. Moreover, by virtue of (C6), (C11) and (C12), we obtain,

$$\begin{aligned}&\left\|g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) - g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right)\right\| \\ &\le 2\left\{\overline{c}_1 r + \overline{c}_2\left(\int_0^t m_1(t,s)\psi_1(r)ds\right) + \overline{c}_3\right\},\end{aligned}$$

$$\|h(t, x_{n_t}) - h(t, x_t)\| \le 2p_0(t)\Omega_0(r),$$

$$\begin{aligned}&\left\|f\left(t, x_{n_t}, \int_0^t e_2(t,s,x_{n_s})ds\right) - f\left(t, x_t, \int_0^t e_2(t,s,x_s)ds\right)\right\| \\ &\le 2\left\{p_1(t)\Omega_1(r) + p_2(t)\Omega_2\left(L_0\psi_2(r)\right)\right\},\end{aligned}$$

$$\begin{aligned}&\Bigg\|AE^{-1}T(t-s)g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) \\ &\qquad -AE^{-1}T(t-s)g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right)\Bigg\| \\ &\le 2\left\{c_1^* r + c_2^*\left(\int_0^t m_1(t,s)\psi_1(r)ds\right) + c_3^*\right\}.\end{aligned}$$

As we know by assumptions (C6)-(C10) and (C12) that

$$g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) \to g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right),$$

$$h(t, x_{n_t}) \to h(t, x_t),$$

$$f\left(t, x_{n_t}, \int_0^t e_2(t,s,x_{n_s})ds\right) \to f\left(t, x_t, \int_0^t e_2(t,s,x_s)ds\right),$$

$$AE^{-1}T(t-s)g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) \to AE^{-1}T(t-s)g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right),$$

as $n \to \infty$, for each $t \in J$. Also, we have

$$
\begin{aligned}
&\|Fx_n - Fx\| \\
&\leq \sup_{t\in J}\Bigg\{\Bigg\|E^{-1}\left[g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) - g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right)\right]\Bigg\| \\
&\quad + \Bigg\|\int_0^t E^{-1}\Bigg[AE^{-1}T(t-s)g\left(s, x_{n_s}, \int_0^s e_1(s,\tau,x_{n_\tau})d\tau\right) \\
&\quad -AE^{-1}T(t-s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)\Bigg]ds\Bigg\| \\
&\quad + \Bigg\|\int_0^t E^{-1}T(t-s)[h(s,x_{n_s}) - h(s,x_s)]ds\Bigg\| \\
&\quad + \Bigg\|\int_0^t E^{-1}T(t-s)\Bigg[f\left(s, x_{n_s}, \int_0^s e_2(s,\tau,x_{n_\tau})d\tau\right) \\
&\quad -f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)\Bigg]ds\Bigg\|\Bigg\} \\
&\leq \left|E^{-1}\right| \Bigg\|g\left(t, x_{n_t}, \int_0^t e_1(t,s,x_{n_s})ds\right) - g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right)\Bigg\| \\
&\quad + \int_0^b \left|E^{-1}\right| \Bigg\|AE^{-1}T(t-s)g\left(s, x_{n_s}, \int_0^s e_1(s,\tau,x_{n_\tau})d\tau\right) \\
&\quad -AE^{-1}T(t-s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)\Bigg\| ds \\
&\quad + \int_0^b \left|E^{-1}\right| \|T(t-s)\| \|h(s,x_{n_s}) - h(s,x_s)\| ds \\
&\quad + \int_0^b \left|E^{-1}\right| \|T(t-s)\| \Bigg\|f\left(s, x_{n_s}, \int_0^s e_2(s,\tau,x_{n_\tau})d\tau\right) \\
&\quad -f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)\Bigg\| ds.
\end{aligned}
$$

Taking limit as $n \to \infty$, we get from the dominated convergence theorem that the right-hand side of the above inequality tends to zero. Therefore, $F$ is continuous, and consequently, $F$ is a completely continuous operator.

Finally the set $\zeta(F) = \{x \in C_b : x = \lambda Fx,\ \lambda \in (0,1)\}$ is bounded. Consequently, by Schaefer's theorem the operator $F$ has a fixed point in $C_b$. Thus, the problem (2.1) has at least one mild solution on $[-r, b]$. □

## 4. Neutral Evolution Integrodifferential Equation

Consider the neutral evolution integrodifferential equation of the form

$$\begin{aligned}
&\frac{d}{dt}\Big[x(t) - g\Big(t, x_t, \int_0^t e_1(t,s,x_s)ds\Big)\Big] \\
&\qquad = A(t)x(t) + h(t,x_t) + f\Big(t, x_t, \int_0^t e_2(t,s,x_s)ds\Big),\ t \in J = [0,b], \quad (4.1) \\
&x_0 = \phi \quad \text{on} \quad [-r,0],
\end{aligned}$$

where $A(t)$ is a closed, densely defined linear operator in a Banach space $X$ with norm $\|.\|$, the functions $g, f : J \times C \times X \to X,\ e_1, e_2 : J \times J \times C \to X$ and $h : J \times C \to X$ are known continuous functions. Here $C = C([-r,0],X)$ is the Banach space of all continuous functions $\phi : [-r,0] \to X$ endowed with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}$. Also, for $x \in C([-r,b],X)$, we have $x_t \in C$ for $t \in [0,b]$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$ and $A(t) : D(A(t)) \to X$ is the infinitesimal generator of an analytic semigroup.

We need the following hypotheses:

(H1) The operator $U(t,s)$ is compact such that for $0 \le s \le t \le b$, $\|U(t,s)\| \le M_1$, for some constant $M_1 \ge 1$.

(H2) There exists a constant $M_2 > 0$ such that $\|A(t)g(t,x,y)\| \le M_2$ for $t \in J,\ x \in C,\ y \in X$.

(H3) The function $g : J \times C \times X \to X$ is completely continuous and, for any bounded set $D$ in $C([-r,b],X)$, the set

$$\left\{t \to g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right) : x \in D\right\}$$

is equicontinuous in $C([0,b],X)$. There exist $c_1 \in (0,1),\ c_2 > 0$ and $c_3 \ge 0$ such that $\|g(t,\phi,y)\| \le c_1\|\phi\| + c_2|y| + c_3$, for all $t \in J$, $\phi \in C,\ y \in X$.

(H4) For each $s \in J,\ x \in C$, the function $e_1(.,s,x) : J \to X$ is completely continuous, the function $e_1(.,.,x) : J \times J \to X$ is strongly measurable and $\{t \to e_1(t,s,x_s)\}$ is equicontinuous in $C([0,b],X)$.

(H5) For each $(t,s) \in J \times J$, the function $e_2(t,s,.) : C \to X$ is continuous and, for each $x \in C,\ e_2(.,.,x) : J \times J \to X$ is strongly measurable.

(H6) For each $t \in J$, $h(t,.) : C \to X$ is continuous and, for each $x \in C$, the function $h(.,x) : J \to X$ is strongly measurable.

(H7) For each $t \in J,\ f(t,.,.) : C \times X \to X$ is continuous and, for each $(x,y) \in C \times X$, the function $f(.,x,y) : J \to X$ is strongly measurable.

(H8) There exist integrable functions $p_i : J \to [0,\infty),\ i = 0,1,2$, such that

$$\begin{aligned}\|f(t,x,y)\| &\leq p_1(t)\Omega_1(\|x\|) + p_2(t)\Omega_2(|y|),\ t \in J,\ x \in C,\ y \in X,\\ \|h(t,x_t)\| &\leq p_0(t)\Omega_0(\|x_t\|),\ 0 \leq t \leq b,\quad x_t \in C,\end{aligned}$$

where $\Omega_i : [0,\infty) \to (0,\infty),\ i = 0,1,2$ are continuously differentiable nondecreasing functions, such that $\lim_{s\to\infty} \Omega_0(s) = \infty,\ \Omega_i',\ i = 0,1,2$ (the first derivative of $\Omega_i$) are also nondecreasing and $\Omega_0'(M_1\|\phi\|) > 0$.

(H9) There exist $m_i : J \times J \to [0,\infty), i = 1,2$, differentiable a.e., with respect to the first variable, such that $\int_0^t m_i(t,s)ds$, $\int_0^t \frac{\partial m_i}{\partial t}(t,s)ds$ are bounded on $J$ and $\frac{\partial m_1}{\partial t}(t,s) \geq 0$, for a.e., $0 \leq s < t \leq b$. Moreover

$$\begin{aligned}|e_1(t,s,x)| &\leq m_1(t,s)\psi_1(\|x\|),\ 0 \leq s < t \leq b,\ x \in C,\\ |e_2(t,s,x)| &\leq m_2(t,s)\psi_2(\|x\|),\ 0 \leq s < t \leq b,\ x \in C,\end{aligned}$$

where $\psi_i : [0,\infty) \to (0,\infty),\ i = 1,2$ are continuous nondecreasing functions.

(H10) Let $p(t) = \max\left\{a(t), b(t), \frac{M_1}{1-c_1}q(t)\right\}$ be such that

$$\begin{aligned}\int_0^b p(s)ds < \int_a^\infty &\Big\{[\psi_1(s) + \Omega_0(s) + \Omega_1(s) + \Omega_2(L_0\psi_2(s))]\\ &\times\left[1 + \frac{\Omega_1'(s)}{\Omega_0'(s)}\right] + \frac{\psi_2(s)}{\Omega_0'(s)}\Omega_2'(L_0\psi_2(s))\Big\}^{-1} ds,\end{aligned}$$

where

$$\begin{aligned}a(t) &= \frac{c_2}{1-c_1}\left\{m_1(t,t) + \int_0^t \frac{\partial m_1}{\partial t}(t,s)ds\right\},\\ b(t) &= m_2(t,t) + \int_0^t \left|\frac{\partial m_2}{\partial t}(t,s)\right| ds,\\ q(t) &= \max\{p_0(t), p_1(t), p_2(t)\}.\end{aligned}$$

$L_0$ is a finite bound for $\int_0^t m_2(t,s)ds$ and $a = \Omega_0^{-1}(\Omega_0(\alpha_0) + \Omega_1(\alpha_0) + \Omega_2(0))$ with $\alpha_0 = \dfrac{1}{1-c_1}[(1+c_1)M_1\|\phi\| + c_3(1+M_1) + M_1M_2b]$ .

**Definition 4.1.** By a mild solution of the Cauchy problem (4.1), we mean a function $x : (-r,b) \to X,\ b > 0$, such that

(i) $x_0 = \phi$;
(ii) the restriction of $x(\cdot)$ to the interval $[0,b)$ is continuous;
(iii) for each $t \in [0,b)$, the function $U(t,s)A(s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)$, $s \in [0,t)$, is integrable and

(iv) the integral equation

$$\begin{aligned} x(t) &= U(t,0)[\phi(0) - g(0,\phi,0)] + g\Big(t, x_t, \int_0^t e_1(t,s,x_s)ds\Big) \\ &\quad + \int_0^t U(t,s)A(s)g\Big(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\Big)ds + \int_0^t U(t,s)h(s,x_s)ds \\ &\quad + \int_0^t U(t,s)f\Big(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\Big)ds,\ t \in J \end{aligned} \tag{4.2}$$

is satisfied.

## 5. Existence Results

**Theorem 5.1.** *Assume that the hypotheses* $(H1) - (H10)$ *hold. Then, the problem* (4.1) *admits a mild solution on* $[-r, b]$.

*Proof.* Let $C_b = C([-r,b] : X)$ endowed with the norm,

$$\|x\|_1 = \sup\{|x(t)| : -r \le t \le b\}.$$

And consider the nonlinear operator equation,

$$x(t) = \lambda F x(t),\ 0 < \lambda < 1, \tag{5.1}$$

where $F : C_b \to C_b$ is given by

$$\begin{aligned} Fx(t) &= U(t,0)[\phi(0) - g(0,\phi,0)] + g\left(t, x_t, \int_0^t e_1(t,s,x_s)ds\right) \\ &\quad + \int_0^t U(t,s)A(s)g\left(s, x_s, \int_0^s e_1(s,\tau,x_\tau)d\tau\right)ds \\ &\quad + \int_0^t U(t,s)h(s,x_s)ds \\ &\quad + \int_0^t U(t,s)f\left(s, x_s, \int_0^s e_2(s,\tau,x_\tau)d\tau\right)ds,\ t \in J. \end{aligned}$$

As in Theorem 3.1, it is easy to prove that $\|x\|_1 \le L$. Next, let us define

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & -r \le t \le 0 \\ U(t,0)\phi(0), & 0 \le t \le b. \end{cases}$$

Observe that, if $\phi \in C$, then, $\widehat{\phi} \in C_b = C([-r, b], X)$. Also, define $x(t) = y(t) + \widehat{\phi}(t),\ -r \le t \le b$ where $y(t)$ verifies

$$\begin{aligned} y(t) &= 0, -r \le t \le 0, \\ y(t) &= -U(t,0)g(0,\phi,0) + \widehat{g}(t) + \int_0^t U(t,s)A(s)\widehat{g}(s)ds \\ &\quad + \int_0^t U(t,s)h(s, y_s + \widehat{\phi}_s)ds \\ &\quad + \int_0^t U(t,s)f\Big(s, y_s + \widehat{\phi}_s, \int_0^s e_2(s,\tau, y_\tau + \widehat{\phi}_\tau)d\tau\Big)ds, \quad 0 \le t \le b, \end{aligned}$$

where $\widehat{g}(t) = g\bigg(t, y_t + \widehat{\phi}_t, \displaystyle\int_0^t e_1(t,s,y_s + \widehat{\phi}_s)ds\bigg)$.

Let $C_b^0 = \big\{y \in C_b : y(t) = 0, \text{ for } -r \le t \le 0\big\}$. We shall consider the operator $F : C_b^0 \to C_b^0$ by

$$\begin{aligned} (Fy)(t) &= 0, -r \le t \le 0, \\ (Fy)(t) &= -U(t,0)g(0,\phi,0) + \widehat{g}(t) + \int_0^t U(t,s)A(s)\widehat{g}(s)ds \\ &\quad + \int_0^t U(t,s)h(s, y_s + \widehat{\phi}_s)ds \\ &\quad + \int_0^t U(t,s)f\Big(s, y_s + \widehat{\phi}_s, \int_0^s e_2(s,\tau, y_\tau + \widehat{\phi}_\tau)d\tau\Big)ds, \quad 0 \le t \le b. \end{aligned}$$

By similar argument as above, one can prove that $F$ is completely continuous. Hence, the set $\zeta(F) = \big\{y \in C_b^0 : y = \lambda F y, \lambda \in (0,1)\big\}$ is bounded, since for every solution $y$ in $\zeta(F)$, the function $x = y + \widehat{\phi}$ is a mild solution of (5.1), for which we have proved that $\|x\|_1 \le L$ and hence

$$\|y\|_1 \le L + \left\|\widehat{\phi}\right\|.$$

Consequently, by Schaefer's theorem the operator $F$ has a fixed point in $C_b^0$. This means that the problem (4.1) has a mild solution. $\square$

## 6. Examples

**Example 6.1.** Consider the following integrodifferential equation of the form

$$\frac{\partial}{\partial t}\left[z(x,t) - z_{xx}(x,t) - p\left(t, z(x,t-r)\right)\right] \tag{6.1}$$
$$= \frac{\partial^2}{\partial x^2} z(x,t) + q\left(t, z(x,t-r), \int_0^t k(t,s,z(x,s-r))ds\right),\ 0 \le x \le \pi,\ t \in J,$$
$$z(0,t) = z(\pi,t) = 0, \quad t \ge 0,$$
$$z(x,t) = \phi(x,t), \quad -r \le t \le 0,$$

where $\phi$ is continuous and $p, q$ and $k$ are continuous functions and satisfy certain smoothness conditions. Take $X = Y = L^2[0,\pi]$ and let $g(t,\omega_t)x = p\left(t, \omega(t-x)\right), e_2(t,s,w_s) = k(t,s,w(s-x))$ and $f(t,\omega_t,v)x = q(t,\omega(t-x),v(x))$, $0 \le x \le \pi$.

Define the operators $A : D(A) \subset X \to Y$ and $E : D(E) \subset X \to Y$ by

$$A\omega = \omega^{''} \text{ and } E\omega = \omega - \omega^{''},$$

where each domain $D(A)$ and $D(E)$ is given by

$$\left\{\omega \in X :\ \omega,\ \omega^{'} \text{ are absolutely continuous, } \omega^{''} \in X,\ \omega(0) = \omega(\pi) = 0\right\}.$$

Then $A$ and $E$ can be written respectively as

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega \in D(A),$$
$$E\omega = \sum_{n=1}^{\infty} (1+n^2)(\omega,\omega_n)\omega_n, \quad \omega \in D(E),$$

where $\omega_n(x) = \sqrt{\frac{2}{\pi}}\ \sin nx,\ n = 1,2,3,...$ is the orthogonal set of eigenvectors of $A$. Furthermore, for $\omega \in X$ we have

$$E^{-1}\omega = \sum_{n=1}^{\infty} \frac{1}{1+n^2}(\omega,\omega_n)\omega_n,$$
$$AE^{-1}\omega = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(\omega,\omega_n)\omega_n,$$
$$T(t)\omega = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right)(\omega,\omega_n)\omega_n.$$

It is easy to see that $AE^{-1}$ generates a strongly continuous semigroup $T(t)$ on $Y$ and $T(t)$ is compact such that $\|T(t)\| \leq N$ and $\left\|AE^{-1}T(t)g(t,\omega)\right\| \leq N_1$ for each $t > 0$.

The function $p : J \times [0,\pi] \to [0,\pi]$ is completely continuous and there exists a constant $n_1 > 0$ such that

$$\|p(t, \omega(t-x))\| \leq n_1.$$

Also, the functions $k : J \times J \times [0,\pi] \to [0,\pi]$ and $q : J \times [0,\pi] \times [0,\pi] \to [0,\pi]$ are measurable and there exist integrable functions $l_1, l_2 : J \to [0,\infty)$ and $l_3 : J \times J \to [0,\infty)$ such that

$$\begin{aligned} \|q(t,v,\omega)\| &\leq l_1(t)\Omega_1(\|v\|) + l_2(t)\Omega_2(\|\omega\|), \\ |k(t,s,\omega)| &\leq l_3(t,s)\psi_1(\|\omega\|), \end{aligned}$$

where $\Omega_i : [0,\infty) \to (0,\infty)$, $i = 1,2$, are continuously differentiable nondecreasing functions, $\psi_1 : [0,\infty) \to (0,\infty)$ are continuous nondecreasing functions.

Let $p(t) = \max\left\{b(t), \left|E^{-1}\right| N q(t)\right\}$ be such that

$$\int_0^b p(s)ds < \int_a^\infty \left\{ [\Omega_1(s) + \Omega_2(L_0\psi_1(s))] \frac{\Omega_1'(s)}{\Omega_0'(s)} + \frac{\psi_1(s)}{\Omega_0'(s)}\Omega_2'(L_0\psi_1(s)) \right\}^{-1} ds,$$

where

$$\begin{aligned} b(t) &= \left\{ l_3(t,t) + \int_0^t \left| \frac{\partial l_3(t,s)}{\partial t} \right| ds \right\}, \\ q(t) &= \max\{l_1(t), l_2(t)\}, \end{aligned}$$

$L_0$ is a finite bound for $\int_0^t l_3(t,s)ds$ and $a = \Omega_0^{-1}(\Omega_1(\alpha_0) + \Omega_2(0))$ with $\alpha_0 = \left|E^{-1}\right| N[\|E\phi(0)\| + n_1] + \left|E^{-1}\right| n_1 + \left|E^{-1}\right| N_1 b$.

Further, all the conditions stated in the Theorem 3.1 are satisfied. Hence the equation (6.1) has a mild solution on $[0,b]$.

**Example 6.2.** Consider the following partial neutral integrodifferential equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left[ z(t,x) + \int_{-h}^t a_1(t,x,s-t) P_1(z(s,x)) ds + \int_0^t \int_{-h}^s k(s-\tau) P_2(z(\tau,x)) d\tau ds \right] \\ &= \frac{\partial^2}{\partial x^2} z(t,x) + a(t,x)z(t,x) + k_0(x)z(t,x) + \int_{-h}^t a_2(t,x,s-t) Q_1(z(s,x)) ds \\ &\quad + \int_0^t \int_{-h}^s k(s-\tau) Q_2(z(\tau,x)) d\tau ds,\ 0 \leq x \leq \pi,\ t \in J = [0,b], \qquad (6.2) \\ z(t,0) &= z(t,\pi) = 0, \quad t \geq 0, \\ z(t,x) &= \phi(t,x), \quad -h \leq t \leq 0, \quad 0 \leq x \leq \pi. \end{aligned}$$

where $a(t,x)$ is continuous on $0 \leq x \leq \pi,\ 0 \leq t \leq b$, $h$ is a positive constant and $\phi$, a continuous function.

Let us take $X = L^2[0,\pi]$ with the norm $|.|_{L^2}$. Put $x_t(s) = z_t(s,x)$,

$$\begin{aligned} f\left(t, x_t, B_2(x_t)\right) &= \int_{-h}^{t} a_2\left(t, x, s-t\right) Q_1\left(z(s,x)\right) ds + B_2(x_t), \\ g\left(t, x_t, B_1(x_t)\right) &= \int_{-h}^{t} a_1\left(t, x, s-t\right) P_1\left(z(s,x)\right) ds + B_1(x_t) \end{aligned}$$

and $\quad h(t,x_t) = k_0(x)z(t,x)$,
where

$$\begin{aligned} B_2(x_t) &= \int_0^t \int_{-h}^s k(s-\tau) Q_2\left(z(\tau,x)\right) d\tau ds, \\ B_1(x_t) &= \int_0^t \int_{-h}^s k(s-\tau) P_2\left(z(\tau,x)\right) d\tau ds. \end{aligned}$$

Let $A : D(A) \subset X \to X$ be the operator defined by $Az = z''$ with the domain $D(A) = \{z \in X :\ z,\ z'$ are absolutely continuous, $z'' \in X,\ z(0) = z(\pi) = 0\}$. Then,

$$Az = \sum_{n=1}^{\infty} n^2 (z, z_n) z_n,\ z \in D(A),$$

where $z_n(x) = \sqrt{\frac{2}{\pi}}\ \sin nx,\ n = 1, 2, 3, ...$ is the orthogonal set of eigenvectors of $A$. It is well known that $A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $X$ and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} (z, z_n) z_n,\ z \in X.$$

Now, we define the operator $A(t)z = Az(x) + a(t,x)z,\ z \in D(A(t)),\ t \geq 0,\ x \in [0,\pi]$, where $D(A(t)) = D(A),\ t \geq 0$. By assuming that $x \to a(t,x)$ is continuous in $t$ and there exists $\rho > 0$ such that $a(t,x) \leq -\rho$ for all $t \in J,\ x \in [0,\pi]$, it follows that the system

$$\begin{aligned} z'(t) &= A(t)z(t),\ t \geq s, \\ z(s) &= x \in X \end{aligned}$$

generates an evolution system $U(t,s)$ as $U(t,s)y = T(t-s)\exp\left(\int_s^t a(\tau,x)d\tau\right) y$, for $y \in X$, where $T(t)$ is the compact analytic semigroup generated by the operator $A(t)$ and $\|U(t,s)\| \leq e^{-(1+\rho)(t-s)}$ for every $t \geq s$.

With this choice of $A(t), f, g$ and $h$, we see that the equation (6.2) can be written in the abstract formulation of (4.1).

Further, all the conditions of the Theorem 5.1 are satisfied. Hence, by Theorem 5.1, the equation (6.2) has a mild solution on $[0, b]$.

## Acknowledgements

One of the authors K. Shri Akiladevi, is thankful to Dr. S. Gnanavel for his help in preparing the manuscript.

## References

[1] I.M. Ananjevskii and V.B. Kolmanovskii, *Stabilization of some nonlinear hereditary mechanical systems*, Nonlinear Anal., **15** (1990), 101-114.

[2] N. Annapoorani and K. Balachandran, *Existence of solutions of abstract partial neutral integrodifferential equations*, Carpathian J. Math., **26** (2010), 134-145.

[3] O. Arino, R. Benkhalti and K. Ezzinbi, *Existence results for initial value problems for neutral functional differential equations*, J. Differential Equations, **138** (1997), 188-193.

[4] K. Balachandran and J.P. Dauer, *Existence of solutions of a nonlinear mixed neutral equations*, Appl. Math. Lett., **11** (1998), 23-28.

[5] K. Balachandran and R. Sakthivel, *Existence of solutions of neutral functional integrodifferential equations in Banach spaces*, Proc. Indian Acad. Sci. (Math. Sci.), **109** (1999), 325-332.

[6] K. Balachandran, D.G. Park and Y.G. Kwun, *Nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces*, Commun. Korean Math. Soc., **14** (1999), 223-231.

[7] K. Balachandran and J.P. Dauer, *Controllability of nonlinear systems in Banach spaces: A survey*, J. Optim. Theory Appl., **115** (2002), 7-28.

[8] K. Balachandran, J.Y. Park and M. Chandrasekaran, *Nonlocal Cauchy Problem for delay integrodifferential equations of Sobolov type in Banach spaces*, Appl. Math. Lett., **15** (2002), 845-854.

[9] K. Balachandran, J.P. Dauer and S. Karunanithi, *Existence results for abstract neutral integrodifferential equations*, Int. J. Pure Appl. Math., **5** (2003), 29-46.

[10] K. Balachandran, G. Shija and J.H. Kim, *Existence of solutions of nonlinear abstract neutral integrodifferential equations*, Comput. Math. Appl., **48** (2004), 1403-1414.

[11] K. Balachandran, N. Annapoorani and J.K. Kim, *Existence of mild solutions of neutral evolution integrodifferential equations*, Nonlinear Funct. Anal. Appl., **16** (2011), 141-153.

[12] G.I. Barenblatt, I.P. Zheltov and I.N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., **24** (1960), 1286-1303.

[13] H. Brill, *A semilinear Sobolev evolution equation in Banach space*, J. Differential Equations, **24** (1977), 412-425.

[14] P.J. Chen and M.E. Gurtin, *On a theory of heat conduction involving two temperatures*, Z. Angew. Math. Phys., **19** (1968), 614-627.

[15] J.P. Dauer and K. Balachandran, *Existence of solutions of nonlinear neutral integrodifferential equations in Banach spaces*, J. Math. Anal. Appl., **251** (2000), 93-105.

[16] J.K. Hale and S.M. Verduyn Lunel, *An Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993.

[17] E. Hernandez and H.R. Henriquez, *Existence results for partial neutral functional integrodifferential equations with unbounded delay*, J. Math. Anal. Appl., **221** (1998), 452-475.
[18] E. Hernandez and H.R. Henriquez, *Existence of periodic solutions of partial neutral functional differential equations with unbounded delay*, J. Math. Anal. Appl., **221** (1998), 499-522.
[19] E. Hernandez and K. Balachandran, *Existence results for abstract degenerate neutral functional differential equations*, Bull. Aust. Math. Soc., **81** (2010), 329-342.
[20] W.H. Kwon, G.W. Lee and S.W. Kim, *Performance improvement using time delays in multivariable controller design*, Internat. J. Control., **52** (1990), 1455-1473.
[21] J.H. Lightbourne III and S.M. Rankin III, *A partial functional differential equation of Sobolev type*, J. Math. Anal. Appl., **93** (1983), 328-337.
[22] S.K. Ntouyas, Y.G. Sficas and P.Ch. Tsamatos, *Existence results for initial value problems for neutral functional differential equations*, J. Differential Equations, **114** (1994), 527-537.
[23] S.K. Ntouyas and P.Ch. Tsamatos, *Global existence for functional integrodifferential equations of delay and neutral type*, Appl. Anal., **54** (1994), 251-262.
[24] S.K. Ntouyas, *Global existence for neutral functional integrodifferential equations*, Nonlinear Anal., **30** (1997), 2133-2142.
[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[26] H. Schaefer, *Über die methode der* a priori *schranken*, Math. Ann., **129** (1955), 415-416.
[27] R.E. Showalter, *Existence and representation theorem for a semilinear Sobolev equation in Banach space*, SIAM J. Math. Anal., **3** (1972), 527-543.
[28] D. Taylor, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, 1952.
[29] T.W. Ting, *Certain non-steady flows of second order fluids*, Arch. Ration. Mech. Anal., **14** (1963), 1-26.